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TOPOLOGICAL ASPECTS OF DYNAMICAL SYSTEMS ON MANIFOLDS

ТОПОЛОГИЧЕСКИЕ АСПЕКТЫ ДИНАМИЧЕСКИХ СИСТЕМ НА МНОГООБРАЗИЯХ

The necessary and sufficient conditions for existence on manifolds of the dynamical systems having non-wandering set consisting of disconnected union of 2-dimensional tori with hyperbolic structure are given.

Дано необхідні та достатні умови існування на многовидах динамічних систем, у яких множина неблукаючих точок складається з незв'язного об'єднання двовимірних торів з гіперболічною структурою.

Let M^n be a smooth closed manifold and $X(M^n)$ be a set of $\mathbb{C}^r$ vector fields on M^n.

Let X be a vector field from $X(M^n)$ having a finite number of singular points and closed orbits. Suppose that the singular points and closed orbits have a hyperbolic structure. Denote by $V(M^n) \subset X(M^n)$ the set of vector fields on M^n which satisfy these conditions. Denote by $N_i(X)$ ($\overline{N}_i(X)$) the number of singular points (closed orbits) of index i. Consider vector fields X and Y from $V(M^n)$. We say that $X > Y$ (greater than) if:

1) $N_i(X) \geq N_i(Y)$, $\overline{N}_i(X) \geq \overline{N}_i(Y)$ for all i;

2) there exists an index i_0 such that $N_{i_0}(X) \geq N_{i_0}(Y)$ or $\overline{N}_{i_0}(X) \geq \overline{N}_{i_0}(Y)$.

Definition. *A vector field $X \in V(M^n)$ will be called minimal if there exists no vector field $Y \in V(M^n)$ such that $X > Y$.*

In general we may describe the set $V(M^n)$ as a connected oriented graph K. A vertex of K is a set of vector fields $\{X_{j\in J}\} \in V(M^n)$ such that $N_i(X_{j_1}) = N_i(X_{j_2})$ and $\overline{N}_i(X_{j_1}) = \overline{N}_i(X_{j_2})$ for all indices i and $j_1, j_2 \in J$. If $X, Y \in V(M^n)$ and $X > Y$ then the vertex $[X]$ representing the field X and the vertex $[Y]$ representing Y are joined by an arc directed from X to Y: $[X] \to [Y]$.

Let denote by $Q(M^n)$ the number of minimal vector fields on M^n. It is known that $Q(M^n)$ is finite. For any vector field from $V(M^n)$ there exists a function of distribution of critical elements:

$$X \xrightarrow{D} \begin{cases} N_i(X), \\ \overline{N}_i(X). \end{cases}$$

The general expression for this function for a minimal vector field from $V(M^n)$ is unknown. But for the minimal Morse-Smale vector field, the function of distribution can be calculated. For example, if M^n is simply-connected and $n \geq 5$ then for the minimal Morse-Smale vector field without closed orbits, the function of distribution on M^n will be as follows

$$X \to \begin{cases} N_i = b_i + q_i + q_{i-1}, \\ \overline{N}_i = 0. \end{cases}$$

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where $b_i = \operatorname{rank} H_i(M^n, \mathbb{Q})$, $q_i = \mu(\operatorname{tors}(H_i(M^n, \mathbb{Z})))$ ($\mu(H)$ is the number of generators of the group H).

Let's consider the case, when the minimal Morse-Smale vector field have only closed orbits. Let $\chi_i(M^n) = \sum_{j=1}^{i} (-1)^{i+j} \operatorname{rank} H_j(M^n, \mathbb{Q})$. We say that the dimension of the manifold M^n is singular if:

$$\chi_{i-1}(M^n) = \chi_{i-1}(M^n) = 0, \quad \chi_i(M^n) = k > 0 \text{ and}$$

$$H_i(M^n, \mathbb{Z}) = \underbrace{\mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2}_{k} \oplus \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_s},$$

where $k_{l_1} > 0$ and k_{l_i} divides $k_{l_{i+1}}$.

Theorem 1. *Let M^n be a closed manifold ($n \geq 5$), $\pi_1(M^n) = 0$. Let's consider on M^n the minimal Morse-Smale vector field X without singular points. Then the function of distribution of X is as follows*

$$D \to \begin{cases} N_i(X) = 0, \\ \overline{N}_i(X) = \rho(\chi_i(M^n)), \text{ if } i \text{ is a nonsingular dimension,} \end{cases}$$

and

$$D \to \begin{cases} N_i(X) = 0, \\ \overline{N}_j(X) = \rho(\chi_j(M^n)) \quad \text{for all} \quad j \neq i-1, \ i+1, \\ \overline{N}_{i-1}(X) = \rho(\chi_{i-1}(M^n)) + 1, \\ \overline{N}_{i+1}(X) = \rho(\chi_{i+1}(M^n)), \end{cases}$$

or

$$D \to \begin{cases} N_i(X) = 0, \\ \overline{N}_j(X) = \rho(\chi_j(M^n)) \quad \text{for all} \quad j \neq i-1, \ i+1, \\ \overline{N}_{i-1}(X) = \rho(\chi_{i-1}(M^n)), \\ \overline{N}_{i+1}(X) = \rho(\chi_{i+1}(M^n)) + 1, \end{cases}$$

if i is a singular dimension, where $\rho(N) = ((N + |N|)/2, N \in \mathbb{Z})$ [1].

.Let M^n be a smooth compact manifold and X be a vector field on M^n. Let $\Omega(X)$ be the set of non-wandering points. The field X admits a Lyapunov's function if there exists such a smooth function $f: M^n \to \mathbb{R}$, that $D(f)_p = 0$, where $p \in \Omega(X)$, $X(f)_p > 0$ for $p \in M^n \setminus \Omega(X)$. Suppose $\Omega(X)$ consists of a disjoint union of k-dimensional tori ($k \geq 2$) with the hyperbolic structure. The restriction of the tangent bundle of M^n to $\Omega(X)$ can be represented as a continuous decomposition into subbundles E^s, E^u, $T(M^n)/\Omega(X) = E^s \oplus E^u$. This decomposition is invariant under the action of the flow X_t of X. Moreover, there are Riemannian metric on M^n and constants $\lambda \in (0,1)$ and C, for which $|X_t(V)| \leq C\lambda^t$, $v \in E^s$, $t \geq 0$; $|X_t(W)| \leq C\lambda^{-t}$, $w \in E^u$, $t \geq 0$. The number S is called the index of a connected component $\Omega_i \in \Omega(X)$, $\Omega(X) = \bigcup_i \Omega_i$ [2].

A function $f: M^n \to \mathbb{R}^1$ will be called the Morse-Bott function if its set of singular points $\Sigma(f)$ consists of a disjoint union of smooth submanifolds $\Sigma(f) = \bigcup_j \Sigma_j$ and, for any arbitrary point $p \in \Sigma$ the restriction of f a small disk transversal to Σ_j in p, is a Morse function, index of which will be called the index of Σ_j.

Theorem 2. *Let M^n $(n \geq 5)$ be a smooth compact manifold. The vector field X on M^n admits a Lyapunov function and has a set of non-wandering points consisting of a disjoint union of 2-dimensional tori with a hyperbolic structure if X satisfies the condition of absence of cycles on $\Omega(X)$ and ranks b_i of homology groups of M^n satisfy the equations:*

$$\sum_{i=1}^{n} (-1)^{i+1}\, i\, b_{n-i} = 0, \quad \sum_{i=2}^{n} (-1)^{i} (i-1) b_{n-i} = 1.$$

For a Lyapunov function, we can choose a Morse-Bott function having indices of critical 2-dimensional tori coinciding with indices of these tori considered as a connected composition of $\Omega(X)$.

Denote by $L_{T^2}(M^n)$ a set of vector fields on M^n satisfying conditions of this theorem and having a finite number of connected components of $\Omega(X)$. Let $\tilde{N}_i(X)$ – the number of tori of index i. Let vector fields X and Y belong to $L_{T^2}(M^n)$. We say that $X > Y$ if $\tilde{N}_i(X) \geq \tilde{N}_i(Y)$ and there exists an index i_0 such that $\tilde{N}_{i_0}(X) > \tilde{N}_{i_0}(Y)$. If, for the vector field X from $L_{T^2}(M^n)$ exist a vector field Y from $L_{T^2}(M^n)$ such that $X > Y$, we will say that X is a minimal vector field.

Theorem 3. *Let M^n $(n \geq 5)$ be a closed simply-connected manifold and $H_k(M^n, \mathbb{Z})$ be a free group for any k. Then if*

$$\sum_{j=0}^{i} (-1)^{i+j} (i+1-j)\, b_j \geq 0$$

for all i, then there exists on M^n a unique minimal vector field from $L_{T^2}(M^2)$. The number of 2-dimensional tori of index i from $\Omega(X)$ is equal to

$$\tilde{N}_i = \sum_{j=0}^{i} (-1)^{i+j} (i+1-j)\, b_j, \quad b_k = \operatorname{rank} H_k(M^n, \mathbb{Z}).$$

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