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MEASURE-VALUED DIFFUSION

МІРОЗНАЧНА ДИФУЗІЯ

We consider the class of continuous measure-valued processes $\{\mu_t\}$ on a finite dimensional Euclidian space X for which $\int f d\mu_t$ is a semimartingale with absolutely continuous characteristics with respect to t for all $f: X \to R$ smooth enough. It is shown that, under some general condition, the Markov process with this property can be obtained as a weak limit for systems of randomly interacting particles that are moving in X along the trajectories of a diffusion process in X as the number of particles increases to infinity.

Розглядається клас неперервних мірозначних процесів $\{\mu_t\}$ на скінченновимірному евклідовому просторі X, для якого $\int f d\mu_t$ — семімартингал з характеристикою, що є абсолютно неперервною відносно t для всіх досить гладких $f: X \to R$. Показано, що при досить загальних умовах марковський процес з цією властивістю може бути отриманий як слабка границя для систем випадково взаємодіючих частинок, що рухаються в X уздовж траєкторій дифузійного процесу в X, коли число частинок зростає до нескінченності.

1. Introduction. The theory of measure-valued stochastic processes was founded by D. A. Dawson [1]. Measure-valued branching processes are the most investigated class of measure-valued processes (see E. B. Dynkin [2]). Some limit theorems on the convergence of measure-valued processes generated by systems of randomly interacting particles were obtained by A. V. Skorokhod [3] and P. Kotelenez [4]. In the article of R. Ya. Maydaniuk and A. V. Skorokhod, quasidiffusion measure-valued processes and limit theorems for such processes were considered.

We consider a finite dimensional space X. Let $\mathcal{B}$ be its Borelian σ-algebra, let $M(X)$ be the space of finite measures on $\mathcal{B}$, and let $C(X)$ be the space of bounded continuous functions $f: X \to R$.

We introduce a metric d_M in $M(X)$ with the following properties:

1) $M(X)$ in this metric is a complete separable locally compact space,

2) $d(m_n, m) \to 0$ iff the sequence of measures m_n weakly converges to the measure m, i.e., $\int f dm_n \to \int f dm$ for all $f \in C(X)$.

We use the notation $\langle m, f\rangle = \int f dm$.

Denote by $\mathcal{M}$ the σ-algebra of Borelian subsets of $M(X)$. We will investigate a special class of continuous Markov processes in the space $(M(X), \mathcal{M})$.

First, we recall the notion of quasidiffusion process introduced in [5].

***Definition* 1.** *A continuous Markov process* μ_t *in* $(M(X), \mathcal{M})$ *is called a quasidiffusion process if there exists a linear subset* $D \subset C(X)$ *satisfying the following conditions:*

QD1) D *is dense in* $C(X)$,

QD2) *for* $f \in D$, *the process* $\langle f, \mu_t\rangle$ *is a continuous semimartingale (with respect to the filtration* $(\mathcal{F}_t)_{t\ge 0}$ *generated by the Markov process), for which its compensator, denote it by* $a(t,f)$, *and the square characteristic of the martingale* $m(t,f) = \langle f, \mu_t\rangle - a(t,f)$, *denote it by* $b(t,f)$, *are absolutely continuous functions with respect to* t:

$$a(t,f) = \int_0^t A(s,f)\,ds, \tag{1}$$

$$b(t,f) = \int B(s,f)\,ds, \tag{2}$$

where $A(s,f)$ *and* $B(s,f)$ *are nonrandom functions of* μ_s.

***Definition* 2.** *A quasidiffusion process* μ_t *in* $(M(X), \mathcal{M})$ *is called a diffusion process if* $D = C^{(2)}(X)$ *is the space of twice continuously differentiable functions from* $C(X)$ *with derivatives that are also from* $C(X)$; *besides, we assume that the functions* $A(s,f)$ *and* $B(s,f)$ *in relations* (1) *and* (2) *are of the form*

$$A(s,f) = A(s,\mu_s,f) =$$

$$= \int [\operatorname{Tr} A_2(s,\mu_s,x)f''(x) + (A_1(s,\mu_s,x), f'(x)) + A_0(s,\mu_s,x)f(x)]\,\mu_s(dx), \tag{3}$$

$$B(s,f) = B(s,\mu_s,f) =$$

$$= \int [(B_2(s,\mu_s,x,\tilde{x})f'(x), f'(\tilde{x})) + (B_1(s,\mu_s,x,\tilde{x}), f'(x))f(\tilde{x}) +$$

$$+ B_0(s,\mu_s,x,\tilde{x})f(x)f(\tilde{x})]\,\mu_s(dx)\mu_s(d\tilde{x}), \tag{4}$$

where

$$A_2\colon R_+\times M(X)\times X \to L(X), \quad B_2\colon R_+\times M(X)\times X^2 \to L(X),$$

$$A_1\colon R_+\times M(X)\times X \to X, \quad B_2\colon R_+\times M(X)\times X^2 \to X,$$

$$A_0\colon R_+\times M(X)\times X \to R, \quad B_2\colon R_+\times M(X)\times X^2 \to R;$$

these functions are measurable on $R_+\times M(X)$ and continuous and bounded in x and $\tilde{x}$.

The goal of the article is to establish the existence of a process for A_0, A_1, A_2, B_0, B_1, and B_2 satisfying some additional conditions. The tool of investigation is some limit theorem for measure-valued continuous processes and processes generated by a system of randomly moving particles in the space X.

2. Compactness of probability distributions in $C_{[0,1]}(M(X))$. Here, we consider conditions under which the distributions for a sequence $\{\mu_n(t), t\in[0,1], n=1,2,\ldots\}$ of continuous measure-valued processes is a compact set in the space $C_{[0,1]}(M(X))$ of all continuous functions $m\colon [0,1]\to M(X)$.

Theorem 1. *The distributions of the processes* $\{\mu_n(t)\}$ *form a compact set iff the following conditions are fulfilled:*

1) *there exists a continuous function* $\rho\colon X\to[1,\infty)$ *for which* $\rho(x)\to+\infty$ *as* $|x|\to+\infty$ *and* $\sup_t\langle\rho,\mu_n(t)\rangle$ *are uniformly bounded in probability;*

2) *for any* $f\in C(X)$ *and* $\varepsilon>0$,

$$\lim_{h\to 0}\sup_n P\left\{\sup_{|t-s|\le h}|\langle\mu_n(s),f\rangle - \langle\mu_n(t),f\rangle| > \varepsilon\right\} = 0.$$

Proof. The necessity of condition 2) follows from the necessary conditions for compactness of the distributions of the processes $\langle\mu_n(t),f\rangle$ in $C_{[0,1]}$ for all $f\in C(X)$. We prove now the necessity of condition 1).

It follows from the compactness of the distributions of processes $\mu_n(\cdot)$ that, for any $\varepsilon>0$ and $\delta>0$, there exists $r>0$ for which

$$P\left\{\sup_t \mu_n(t, X \setminus B_r(0) > \varepsilon\right\} < \delta, \quad B_r(0) = \{x: |x| \le r\},$$

for all n. If r_k satisfies the inequality

$$P\left\{\sup_t \mu_n(t, X \setminus B_{r_k}(0) > \frac{1}{2^k}\right\} < \frac{1}{2^k}$$

for all n and $\rho(x) \le k$ for $|x| \le r_k$, then 1) is fulfilled.

To prove the sufficiency of conditions 1) and 2), we consider measures $\mu_n^*(t)$ for which $\langle \mu_n^*(t), \lambda \cdot f\rangle$, where $\lambda \in C(X)$, $\lambda(x) = 1$ for $x \in B_r(0)$, $\lambda(x) = 0$ for $x \in X \setminus B_{2r}(0)$, $\lambda(x) \in [0, 1]$ is valid.

It is easy to check that the distributions of $\mu_n^*(t)$, $n = 1, 2, \ldots$, form a compact set of measures because $\mu_n^*(t)$ are measures on the compact $B_{2r}(0)$. Using condition 1), we can choose λ in such way that

$$P\left\{\sup_t \operatorname{Var}\left(\mu_n(t) - \mu_n^*(t)\right) > \varepsilon\right\} < \delta$$

for given $\varepsilon > 0$ and $\delta > 0$. This completes the proof.

Corollary. *Let $\{\mu_n(t)\}$ be a sequence of measure-valued diffusion processes for which their diffusion characteristics $A^n(s, f)$ and $B^n(s, f)$ are represented by formulas* (3) *and* (4) *with functions A_0^n, A_1^n, A_2^n and B_0^n, B_1^n, B_2^n instead of A_0, A_1, A_2 and B_0, B_1, B_2. Assume that the following conditions are fulfilled:*

(i) the distributions of $\mu_n(o)$ are compact in $M(X)$;

(ii) there exists a continuous function $\rho: X \to [1, \infty)$ such that $\rho(x) \to \infty$ as $|x| \to \infty$, and $\rho''(x)$ are continuous and bounded and

$$\int [\operatorname{Tr} A_2^n(s, m, x)\rho''(x) + (A_1^n(s, m, x), \rho'(x)) +$$

$$+ A_0^n(s, m, x)\rho(x)]\, m(dx)\langle m, f\rangle + \iint [(B_2^n(s, m, x, \tilde{x})\rho'(x), \rho'(x)) +$$

$$+ (B_1^n(s, m, x, \tilde{x}), \rho'(x))\rho(\tilde{x}) +$$

$$+ B_0^n(s, m, x, \tilde{x})\rho(x)\rho(\tilde{x})]\, m(dx) m(d\tilde{x}) \le C\langle m, \rho\rangle^2$$

for all $s \in R_+$ and $m \in M(X)$; here, C is a constant;

(iii) for any $\varphi \in C^{(2)}(X)$, there exists a constant C_φ for which

$$|A^n(s, m, \varphi)| \le C_\varphi\langle m, \rho\rangle, \quad |B^n(s, m, \varphi)| \le C_\varphi\langle m, \rho\rangle^2.$$

Then the distributions of $\mu_n(\cdot)$ are compact in $C_{[0,1]}(M(X))$.

3. Diffusion processes generated by a system of randomly moving particles. Let $x_1(t), \ldots, x_N(t)$ be a stochastic continuous X-valued process; we refer to $x_k(t)$ as to the trajectory of the kth particle of a system of N particles that are moving in X. We assume that $x_k(t)$ is a semimartingale:

$$x_k(t) = x_k(0) + \alpha_k(t) + m_k(t), \tag{5}$$

where $\alpha_k(t)$ is absolutely continuous with respect to t and the square characteristic of the martingale $m_k(t)$, denote it by $\beta_k(t)$, is also absolutely continuous with respect to t. Note that $\alpha_k(t)$ is an X-valued process and $\beta_k(t)$ is an $L(X)$-valued process. Denote by $\beta_{kj}(t)$ the mutual characteristic of the martingales $m_k(t)$ and $m_j(t)$. We have the relations

$$\alpha_k(t) = \int_0^t A_k(s)\,ds, \tag{6}$$

$$\beta_{kj}(t) = \int_0^t B_{kj}(s)\,ds; \tag{7}$$

the functions $A_k(s)$ and $B_{kj}(s)$ are supposed to be adapted and predictable with respect to some filtration $(\mathcal{F}_s)_{s\geq 0}$.

Introduce the measure μ_t^N determined by relation

$$\langle \mu_t^N, f\rangle = \frac{1}{N}\sum_{k=1}^N f(x_k(t)), \tag{8}$$

$f \in C(X)$. Using Itô's formula for $f \in C^{(2)}(X)$, we can write

$$d\langle \mu_t^N, f\rangle = \frac{1}{N}\sum_{k=1}^N \left[\left(f'(x_k(t)), A_k(t)\right) + \frac{1}{2}\mathrm{Tr}\, f''(x_k(t))B_{kk}(t)\right]dt +$$

$$+\frac{1}{N}\sum_{k=1}^N \left(f'(x_k(t)), dm_k(t)\right). \tag{9}$$

Denote

$$\hat{A}(t,f) = \frac{1}{N}\sum_{k=1}^N \left[\left(f'(x_k(t)), A_k(t)\right) + \frac{1}{2}\mathrm{Tr}\, f''(x_k(t))B_{kk}(t)\right].$$

Then

$$m(f,t) = \langle \mu_t^N, f\rangle - \int_0^t \hat{A}(s,f)\,ds \tag{10}$$

is a martingale and its square characteristics is

$$\langle m(f,\cdot)\rangle_t = \int_0^t \hat{B}(s,f)\,ds = \int_0^t \frac{1}{N^2}\sum_{k,j=1}^N \left(B_{kj}(s)f'(x_k(s)), f'(x_j(s))\right)ds. \tag{11}$$

Assume that

$$A_k(t) = \tilde{A}\left(t, \mu^N(t), x_k(t)\right),$$

$$B_{kj}(t) = \tilde{B}\left(t, \mu^N(t), x_k(t), x_j(t)\right),$$

where

$$\tilde{A} : R_+ \times M(X) \times X \to X,$$

$$\tilde{B} : R_+ \times M(X) \times X^2 \to L(X).$$

Then μ_t^N is a measure-valued diffusion process for which

$$A_0(t, m, x) = 0, \quad A_1(t, m, x) = \tilde{A}(t, m, x), \quad A_2(t, m, x) = \tilde{B}(t, m, x, x),$$

$$B_0(t, m, x, \tilde{x}) = 0, \quad B_1(t, m, x, \tilde{x}) = 0, \quad B_2(t, m, x, \tilde{x}) = \tilde{B}(t, m, x, \tilde{x}).$$

Under which conditions does there exist a system $\{x_1(t), \ldots, x_N(t)\}$ that satisfies the required conditions?

The necessary condition is:

I. $\tilde{B}(t, m, x, \tilde{x})$ is a symmetric operator, $\tilde{B}(t, m, x, \tilde{x}) = \tilde{B}(t, m, \tilde{x}, x)$, and, for any $z_1, \ldots, z_N \in X$ and $x_1, \ldots, x_N \in X$,

$$\sum_{i,j=1}^{N} (\tilde{B}(t, m, x_i, x_j) z_i, z_j) \geq 0. \tag{12}$$

It follows from (12) that there exists a set of operators $\Lambda_{ik} = \Lambda_{ik}(t, m, x_1, \ldots, x_N)$, $i, k \in \overline{1, N}$, for which

$$\Lambda_{ik} = \Lambda^*_{ki}, \qquad \sum_{k=1}^{N} \Lambda_{ik} \Lambda_{ki} = \tilde{B}(t, m, x_i, x_j).$$

We assume that, in addition to condition I, the following condition holds:

II. There exists $\delta > 0$ for which

$$\sum_{i,j>1} (\tilde{B}(t, m, x_i, x_j) z_i, z_j) \geq \delta \sum_{i=1}^{N} (z_i, z_i) \tag{13}$$

for all $t \in R_+$, $m \in M(X)$, and $x_1, \ldots, x_N \in M(X)$, and the function

$$F(t, x_1, \ldots, x_N, x_{N+1}, x_{N+2}) = \tilde{B}\left(t, \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k}, x_{N+1}, x_{N+2}\right), \tag{14}$$

where δ_x is the measure for which $\langle \delta_x, f \rangle = f(x)$, satisfies the conditions

a) $$\| F(t, x_1, \ldots, x_N, x_{N+1}, x_{N+2}) \| \leq K\left(1 + \sum_{k=1}^{N+2} |x_k|^2\right), \tag{15}$$

where $K > 0$ is a constant;

b) for any $C > 0$, there exists $l_c > 0$ for which

$$\| F(t, x_1, \ldots, x_N, x_{N+1}, x_{N+2}) - F(t, \tilde{x}_1, \ldots, \tilde{x}_N, \tilde{x}_{N+1}, \tilde{x}_{N+2}) \| \leq$$

$$\leq l_C \sum_{k=1}^{N+2} |x_k - \tilde{x}_k|, \tag{16}$$

if $|x_k| \leq C$, $k = 1, 2, \ldots, N+2$.

III. The function

$$G(t, x_1, \ldots, x_N, x_{N+1}) = \tilde{A}\left(t, \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k}, x_{N+1}\right) \tag{17}$$

satisfies the conditions

c) $$| G(t, x_1, \ldots, x_N, x_{N+1}) | \leq K\left(1 + \sum_{k=1}^{N+1} |x_k|\right), \tag{18}$$

d) $$|G(t, x_1, \ldots, x_N, x_{N+1}) - G(t, \tilde{x}_1, \ldots, \tilde{x}_N, \tilde{x}_{N+1})| \le l_C \sum_{k=1}^{N+1} |x_k - \tilde{x}_k| \quad (19)$$

if $|x_k| \le C$, $k = 1, 2, \ldots, N+1$, and K and l_C are the same as in II.

It follows from M. Freidlin [6] that, under condition II, the function $\Lambda_{ik}(t, m, x_1, \ldots, x_N)$ satisfies the conditions:

e) there exists a constant K_1 for which

$$\left\| \Lambda_{ij}\left(t, \frac{1}{N}\sum_{k=1}^{N} \delta_{x_k}, x_1, \ldots, x_N\right)\right\| \le K_1\left(1 + \sum_{k=1}^{N}|x_k|\right), \quad i, j \in \overline{1, N}, \quad (20)$$

f) for any C, there exists a constant $\tilde{l}_C$ for which

$$\left\| \Lambda_{ij}\left(t, \frac{1}{N}\sum_{k=1}^{N} \delta_{x_k}, x_1, \ldots, x_N\right) - \Lambda_{ij}\left(t, \frac{1}{N}\sum_{k=1}^{N} \delta_{\tilde{x}_k}, \tilde{x}_1, \ldots, \tilde{x}_N\right)\right\| \le$$

$$\le \tilde{l}_C \sum_{k=1}^{N} |x_k - \tilde{x}_k|, \quad i, j \in \overline{1, N}, \quad (21)$$

if $|x_k| \le C$, $k = 1, 2, \ldots, N$.

Let $w_1, \ldots, w_N$ be independent Wiener processes in X, let $Ew_k(t) = 0$, and let $E(w_k(t), z)^2 = |z|^2$, $z \in X$, $k = 1, \ldots, N$.

We consider the system of stochastic differential equations

$$dx_k(t) = G(t, x_1(t), \ldots, x_N(t), x_k(t))\,dt +$$

$$+ \sum_{i=1}^{N} \Lambda_{ki}\left(t, \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j(t)}, x_1(t), \ldots, x_N(t)\right) dw_k(t). \quad (22)$$

It follows from conditions c), d), e), and f) that system (22) has a unique solution for any initial condition.

Theorem 2. *Let the functions* $\tilde{A}(t, m, x)$ *and* $\tilde{B}(t, m, x, \tilde{x})$ *satisfy conditions* I, II, *and* III *for any* $N \ge 1$. *Then for any* $m \in M(X)$, *there exists a measure-valued diffusion process* μ_t *for which*

$$A_0(t, m, x) = B_0(t, m, x, \tilde{x}) = 0,$$

$$A_1(t, m, x) = \tilde{A}(t, m, x), \quad B_1(t, m, x, \tilde{x}) = 0,$$

$$A_2(t, m, x) = \tilde{B}(t, m, x, x), \quad B_2(t, m, x, \tilde{x}) = \tilde{B}(t, m, x, \tilde{x}).$$

Proof. We consider the sequence μ_t^N of measure-valued processes for which

$$\langle \mu_t^N, f\rangle = \frac{1}{N}\sum_{k=1}^{N} f(x_k^N(t)),$$

where $x_1^N(t), \ldots, x_N^N(t)$ is the solution of system (22) with the initial conditions $x_1^N(0), \ldots, x_N^N(0)$ for which

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(x_k^N(0)) = \langle m, f\rangle$$

for $f \in C(X)$. It follows from Theorem 1 and the corollary that distributions of $\{\mu_t^N, N = 1, 2, \ldots\}$ are compact in $C_{[0,1]}(M(X))$.

4. Pure birth and death processes. We assume that

$$\mu_t(A) = \int_A u(t, x) m(dx)$$

and $u(t, x)$ is a random function from $R_+ \times X$ to R_+ satisfying the following conditions:

4.1. $u(t, x)$ is measurable in t and x; it is a continuous semimartingale in t;

4.2.
$$du(t, x) = v(t, x) dt + dw(x, t),$$

$$v(t, x) = \tilde{V}(t, u(t, \cdot), x),$$

$w(x, t)$ is a continuous martingale in t, and the mutual square characteristic for $w(x, t)$ and $w(\tilde{x}, t)$ is

$$\int_0^t \tilde{W}(s, u(s, \cdot), x, \tilde{x}) ds.$$

We assume that the functions $\tilde{V}$ and $\tilde{W}$ can be represented in the form

$$\tilde{V}(t, u(t, \cdot), x) = A_0\Big(t, \int u dm, x\Big), \qquad \tilde{W}(t, u(t, \cdot), x, \tilde{x}) = B_0\Big(t, \int u dm, x, \tilde{x}\Big), \tag{23}$$

where $\int u dm$ denotes the measure

$$\int u dm(C) = \int_C u dm.$$

It is easy to check that μ_t is a measure-valued diffusion process for which the functions A_0 and B_0 satisfy relation (23) and $A_1 = B_1 = 0$, $A_2 = B_2 = 0$.

The processes of general forms can be constructed by combination of the processes considered in Secs. 3 and 4.

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