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ON HAUSDORFF–YOUNG INEQUALITIES FOR QUANTUM FOURIER TRANSFORMATIONS*

ПРО НЕРІВНОСТІ ХАУСДОРФА–ЮНГА ДЛЯ КВАНТОВОГО ПЕРЕТВОРЕННЯ ФУР'Є

The classical Hausdorff–Young inequality for the Fourier transformation is generalized to various quantum contexts involving noncommutative L^p-spaces based on translation invariant traces.

Класична нерівність Хаусдорфа–Юнга для перетворення Фур'є узагальнюється в багатьох квантових контекстах, що включають некомутативні L^p-простори, засновані на інваріантних відносно зсуву слідах.

1. Introduction. The classical Hausdorff–Young inequality [1] may be regarded as the estimate

$$\| \mathcal{F} \|_{q \to p} \leq (2\pi)^{1/2 - q^{-1}} \tag{1}$$

for the bound of the Fourier transformation

$$\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R}); \quad \mathcal{F} f(x) = (2\pi)^{-1/2} \int e^{-ixy} f(y)\, dy$$

regarded as an operator form $L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ to $L^p(\mathbb{R})$, where $1 \leq q \leq 2$ and $p^{-1} + q^{-1} = 1$. It is known that the estimate is not exact [2] and that the exact bound is achieved by Gaussian functions and (when $q < 2$) only by these [3].

The inequality (1) can be proved [3] as an application of the Riesz–Thorin interpolation principle, starting with Fourier–Plancherel isometry property

$$\| \mathcal{F} \|_{2 \to 2} = 1 \tag{2}$$

and the obvious estimate

$$\| \mathcal{F} \|_{1 \to \infty} \leq (2\pi)^{-1/2}. \tag{3}$$

This paper concerns noncommutative analogs of this Fourier transformation in which the rôle of the translation-invariant (Lebesgue) integral on either the initial or final space, or both, is played by a "translation-invariant trace" on a von Neumann algebra generated by a representation of quantum mechanical commutation relations or a generalization thereof. We shall see that, in all cases, there is an isometry property analogous to (1) for an appropriate normalization of the trace, and an estimate analogous to (2) allowing deduction of a Hausdorff–Young inequality from the abstract Calderon–Lyons form of the interpolation principle.

The simplest case is that of the so-called Fourier–Weyl transformation [4, 5]; this is considered in Sec. 2 together with its inverse. In Sec. 3 we consider the "canonical Fourier transformation" [6] in which both the initial and the final spaces are quantized. In Sec. 4 we consider further generalizations in which the translation-invariance of the trace enters non-trivially, based on the ideas of [7]. Finally, in Sec. 5, we consider the problem of how to define "Gaussians" in noncommutative contexts, and the question of whether in such contexts, Gaussians and only Gaussians achieve the bounds $\| \mathcal{F} \|_{q \to p}$.

2. The Fourier–Weyl transformation. Let p and q be self-adjoint operators acting in a Hilbert space $\mathcal{H}$ and satisfying the Heisenberg commutation relation

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The inverse of the Fourier–Weyl–Plancherel transformation acts on the space $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ as

$$\mathcal{F}^{-1}: f \mapsto (2\pi)^{-1/2}\int e^{i(xp+yq)} f(x,y)\,dx\,dy.$$

Evidently the bound

$$\|\mathcal{F}^{-1}f\|_\infty = (2\pi)^{-1/2}\left\|\int e^{i(xp+yq)} f(x,y)\,dx\,dy\right\|_\infty \le$$

$$\le (2\pi)^{-1/2}\int \| e^{i(xp+yq)} f(x|y)\|\,dx\,dy = (2\pi)^{-1/2}\int |f(y,x)|\,dx\,dy$$

so that

$$\|\mathcal{F}^{-1}\|_{1\to\infty} \le (2\pi)^{-1/2}.$$

Evidently, by the same argument as was used for Theorem 1, we have the Hausdorff–Young inequality

$$\|\mathcal{F}^{-1}\|_{q\to p} \le (2\pi)^{1/2-q^{-1}}.$$

3. The canonical Fourier transformation [5]. Let us now introduce two canonical pairs, each satisfying (4) in the rigorous form (5) in the Hilbert space $\mathcal{H}\otimes\mathcal{H}$, defined by

$$p_1 = p\otimes 1, \quad q_1 = q\otimes 1, \quad p_2 = 1\otimes p, \quad q_2 = 1\otimes q.$$

Strictly speaking the pairs (p_j, q_j), $j=1, 2$, should be defined as the infinitesimal generators of the two pairs of one parameter unitary groups got by ampliation from $U=(e^{ixp}, x\in\mathbb{R})$, $V=(e^{ixq}, x\in\mathbb{R})$. Then in the Schrödinger case $\mathcal{H}=L^2(\mathbb{R})$, $\mathcal{H}\otimes\mathcal{H}=L^2(\mathbb{R}^2)$, the operator $p_1q_2-q_1p_2$ can be defined as the angular momentum, that is as the infinitesimal generator of the one parameter unitary group $(W_t\colon t\in\mathbb{R})$ where

$$W_t f(x,y) = f(\cos tx - \sin ty, \sin tx + \cos ty) \quad (f\in L^2(\mathbb{R}^2), x, y\in\mathbb{R}).$$

We may also define $p_1p_2+q_1q_2$ as $(1\otimes S)(p_1q_2-q_1p_2)(1\otimes S)^{-1}$ where S is a unitary operator on $\mathcal{H}=L^2(\mathbb{R})$ such that

$$SqS^{-1} = p, \quad SpS^{-1} = -q.$$

the existence of which follows from the von Neumann uniqueness theorem (in fact we can take S to be the one-dimensional Fourier–Plancherel transformation). It is clear that $p_1q_2-q_1p_2$ and $p_1p_2+q_1q_2$ are well-defined self-adjoint operators, even in the case of general $\mathcal{H}$ where we use the von Neumann uniqueness theorem. The operator $p_1q_2+q_1p_2$ can be defined similarly, in the Schrödinger case as the infinitesimal generator of the one parameter unitary group $(W_t\colon t\in\mathbb{R})$ given by

$$W_t f(x,y) = f(\cosh tx + \sinh ty, \sinh tx + \cosh ty), \quad f\in L^2(\mathbb{R}^2),\ x, y\in\mathbb{R}.$$

A bounded operator $T\in B(\mathcal{H}\otimes\mathcal{H})$ is said to have a *partial trace* over the first copy of $\mathcal{H}$ if the linear functional

$$L^1(N)\ni S \mapsto \mathrm{tr}_{\mathcal{H}\otimes\mathcal{H}} T(1\otimes S)$$

is bounded, in which case there exists a unique bounded operator $\mathrm{tr}_1 T\in N$, such that, for all $S\in L^1(N)$,

$$\mathrm{tr}_{\mathcal{H}\otimes\mathcal{H}} T(1\otimes S) = \mathrm{tr}_{\mathcal{H}}((\mathrm{tr}_1 T)S).$$

For fixed $\alpha \in (0, \pi)$, the *canonical Fourier transform* of $X \in L^1(N)$ is defined

$$\hat{X} = \sin\alpha \, \mathrm{tr}_1 \, e^{-i\alpha(p_1 p_2 + q_1 q_2)} X \otimes 1.$$

It is shown in [5] that the partial trace exists, and that the map $X \mapsto \hat{X}$ extends to a unique isometry $\mathcal{F}$ from $L^2(N)$ onto itself whose inverse map is found by replacing $-i$ by i in the defining formula. Moreover the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^2) & \xrightarrow{\mathcal{F}_\alpha} & L^2(\mathbb{R}^2) \\ \uparrow & & \uparrow \\ L^2(N) & \xrightarrow{\mathcal{F}} & L^2(N) \end{array} \tag{7}$$

commutes, in which the vertical maps are both the Fourier–Weyl–Plancherel transformation and the upper horizontal map $\mathcal{F}_\alpha$ is a Fourier–Plancherel transformation extending the map

$$L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \ni f \mapsto \hat{f};$$

$$\hat{f}(x_1, x_2) = \theta(2\pi)^{-1} \int_{\mathbb{R}^2} e^{-i\theta(x_1 y_1 + x_2 y_2)} f(y_1, y_2)\, dy_1 dy_2$$

and θ is a certain non-zero real number depending on α.

Let us estimate the bound of the canonical Fourier transform as a map from $L^1(N)$ to $L^\infty(N)$. With $X \in L^1(N)$ and $\hat{X} = \mathcal{F}X$ we have

$$\|\hat{X}\|_\infty = \sup\{|\mathrm{tr}\, \hat{X} T|,\ T \in L^1(N),\ \|T\|_1 \le 1\} =$$

$$= \sup\{\sin\alpha \,|\, \mathrm{tr}_{\mathcal{H}\otimes\mathcal{H}}\, e^{i\alpha(p_1 p_1 + q_2 q_2)} X \otimes T|\} =$$

$$= \sup\{\sin\alpha\, |X \otimes T|\} = \sin\alpha \,\|X\|_1.$$

We thus obtain the estimate $\|\mathcal{F}\|_{1\to\infty} \le \sin\alpha$. Combining this with the non-commutative L^2 isometry property $\|\mathcal{F}\|_{2\to 2} = 1$ we may again use the abstract Calderon–Lyons interpolation principle [1] to conclude:

Theorem 2. *For* $1 \le q \le 2$ *and* $p^{-1} + q^{-1} = 1$, *the canonical Fourier transformation* $\mathcal{F}$ *maps* $L^q(N) \cap L^1(N)$ *into* $L^p(N)$ *and the bound of its restriction to this domain satisfies*

$$\|\mathcal{F}\|_{q\to p} \le (\sin\alpha)^{-1/2 - q^{-1}}. \tag{8}$$

It is evident that exactly the same estimate is obtained if the exponent $p_1 p_2 + q_1 q_2$ in the definition of the canonical Fourier transform is replaced by the "angular momentum" $p_1 q_2 - q_1 p_2$.

We may also replace the exponent by the operator $p_1 q_2 + q_1 p_2$. In this case, to obtain L^2-isometry we must re-define the canonical Fourier transformation to be

$$\hat{X} = \sinh\alpha \, \mathrm{tr}_1 \, e^{-i\alpha(p_1 q_2 + q_1 p_2)} X \otimes 1.$$

For the diagram (7) to remain commutative we must replace the classical Fourier transform $\mathcal{F}_\alpha$ by one of symplectic type in which the exponent has the form $\theta(x_1 y_2 - x_2 y_1)$. Evidently (8) becomes

$$\|\mathcal{F}\|_{q\to p} \le (\sinh\alpha)^{-1/2 - q^{-1}}.$$

4. Generalizations. A canonical pair (p, q) satisfying (4) generates a unitary multiplier-representation

$$\mathbb{R}^2 \ni (x, y) \mapsto W_{x,y} = e^{i(xp+yq)}$$

of the additive group $\mathbb{R}^2$; we have

$$W_{x,y}W_{x',y'} = \omega(x, y, x', y')\, W_{x+x', y+y'}$$

where the multiplier $\omega: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{T}$ is given by

$$\omega(x, y, x', y') = \exp\left(\frac{1}{2}i(xy' - yx')\right). \tag{9}$$

More generally we may consider unitary multiplier representations $V \ni v \mapsto W_v$; $W_u W_v = \omega(u, v)\, W_{u+v}$ of the additive group of a real finite-dimensional vector space V; here the W_u are unitary operators in a Hilbert space $\mathcal{H}$ and the multiplier ω is a measurable function from $V \times V$ to $\mathbb{T}$ satisfying the 2-cocycle condition

$$\omega(u, v + w)\omega(v, w) = \omega(u, v)\omega(u + v, w).$$

A set of canonical forms for such a 2-cocycle, under the cohomologically natural equivalence relation $\sim$, where $\omega \sim \omega'$ iff there exists a measurable function $f: V \to \mathbb{T}$ such that

$$\omega'(u, v) = \omega(u, v)\frac{f(u)f(v)}{f(u+v)} \quad \forall\, u, v \in V,$$

is obtained by taking $\omega = \exp i\mathcal{M}$ where $\mathcal{M}: V \times V \to \mathbb{R}$ is a skew-symmetric bilinear form on V. The form $\mathcal{M}$ may itself be taken in the canonical form

$$\mathcal{M}(u, v) = \sum_{j=1}^{r} (x_{2j-1}y_{2j} - x_{2j}y_{2j-1})$$

where u, v have coordinates $x_1, \ldots, x_{2r}, x_{2r+1}, \ldots, x_n, y_1, \ldots, y_{2r}, y_{2r+1}, \ldots, y_n$ in some basis of V. If $\mathcal{M}$ is degenerate, that is $n > 2r$, the von Neumann uniqueness theorem breaks down and there are many unitarily inequivalent irreducible ω-representations. To generalise the Fourier–Weyl and related transformations to this context we first introduce the *regular* ω-representation R_ω defined on the Hilbert space $L^2(V)$ of square-integrable (with respect to Haar–Lebesgue measure V) functions by

$$R_\omega(u)f(v) = \omega(v, u)f(v + u).$$

There is a natural action γ of the dual space V' of V (regarded as an additive group) on the von Neumann algebra N_ω generated by R_ω given by $\tau_{v'}T = M_{\gamma_{v'}}^{-1}TM_{\gamma_v}$ where $M_{\gamma_{v'}}$ is the operator on $L^2(N)$ of multiplication by the character $\gamma_{v'} = e^{iv'(\cdot)}$, $v' \in V'$. That is indeed an automorphism (in general not inner since $M_{\gamma_v} \notin N_\omega$) follows from the fact that $\tau_{v'}R_\omega(v) = e^{iv'(v)}R_\omega(v)$, in consequence of which it is natural to call γ the *translation* action.

It can then be shown [7] that

(1) there exists a translation trace tr_ω on N_ω which is unique to within normalization; we may call tr_ω the *Haar* trace;

(2) for a given normalization of the Haar measure on V there is a normalization of the Haar trace such that the map $\mathcal{F}_\omega$ defined on the noncommutative L^p space $L^1(N_\omega)$ of tr_ω by

$$\mathcal{F}_\omega: F \mapsto f; \quad f(v) = \tau(R_b(v)F)$$

extends uniquely to an isometry from $L^2(N_\omega)$ to $L^2(V)$, whose inverse acts on $L^1(V)$ as

$$\mathcal{F}_\omega^{-1} : f \mapsto F = \int R_\omega(u)^* f(v)\, dv.$$

When $V = \mathbb{R}^2$ and ω is the multiplier (9) of the Heisenberg–Weyl commutation relation $\mathcal{F}_\omega$ becomes the Fourier–Weyl transformation to within a normalization. When $V = \mathbb{R}$ is one-dimensional and ω is the trivial multiplier identically equal to 1, it can be shown that $\mathcal{F}_\omega$ is unitarily equivalent to the Fourier–Plancherel transformation from $L^2(\mathbb{R}')$ to $L^2(\mathbb{R})$ [7]. The general case is in effect a direct sum of copies of these two cases. Thus generally $\mathcal{F}_\omega$ will map $L^q(N_\omega) \cap L^1(N_\omega)$ into $L^p(V)$ (where $1/p + 1/q = 1$) and its inverse will map $L^p(V) \cap L^1(V)$ into $L^q(N_\omega)$, and both transformations will satisfy an inequality of Hausdorff–Young type. Details will be published elsewhere.

We may also consider generalizations of the canonical Fourier transformation in which the Heisenberg–Weyl multiplier (9) is replaced by a more general multiplier in either of its two rôles, for the initial and for the final space. For simplicity we retain the Heisenberg–Weyl multiplier on the initial space $\mathbb{R}^2$. The final space must then also be $\mathbb{R}^2$.

Assuming first that the multiplier on the final space is also the Heisenberg–Weyl multiplier, we may define a generalized canonical Fourier transformation $\mathcal{F}$ by

$$\hat{X} = \operatorname{tr}_1\left(\exp\left\{i(p_1, q_1)\begin{bmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{bmatrix}\begin{pmatrix}p_2\\ q_2\end{pmatrix}\right\}X_1\right)$$

where the partial trace is normalized by the requirement of L^2-isometry and

$$\theta = \begin{bmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{bmatrix}$$

is a real non-singular 2×2 matrix. The theory of such a transformation is reduced to canonical forms by applying independent linear canonical transformations

$$(p_j, q_j) \mapsto (p_j, q_j)\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{pmatrix}, \quad \alpha_j\delta_j - \beta_j\gamma_j = 1, \quad j = 1, 2.$$

In this way we find the canonical forms

$$\begin{bmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{bmatrix} = \alpha\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \quad \alpha\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

which were considered in Sec. 3.

Similarly, if we take the multiplier on the final space $\mathbb{R}^2$ to be trivial, then it is easily seen that the generalized canonical Fourier transformation must reduce, after linear canonical transformation in the initial space, and non-singular linear change of variables in the second space, to the Fourier–Weyl transformation. More exotic possibilities exist in higher dimensions.

5. Gaussian as maximizers. It is known [2] that the Hausdorff–Young inequality does not give the exact bound $\|\mathcal{F}\|_{q\to p}$ in the case of the Fourier transformation from $L^q(\mathbb{R})$ to $L^p(\mathbb{R})$.

Indeed, if f_t denotes the Gaussian

$$f_t(x) = e^{-e^{-t}x^2/2}, \quad t \in \mathbb{R},$$

then $\mathcal{F} f_t = e^{t/2} f_{-t}$. Then we have

$$\| \mathcal{F} f_t \|_p = e^{-t/2}\left[\int e^{-e^{-t} qpx^2/2} dx\right]^{1/p} = e^{-t/2}\left(\frac{e^{t/2}}{\sqrt{p}}\right)^{1/p} \pi^{p/2}$$

whereas

$$\|f_t\|_q = \left(\frac{e^{-t/2}}{\sqrt{q}}\right)^{1/q} \pi^{q/2}$$

whence

$$\| \mathcal{F} \|_{q\to p} \geq \frac{\| \mathcal{F} f_t \|_p}{\| f_t \|_q} = \frac{\pi^{p/2} q^{q/2}}{\pi^{q/2} p^{p/2}}.$$

It is known [8] that this estimate for the bound $\| \mathcal{F} \|_{q\to p}$ is exact and that the Gaussian functions are the only elements of $L^q(\mathbb{R})$ which achieve it.

It is natural to conjecture that similar results hold in noncommutative contexts. It is well known that "quantum Gaussians", which may be defined naturally as limit density operators (that is, noncommutative Radon–Nykodim derivatives with respect to the "Haar trace") for the quantum mechanical central limit theorem [9] for canonical pairs (p, q), have Fourier–Weyl transforms of Gaussian type; thus in the case of the Fourier–Weyl transformation "Gaussian" transforms to Gaussians. However the transform Gaussians are not arbitrary; the matrices of their quadric forms, as well as being positive definite, must be consistent with the restrictions placed by the uncertainly principle on the covariance matrix of (p, q). The inverse Fourier–Weyl transform of a Gaussian whose matrix violates this constraint is found to be a "Gaussian" of the form

$$\exp\left\{-\frac{1}{2}\beta(p, q)\theta\begin{pmatrix} p \\ q \end{pmatrix}\right\}$$

where θ is a positive definite Hermitian matrix of unit determinant, but the "temperature" is now complex. Thus in formulating appropriate conjectures it is evident that a slightly wider class of "Gaussian" than that of limit states of the central limit theorem will be needed. Within such a context it is conjectured that, for the Fourier–Weyl, inverse Fourier–Weyl and canonical Fourier transformations and their generalizations, Gaussians and (if $q < 2$) only Gaussians are maximizers, attaining the $L^q \to L^p$ bounds for all q with $1 \leq q \leq 2$, and thus permitting the exact calculation of these bounds.

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