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PROPERTIES OF RESTRICTIONS OF AN OPERATOR OF MULTIPLICATION BY A CONTINUOUS FUNCTION

For an operator A of multiplication by a continuous function $a(t)$ in the Hilbert space $L_2[0, b] = H$, we give a description of two sets of infinite-dimensional subspaces which have infinite codimensions: $I(A) = \{N \subset H\colon A/N$ is an isomorphism$\}$, $K(A) = \{M \subset H\colon A/M$ is a compact mapping$\}$. As an application, we consider the problem of determining whether the sequence $\{a(t)e_n(t)\}$, where $\{e_n(t)\}$ is an orthonormal basis in $L_2[0, b]$, will be unconditional basis.

Для оператора A множення на неперервну функцію $a(t)$ в просторі $L_2[0, b] = H$ дано опис двох множин нескінченновимірних підпросторів нескінченної корозмірності: $I(A) = \{N \subset H\colon$ A/N — ізоморфізм$\}$, $K(A) = \{M \subset H\colon A/M$ — компактне відображення$\}$. Як приклад розглянуто питання про безумовну базисність послідовності $\{a(t)e_n(t)\}$, де $\{e_n(t)\}$ — ортонормована послідовність в $L_2[0, b]$.

Let H be a Hilbert space and $N \subset H$ a closed linear subspace of H. We will say that N is an essential subspace of H if $\dim N = \operatorname{codim} N = \infty$. By $S(N)$ we denote the unit sphere of N.

Let A be a linear bounded operator acting in H. Suppose that A is non-compact and it has the non-closed range

$$R(A) = \{y \in H : y = Ax \ \text{ for some } \ x \in H\}.$$

It is well known that there exists an essential subspace N of H such that the restriction A/N of A on N is an isomorphism, i.e. $\inf\limits_{x \in S(N)} \|Ax\| > 0$ ([1]).

We will denote the set of essential subspaces with this property by $I(A)$. Also, we introduce into consideration the set of subspaces

$$K(A) = \{M : M \subset H \ \text{ is an essential subspace of } H \text{ such that } A/N \text{ is compact}\}.$$

$K(A) \neq \varnothing$ (see [2, p. 48]). We will describe the set $I(A)$ and $K(A)$ for an operator of multiplication by a continuous function $a(t)$ ($t \in [0, b]$) acting in the Hilbert space $L_2[0, b]$. In this case we will also show that some given subspaces belong to $I(A)$. This allows to conclude that the sequence $\{a(t)e_n(t)\}$ is an unconditional basic sequence if $\{e_n(t)\}$ is an orthogonal basis of $N \in I(A)$. Acting in this manner we will investigate the basic properties of the sequences $\{a(t)\cos 2\pi nt\}$ and $\{a(t)\sin 2\pi nt\}$ in the space $L_2[0, 1]$.

1. Let $a(t)$ ($t \in [0, b]$) be a continuous function such that the closed set

$$\gamma(a(t)) = \{t \in [0, b] : a(t) = 0\}$$

is non-empty. Then the linear bounded operator

$$(Af)(t) = a(t)f(t)$$

is non-compact and it has non-closed range in the Hilbert space $L_2[0, b]$. Let σ be a closed subset of $[0, b]$ with nonzero Lebesgue measure and $\varphi_\sigma(t)$ the characteristic function of σ. We denote by N_σ a subspace of $L_2[0, b]$,

$$N_\sigma = \{g(t) = \varphi_\sigma f(t) : f(t) \in L_2[0, b]\}.$$

It is obvious that if $\sigma \subset [0, b] \setminus \gamma(a(t)) = \operatorname{supp} a(t)$, then $N_\sigma \in I(A)$. Using the set of subspaces N_σ we describe the set $I(A)$ in the whole. Directly from the definitions we derive.

Proposition 1. *The following are equivalent*

1. $N \in I(A)$.

2. *There exists a closed subset* $\sigma \subset \operatorname{supp} a(t)$ *such that for some* $\varepsilon > 0$ *and every* $f \in S(N)$,

$$\| \varphi_\sigma(t) \| > \varepsilon.$$

3. *There exists a closed subset* $\sigma \subset \operatorname{supp} a(t)$ *such that the orthogonal projection* P_σ *onto subspace* N_σ *induces an isomorphism on* N.

2. Now, we consider the subspace $N = [\cos 2\pi nt]_{n=1}^{\infty}$ ($N = [\sin 2\pi nt]_{n=1}^{\infty}$) generated by the sequence $\{\cos 2\pi nt\}_1^{\infty}$ ($\{\sin 2\pi nt\}_1^{\infty}$) in the space $L_2[0,1]$. For which function $a(t)$ does N belong to $I(A)$?

Proposition 2. *Let* $N = [\cos 2\pi nt]_{n=1}^{\infty} \subset L_2[0,1]$ ($N = [\sin 2\pi nt]_{n=1}^{\infty} \subset L_2[0,1]$). $N \in I(A)$ *if and only if for every* $t_0 \in \gamma(a(t))$ *such that* $0 \le t < 1/2$ *the following condition holds*

$$a(1 - t_0) \neq 0. \tag{1}$$

(If $a(t_0) \neq 0$ *for every* $0 \le t_0 < 1/2$, *then we have to understand* (1) *in such a way that* $a(t)$ *may vanish at any point* $[1/2, 1]$).

***Corollary* 1.** *Let* $a(t)$ ($t \in [0,1]$) *be a continuous function such that the condition* (1) *holds. Then the sequence* $\{a(t)\cos 2\pi nt\}_{n=1}^{\infty}$ ($\{a(t)\sin 2\pi nt\}_{n=1}^{\infty}$) *is an unconditional basic sequence in* $L_2[0,1]$.

***Remark* 1.** Specifically, the sequence $\{t\cos 2\pi nt\}_{n=1}^{\infty}$ ($\{t\sin 2\pi nt\}_{n=1}^{\infty}$) is an unconditional basic sequence in $L_2[0,1]$. In contrast with the properties of the sequence $\{te_n(t)\}$, where $\{e_n(t)\}_{n=1}^{\infty}$ ($e_1(t) = 1$, $e_2(t) = \cos 2\pi t$, $e_3(t) = \sin 2\pi t, \ldots$) is the trigonometric basis of $L_2[0,1]$. Namely, it is easy to show that $\{te_n(t)\}$ is even not a deficient minimal sequence in $L_2[0,1]$ (i.e. it cannot be transformed into a minimal sequence by omitting a finite number of its elements, [3, p. 121]).

3. Let us consider the operator T of multiplication by $a(t) = t$ in $L_2[0,1]$,

$$(Tf)(t) = tf(t).$$

Let $\{e_n(t)\}_{n=1}^{\infty}$ be the trigonometric basis in $L_2[0,1]$, $\{n_k\}$ a subsequence of the sequence of natural numbers. It is reasonable to ask for which $\{n_k\}$ the subspace $N = [e_{n_k}]_{k=1}^{\infty}$ belongs to $I(T)$. We have noted that if $n_k = 2k - 1$, or $n = 2k$ ($k = 1, 2, \ldots$), then $N \in I(T)$. There is another class of subspaces which belong to $I(T)$. We recall that a sequence $\{n_k\}$ of natural numbers is called lacunary if

$$\inf_k \frac{n_{k+1}}{n_k} = \lambda > 1.$$

Proposition 3. *Let* $\{n_k\}$ *be a lacunary sequence of trigonometric basis* $\{e_n(t)\}$ *in* $L_2[0,1]$. *Then* $N = [e_{n_k}]_{k=1}^{\infty} \in I(T)$.

4. Let $\{\alpha_n\} \subset [0,1]$, $\lim\limits_{n \to \infty} \alpha_n = 0$, be a decreasing sequence of numbers and $\{f_n(t)\} \subset L_2[0,1]$ a sequence of functions such that

$$\|f_n\| = 1, \text{ and } f_n(t) = 0, \text{ if } t \in [\alpha_{n+1}, \alpha_n]. \quad (2)$$

Put $M = [f_n]_{n=1}^{\infty}$. It is easy to see that $M \in K(T)$, where T is the operator of multiplication by t, i.e. $(Tf)(t) = tf(t)$. It appears that a description of the set $K(T)$ can be obtained by using sequences possessing property (2).

Proposition 4. *Let M be an essential subspace of $L_2[0, 1]$. The following are equivalent:*

1. $M \in K(T)$.

2. *For every orthogonal sequences* $\{e_n(t)\} \subset M$

$$\lim_{n\to\infty} \int_{\delta}^{1} |e_n(t)|^2 dt = 0,$$

where δ is any number such that $0 < \delta < b$.

3. *For every orthogonal sequences $\{e_n(t)\} \subset M$ there exists a sequence of functions $\{f_k\}$ possessing property* (2) *such that, for some subsequence $\{e_{n_k}(t)\}$ of $\{e_n(t)\}$,* $\lim_{k\to\infty} \| e_{n_k} - e_k \| = 0$.

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