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ON DIRECTIONAL MONOGENEITY SETS

ПРО НАПРЯМЛЕНІ МОНОГЕННІ МНОЖИНИ

We introduce and investigate some new differential properties of functions by using geometrical properties of directional monogeneity sets.*

З використанням геометричних властивостей напрямлених моногенних множин встановлюються та вивчаються деякі нові диференціальні властивості функцій.

1. Preliminaries. Let $w = F(z)$ be a function from the closed upper half plane H of the complex z-plane into a second countable topological space W. For any point x on the real line of the z-plane $(x = \operatorname{Re} z)$, the directional monogeneity set $\mathfrak{M}_x^\theta(F)$ of F at x in direction θ is defined as the directional cluster set

$$C\left(\frac{F(x+h) - F(x)}{h}, x, \theta\right)$$

at x in the direction θ.

The essential monogeneity set Ess. $\mathfrak{M}_x^\theta(F)$ of F at x in the direction θ is defined as a set of derived numbers ζ of the function F that satisfies the following condition: For every open set U containing the point $w = F(z)$, the set $F^{-1}(U) \cap L_\theta(x)$ has positive upper density at x.

For each point x on the real line R and $h > 0$, let

$$S(x, h) = \{z\colon z \in H^0, \ |z - x| < h\},$$

and for each direction θ, $0 < \theta < \pi$, let

$$L_\theta(x) = \{z\colon z \in H^0, \ \arg|z - x| = \theta\},$$

and

$$L_\theta(x, h) = S(x, h) \cap L_\theta(x).$$

Set $E \subset H$. Then a point $x \in R$ is called a first-category point of E if and only if, for every $h > 0$, the set $S(x, h) \cap E$ is of the first category in E. A point $x \in R$ is called a second-category point of E if and only if it is not a first-category point of E.

The set of all first (second)-category points of E will be denoted by E $(E_{11}$, respectively).

A point $x \in R$ is called a directional first-category point of E in direction θ if and only if, for every $h > 0$, the set $L_\theta(x, h) \cap E$ is of the first category as a linear set.

A point $x \in R$ is called a directional second-category point of E in direction θ if and only if it is not a directional first-category point of E in direction θ.

The set of all directional first (second)-category point of E in direction θ will be denoted by $E_1(\theta)$ $(E_{11}(\theta))$, respectively.

The qualitative directional monogeneity set Qual. $\mathfrak{M}_x^\theta(F)$ of F at x in direction θ is defined as a set of derived numbers ζ of the function F that satisfies the following condition: For every open set U containing w, the set $[F^{-1}(U)]_{11}(\theta)$ contains the point x.

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* For the definition of monogeneity set of a given function at some point of its domain of definition, see [1].

For a fixed direction ψ, $\psi \in (0, \pi)$, let $\theta(x)$, $x = \operatorname{Re} z$, $z \in H$, denote the set of all directions $\theta \in (0, \pi)$ in which the directional monogeneity set of F at x in direction θ does not contain the qualitative directional monogeneity set of F at x in direction ψ.

Let $\Delta(x)$ denote the set of all directions $\theta \in (0, \pi)$ in which the directional monogeneity set of F at x in direction ψ does not contain the qualitative directional monogeneity set of F at x in directional θ.

It is known [12] that if F is a continuous function from H to a topological space W with a countable basis and $\psi \in (0, \pi)$ is a fixed direction, then for every $x \in R$ except a first-category set of measure zero on R, the set

$$\{\theta\colon\ 0 < \theta < \pi,\ \text{Ess.}\,\mathfrak{M}_x(F, \psi) \not\subset \mathfrak{M}_x(F, \theta)\}$$

is of the first category. If F is measurable and $\psi \in (0, \pi)$ is a fixed direction, then except a set of measure zero on R, the set

$$\{\theta\colon\ 0 < \theta < \pi,\ \text{Ess.}\,\mathfrak{M}_x(F, \theta) \not\subset \mathfrak{M}_x(F, \psi)\}$$

is of measure zero.

2. Results. Let $E \subset H$ and $x \in R$.

Let

$$O(E, x) = \{\theta\colon\ 0 < \theta < \pi,\ x \notin \overline{E \cap L_\theta(x)}\}.$$

For a fixed positive integer n and rational r, s, $0 < r < s < \pi$, we also define

$$O_n(E, x) = \{\theta\colon\ 0 < \theta < \pi,\ E \cap L_\theta(x, n^{-1}) = \varnothing\},$$

and

$$O_{nrs}(E, x) = O_n(E, x) \cap (r, s).$$

Then, clearly,

$$O(E, x) = \bigcup_n \bigcup_r \bigcup_s O_{nrs}(E, x).$$

We quote below the Kuratowski – Ulam theorem in polar coordinates [2].

Theorem P. *If $E \subset H$ is a plane set of the first category, then for a fixed point $x \in R$, $L_\theta(x) \cap E$ is a linear set of the first category for all directions θ except a set of the first category in $(0, \pi)$.*

We now prove the following statement.

Lemma 1. *If $G \subset H$ is open and $P \subset H$ is a set of the first category, then for every $x \in R$ there exists a set of the first category $Q(x) \subset (0, \pi)$ such that*

$$O(G \Delta P, x) \subset O(G, x) \cup Q(x).$$

Proof. Let

$$Q(x) = \{\theta\colon\ 0 < \theta < \pi,\ P \cap L_\theta(x) \text{ is of the second category in } L_\theta(x)\}.$$

In virtue of Theorem P, this implies that $Q(x)$ is of the first category in $(0, \pi)$. Let $\theta \in O(G \Delta P, x) \cap CO(x)$. Then there exists a positive integer n such that

$$L_\theta(x, n^{-1}) \cap (G \Delta P) = \varnothing \tag{1}$$

and

$$P \bigcap L_\theta(x) \text{ is of the first category in } L_\theta(x). \tag{2}$$

In virtue of (1) and (2) and the fact that G is open, we have

$$L_\theta(x, n^{-1}) \bigcap G = \varnothing.$$

Hence, $\theta \in O(G, x)$ and

$$O(G\Delta P, x) \bigcap CQ(x) \subset O(G, x), \quad \text{i.e.} \quad O(G\Delta P, x) \subset O(G, x) \bigcup Q(x).$$

Lemma 2. *If the set $E \subset H$ has the Baire property and $\psi \in (0, \pi)$ is a fixed direction, then the set*

$$B = \{x\colon x \in E_{11}(\psi),\ O(E, x) \text{ is of the second category in } (0, \pi)\}$$

is of the first category in R.

Proof. Let $E = G\Delta P$, where G is open and P is of the first category. Clearly,

$$E_{11}(\psi) \subset G_{11}(\psi) \bigcup P_{11}(\psi). \tag{3}$$

It follows from Lemma 1 that, for every $x \in R$, there exists a set of the first category $Q(x) \subset (0, \pi)$ such that

$$O(E, x) \subset O(G, x) \bigcup Q(x). \tag{4}$$

Let

$$A = \{x\colon x \in G_{11}(\psi),\ O(G, x) \text{ is of the second category in } (0, \pi)\}.$$

Let $x \in B$. Then $O(G, x)$ is of the second category. By virtue of (3), we have $x \in G_{11}(\psi)$ or $x \in P_{11}(\psi)$. In the first case, $x \in A$. Thus,

$$B \subset A \bigcup P_{11}(\psi).$$

The set A is of the first category by Lemma 1. The set $P_{11}(\psi)$ is of the first category in virtue of the Kuratowski – Ulam theorem [3, p. 56]. Hence, the set B is of the first category.

Let W be a second countable topological space and let $\psi \in (0, \pi)$ be a fixed direction.

Theorem 1. *If $F\colon H \to W$ has the Baire property, then, for every $x \in R$ except a set of the first category in R, the set*

$$O(x) = \{\theta\colon 0 < \theta < \pi,\ \text{Qual.}\, \mathfrak{M}_x(F, \psi) \not\subset \mathfrak{M}_x(F, \theta)\}$$

is of the first category.

Proof. Let $\{V_n\}$ be a countable basis for the topology of W.

Let

$$B_n = [x\colon x \in E_{n11}(\psi),\ O(E_n, x) \text{ is of the second category in } (0, \pi)]$$

and let

$$D = [x\colon O(x) \text{ is of the second category in } (0, \pi)].$$

So, if $x_0 \in D$, then, by (4), there is at least one n_0 such that $O(E_{n0}, x, \psi)$ is a second category set. By the definition of $O(E_{n0}, x_0, \psi)$, x_0 belongs to $E_{n0\,11}(\psi)$ and the set $O(E_{n0}, x_0)$ is of the second category. Therefore, $x_0 \in B_{n0}$. Hence,

$$D \subset \bigcup_n B_n.$$

$$S_n = \{x\colon\ x \notin \overline{L_\psi(x) \cap E_n},\ K(E_n, x) \text{ is of the second category in } (0, \pi)\}$$

and let

$$T = \{x\colon\ \Delta(x) \text{ is of the second category in } (0, \pi)\}.$$

So, if $x_0 \in T$, then $\Delta(x_0)$ is of the second category in $(0, \pi)$. Hence, by (13), there is at least one n_0 such that $K(E_{n0}, x_0, \psi)$ is of the second category, and so, by the definition of $K(E_{n0}, x_0, \psi)$, $x_0 \notin \overline{L_\psi(x_0) \cap E_{n0}}$, and $K(E_{n0}, x_0)$ is of the second category. Therefore, $x_0 \in S_{n0}$. Hence,

$$T \subset \bigcup_n S_n.$$

Since the sets S_n are of the first category for all n, by Lemma 4, the set T is of the first category. This prove the theorem.

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