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EXPANSIONS FOR THE FUNDAMENTAL HERMITIAN INTERPOLATING POLYNOMIALS IN TERMS OF CHEBYSHEV POLYNOMIALS

РОЗКЛАДИ ФУНДАМЕНТАЛЬНИХ ІНТЕРПОЛЯЦІЙНИХ ПОЛІНОМІВ ЕРМІТА В ТЕРМІНАХ ПОЛІНОМІВ ЧЕБИШОВА

We obtain explicit expansions of the fundamental Hermitian interpolating polynomials in terms of Chebyshev polynomials in the case where the nodes considered are either the zeros of $n+1$ degree Chebyshev polynomial or the extreme points of n degree Chebyshev polynomial.

Одержано явні розклади фундаментальних інтерполяційних поліномів Ерміта в термінах поліномів Чебишова, коли вузлами інтерполяції є або нулі полінома Чебишова степеня $n+1$, або екстремальні точки полінома Чебишова степеня n.

1. Introduction. Let $n \in \mathbb{N}$ be an integer and $\xi \in [-1, 1]$. For $n \in \mathbb{N}$, the first and second kind Chebyshev polynomials $T_n(\xi)$ and $U_n(\xi)$ of degree n are defined by:

$$T_n(\xi) = \cos(n \arccos \xi), \tag{1}$$

$$U_n(\xi) = \begin{cases} (-1)^n(n+1), & \text{if } \xi = -1; \\ \dfrac{\sin((n+1)\arccos \xi)}{\sqrt{1-\xi^2}}, & \text{if } -1 < \xi < 1; \\ n+1, & \text{if } \xi = 1. \end{cases} \tag{2}$$

If $f \in C^{(m)}_{[a,b]}$, $m \in \mathbb{N}$, then the general interpolation operator is given as:

$$P[f; x] = \sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij}(x) f^{(j)}(x_i), \tag{3}$$

where the subscripts denote differentiation, the x_i are constants and A_{ij} are polynomials in x.

When $m = 0$, we replace $A_{i0}(0)$ by $l_{n,i}(x)$ in order to confirm to standard notation thus (3) becomes:

$$P[f; x] =: \mathcal{L}_n[f; x] = \sum_{i=0}^{n} l_{n,i}(x) f(x_i), \tag{4}$$

where

$$l_{n,i}(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)}, \qquad \omega_{n+1}(x) = \prod_{i=0}^{n} (x - x_i). \tag{5}$$

$l_{n,i}(x)$ is called the fundamental Lagrangian polynomial.

V. K. Dzyadyk in [1] expanded for any $n \in \mathbb{N}$ the fundamental Lagrangian polynomial $l_{n,i}(\xi)$, $i = \overline{0, n}$, built by the nodes considered are either the zeros of $(n+1)$ degree Chebyshev polynomial or the extreme points of n degree Chebyshev polynomial.

In (3), if $m = 1$, and in particular, we suppose that the first derivative as well as the function is known at $(n+1)$ of $(n+1)$ mesh points $\{x_i\}$, $i = \overline{0, n}$, so in place of (3), we have then [2, 3]:

$$P[f;x] =: \mathcal{H}_{2n+1}[f;x] = \sum_{i=0}^{n} h_{n,i}(x) f(x_i) + \sum_{i=0}^{n} \bar{h}_{n,i}(x) f'(x_i), \tag{6}$$

where $h_{n,i}(x)$, $\bar{h}_{n,i}(x)$ are called the fundamental Hermitian polynomials and given by

$$h_{n,i}(x) = \left[1 - l'_{n,i}(x_i)(x-x_i)\right] l_{n,i}^2(x), \qquad \bar{h}_{n,i}(x) = (x-x_i) l_{n,i}^2(x). \tag{7}$$

In this paper, we obtain explicit expansions for $h_{n,i}(x)$, $\bar{h}_{n,i}(x)$ in terms of $T_n(x)$ when the nodes considered are either the zeros of T_{n+1} (Section 2) or the extreme points of T_n (Section 3).

Definition 1. *The function*

$$D_n(x) := \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(n+1/2)x}{2\sin(x/2)}, \tag{8}$$

is called the Dirichlet kernel of order n.

Definition 2. *The function*

$$F_n(x) := \frac{D_0(x) + D_1(x) + \ldots + D_{n-1}(x)}{n}, \tag{9}$$

is called the Fejer kernel of order n.

From (8), it is easy to show that:

$$F_n(t) = \frac{1}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \cos kt \tag{10}$$

and, that

$$F_n(t) = \frac{1-\cos nt}{4n\sin^2(t/2)} = \frac{\sin^2(nt/2)}{2n\sin^2(t/2)}. \tag{11}$$

Definition 3. *By using the Fejer kernel* (9) *define*

$$G_n(x) = 2nF_n(x) - \sin^2\frac{nx}{2}. \tag{12}$$

Substituting $F_n(x)$ *by* (10) *and by the summation form* (11) *we get*

$$G_n(x) = \frac{\sin^2(nx/2)}{\tan^2(x/2)} = \left(n - \frac{1}{2}\right) + 2\sum_{k=1}^{n-1} (n-k)\cos kx + \frac{1}{2}\cos nx. \tag{13}$$

Definition 4. *Define*

$$Q_n(x) = \frac{\sin^2(nx/2)}{\tan(x/2)}. \tag{14}$$

It is easy to show that:

$$\sum_{k=1}^{n} -2\sin kx \sin\frac{x}{2} = \sum_{k=1}^{n} \left\{\cos\left(k+\frac{1}{2}\right)x - \cos\left(k-\frac{1}{2}\right)x\right\}.$$

Therefore:

$$\sum_{k=1}^{n} \sin kx = \frac{\sin(nx/2)\sin((n+1)/2)x}{\sin(x/2)} = \frac{\sin^2(nx/2)}{\tan(x/2)} + \sin\frac{nx}{2}\cos\frac{nx}{2}. \tag{15}$$

So, from (15) and Definition 4, we have

$$Q_n(x) = \frac{\sin^2(nx/2)}{\tan(x/2)} = \sum_{k=1}^{n-1} \sin kx + \frac{1}{2}\sin nx. \tag{16}$$

2. Fundamental Hermitian interpolating polynomials with respect to the extreme points of the n degree Chebyshev polynomials. In this Section we give a useful formulae to the fundamental Hermite polynomials $h_{2n+1}(\xi)$ and $\overline{h}_{2n+1}(\xi)$ given by (7) on the closed interval $[-1, 1]$, and by taken the nodes:

$$x_i = \xi_j^* = -\cos\frac{j\pi}{n}. \tag{17}$$

We have that the nodes (17) are the zeros of the polynomial $\mathring{U}_{n+1}(\xi) = (1-\xi^2)U_{n-1}(\xi) = \sqrt{1-\xi^2}\sin(n\arccos\xi)$ and

$$\omega_{n+1}(\xi) = (\xi-\xi_0^*)(\xi-\xi_1^*)\ldots(\xi-\xi_n^*) =$$

$$= C_n\mathring{U}_{n+1}(\xi) = C_n\sqrt{1-\xi^2}\sin(n\arccos\xi), \quad C_n = -2^{(1-n)}. \tag{18}$$

By differentiating (18) two times we get

$$\omega'_{n+1}(\xi) = -C_n\left[\frac{\xi}{1-\xi^2}\mathring{U}_{n+1}(\xi) + nT_n(\xi)\right] \tag{19}$$

and

$$\omega''_{n+1}(\xi) = -C_n\left[\frac{-n\xi T_n(\xi)}{1-\xi^2} + \frac{\xi}{\left(1-\xi^2\right)^2}\mathring{U}_{n+1}(\xi) + \frac{n^2\mathring{U}_{n+1}(\xi)}{1-\xi^2}\right]. \tag{20}$$

So, for $j = 1, 2, \ldots, n-1$ we get

$$\omega'_{n+1}(\xi_j) = C_n(-1)^{n-j+1}n, \quad \omega''_{n+1}(\xi_j) = C_n\frac{(-1)^{n-j}n\xi_j}{1-\xi_j^2}. \tag{21}$$

Now, for $j = 0$ and form (2) we get

$$\omega'_{n+1}(\xi_0) = \omega'_{n+1}(-1) = -C_n[-U_{n-1}(-1) + nT_n(-1)] = C_n(-1)^{n+1}(2n). \tag{22}$$

Hence, by a simple manipulation we have

$$\lim_{\xi\to 1^+}\frac{\omega'_{n+1}(\xi)-\omega'_{n+1}(-1)}{\omega_{n+1}(\xi)-\omega_{n+1}(-1)} = \lim_{\xi\to -1^+}\frac{\omega'_{n+1}(\xi)+(-1)^2 2nC_n}{\omega_{n+1}(\xi)-0} =$$

$$= \lim_{\xi\to 1^+}\frac{\left[\frac{-\xi}{\sqrt{1-\xi^2}}\sin(n\arccos\xi) - n\cos(n\arccos\xi) + 2n(-1)^n\right]}{\sqrt{1-\xi^2}\sin(n\arccos\xi)} =$$

$$= \lim_{\xi\to 0^+}\frac{\left[-(-1)^n\frac{\cos\varepsilon}{\sin\varepsilon}\sin n\varepsilon - (-1)^n n\cos n\varepsilon + 2n(-1)^n\right]}{-(\sin\varepsilon)\,(-1)^n\sin n\varepsilon} = -\frac{1}{3}(1+2n^2). \tag{23}$$

Similarly, for $j = n$ we have

$$\omega'_{n+1}(\xi_n) = \omega'_{n+1}(1) = -C_n[U_{n-1}(1) + nT_n(1)] = -2nC_n. \tag{24}$$

So,

$$\lim_{\xi\to 1^-}\frac{\omega'_{n+1}(\xi)-\omega'_{n+1}(1)}{\omega_{n+1}(\xi)-\omega_{n+1}(1)} =$$

$$= \lim_{\xi \to 1^-} \frac{\left[\dfrac{-\xi}{\sqrt{1-\xi^2}} \sin(n \arccos \xi) - n \cos(n \arccos \xi) + 2n \right]}{\sqrt{1-\xi^2} \sin(n \arccos \xi)} =$$

$$= \lim_{\xi \to 0^+} \frac{\left[\dfrac{-\cos \varepsilon}{\sin \varepsilon} \sin n\varepsilon - n \cos n\varepsilon + 2n \right]}{\sin \varepsilon \sin n\varepsilon} = \frac{1}{3}(1+2n^2). \tag{25}$$

From (21), (23) and (25) we get

$$\frac{\omega''_{n+1}(\xi_j)}{\omega'_{n+1}(\xi_j)} = \begin{cases} \dfrac{-\xi_j}{1-\xi_j^2}, & \text{if } j = 1, 2, \ldots, n-1; \\ \dfrac{-1}{2} \cos \dfrac{j\pi}{n}(1+2n^2), & \text{if } j = 0, j = n. \end{cases} \tag{26}$$

Now, by differentiating the fundamental Lagrangian polynomials (5), then using L'Hospital's rule, we get

$$\lim_{\xi \to \xi_j} l'_j(\xi) = \frac{1}{\omega'_{n+1}(\xi_j)} \lim_{\xi \to \xi_j} \frac{(\xi - \xi_j)\omega'_{n+1}(\xi) - \omega_{n+1}(\xi)}{(\xi - \xi_j)^2} = \frac{1}{2} \frac{\omega''_{n+1}(\xi_j)}{\omega'_{n+1}(\xi_j)}. \tag{27}$$

Theorem 1. *The fundamental Hermite polynomial* $h_{2n+1,i}(\xi) = h^*_{2n+1,i}(\xi)$ *constructed on the closed interval* $[-1, 1]$ *by the nodes* $\xi_j^* = -\cos j\pi / n$, $j = \overline{0, n}$, *can be written in the form:*

I. *For* $i = 1, 2, \ldots, n-1$:

$$h^*_{2n+1,i}(\xi) =$$

$$= \frac{1}{n^2} \Bigg\{ \left(n - \frac{1}{1-c_{2i}} \right) + c_i \left[\frac{1}{1-c_{2i}} - 2n + 1 \right] T_1(\xi) + \sum_{k=2}^{2n-2} (-1)^k (2n-k) c_{ik} T_k(\xi) -$$

$$- c_i \left[1 + \frac{1}{2(1-c_{2i})} \right] T_{2n-1}(\xi) + \frac{1}{1-c_{2i}} T_{2n}(\xi) - \frac{c_i}{2(1-c_{2i})} T_{2n+1}(\xi) \Bigg\},$$

$$c_i = \cos\left(\frac{i\pi}{n} \right).$$

II. *And, for* $i = 0$, $i = n$:

$$h^*_{2n+1,i}(\xi) =$$

$$= \frac{1}{48n^2} \Bigg\{ 4(n^2 + 6n - 1) - 2(2n^2 + 24n - 11) c_i T_1(\xi) + 24 \sum_{k=2}^{2n-2} (-1)^k (2n-k) c_{ik} T_k(\xi) +$$

$$+ (2n^2 - 23) c_i T_{2n-1}(\xi) - 4(n^2 - 1) T_{2n}(\xi) + (2n^2 + 1) c_i T_{2n+1}(\xi) \Bigg\}.$$

Proof. I. From (7), (27) we get

$$h^*_{2n+1,i} = \left[1 - \frac{\omega''_{n+1}(\xi_i)}{\omega'_{n+1}(\xi_i)} (\xi - \xi_i) \right] l^2_{n,i}(\xi). \tag{28}$$

And, from (5), (18), (21) and (26) we get

$$h^*_{2n+1,i}(\xi) = \left[1 + \frac{\xi_i}{1-\xi_i^2} (\xi - \xi_i) \right] \frac{(1-\xi^2) \sin^2(n \arccos \xi)}{(\xi - \xi_i)^2 n^2}. \tag{29}$$

Put $\xi = -\cos t$ and $\xi_i = -\cos t_i$, $t_i = \frac{i\pi}{n}$.

Therefore (29) becomes:

$$h^*_{2n+1,i}(\xi) = \left[1 + \frac{\cos t_i(\cos t - \cos t_i)}{\sin^2 t_i}\right]\frac{\sin^2 nt \sin^2 t}{n^2[\cos t - \cos t_i]^2} =$$

$$= \frac{1}{n^2}\left[\left(\frac{\sin t}{\cos t - \cos t_i}\right)^2 + \left(\frac{\cos t_i}{\cos t - \cos t_i}\right) - \frac{\cos t_i}{\sin^2 t_i}(\cos t + \cos t_i)\right]\sin^2 nt. \quad (30)$$

From trigonometry we have

$$\frac{\sin t}{\cos t - \cos t_i} = -\frac{1}{2}\left[\cot\frac{t+t_i}{2} + \cot\frac{t-t_i}{2}\right],$$

$$\frac{\cos t_i}{\cos t - \cos t_i} = -\frac{1}{2}\left[1 + \cot\frac{t+t_i}{2}\cot\frac{t-t_i}{2}\right],$$

$$\text{and} \quad \sin^2 nt = \sin^2 n(t \pm t_i). \quad (31)$$

Using (13) and (31) in (30) we get

$$h^*_{2n+1,i}(\xi) =$$

$$= \frac{1}{n^2}\left\{\frac{1}{4}\left[\cot^2\left(\frac{t+t_i}{2}\right) + \cot^2\left(\frac{t-t_i}{2}\right)\right] - \frac{\cos t_i}{\sin^2 t_i}\cos t + \left[\frac{1}{2} - \frac{1}{\sin^2 t_i}\right]\right\}\sin^2 nt =$$

$$= \frac{1}{n^2}\left\{\frac{1}{4}[G_{2n}(t+t_i) + G_{2n}(t-t_i)] - \frac{\cos t_i}{\sin^2 t_i}\cos t \sin^2 nt + \left[\frac{1}{2} - \frac{1}{\sin^2 t_i}\right]\right\}\sin^2 nt =$$

$$= \frac{1}{n^2}\left\{\left[n - \frac{1}{2\sin^2 t_i}\right] + (2n-1)\cos t \cos\left(\frac{i\pi}{n}\right) + \sum_{k=2}^{2n-2}(2n-k)\cos kt_i \cos kt + \right.$$

$$+ \cos\left(\frac{i\pi}{n}\right)\cos(2n-1)t + \frac{\cos 2nt}{2\sin^2 t_i} +$$

$$\left. + \frac{\cos t_i}{4\sin^2 t_i}[\cos(2n-1)t + \cos(2n+1)t - 2\cos t]\right\}. \quad (32)$$

Substituting by, $t = \pi - \arccos\xi$ and $c_i = \cos(i\pi/n)$ in (II.2.16) we get directly I of Theorem 1.

II. When $i = 0$. From (28), (26), (5), and (18) we get

$$h^*_{2n+1,0} = \left[1 + \frac{1}{3}(2n^2+1)(\xi+1)\right]\frac{(1-\xi^2)\sin^2(n\arccos\xi)}{(\xi+1)^2(4n^2)} =$$

$$= \frac{1}{4n^2}\left\{\left[\frac{\sin t}{1-\cos t}\right]^2 + \left[\frac{2n^2+1}{3}\right]\left[\frac{\sin^2 t}{1-\cos t}\right]\right\}\sin^2 nt =$$

$$= \frac{1}{4n^2}\left\{\left(2n - \frac{1}{2}\right) + 2\sum_{k=1}^{2n-1}(2n-k)\cos kt + \frac{1}{2}\cos 2nt + \right.$$

$$\left. + \frac{1+2n^2}{6}\left[1 + \cos t - \cos 2nt - \frac{1}{2}[\cos(2n-1)t + \cos(2n+1)t]\right]\right\}. \quad (33)$$

When $i = n$ we have

$$h^{\bullet}_{2n+1,n} = \left[1 - \frac{1}{3}(2n^2+1)(\xi-1)\right]\frac{(1-\xi^2)\sin^2(n\arccos\xi)}{(\xi-1)^2(4n^2)} =$$

$$= \frac{1}{4n^2}\left\{\left[\frac{\sin t}{1+\cos t}\right]^2 + \left[\frac{2n^2+1}{3}\right]\left[\frac{\sin^2 t}{1+\cos t}\right]\right\}\sin^2 nt =$$

$$= \frac{1}{4n^2}\left\{\left(2n-\frac{1}{2}\right) + 2\sum_{k=1}^{2n-1}(2n-k)(-1)^k\cos kt + \frac{1}{2}\cos 2nt + \right.$$

$$\left. + \frac{1+2n^2}{6}\left[1 - \cos t - \cos 2nt + \frac{1}{2}[\cos(2n-1)t + \cos(2n+1)t]\right]\right\}. \qquad (34)$$

So, from (33) and (34), if $i = 0$ or $i = n$ we have

$$h^{\bullet}_{2n+1,i} =$$

$$= \frac{1}{4n^2}\left\{\left(2n-\frac{1}{2}\right) + 2(2n-1)\cos\frac{i\pi}{n}\cos t + 2\sum_{k=2}^{2n-2}(2n-k)(-1)^k\cos\frac{ik\pi}{n}\cos kt + \right.$$

$$+ 2\cos\frac{i\pi}{n}\cos(2n-1)t + \frac{1}{2}\cos 2nt +$$

$$\left. + \frac{1+2n^2}{6}\left[1 + \cos\frac{i\pi}{n}\cos t - \cos 2nt - \frac{1}{2}\cos\frac{i\pi}{n}[\cos(2n-1)t + \cos(2n+1)t]\right]\right\} =$$

$$= \frac{1}{48n^2}\left\{4(n^2+6n-1) + 2(2n^2+24n-11)\cos\frac{i\pi}{n}\cos t + \right.$$

$$+ 24\sum_{k=2}^{2n-1}(2n-k)\cos\frac{ik\pi}{n}\cos kt - (2n^2-23)\cos\frac{i\pi}{n}\cos(2n-1)t -$$

$$\left. - 4(n^2-1)\cos 2nt - (2n^2+1)\cos\frac{i\pi}{n}\cos(2n+1)t\right\}. \qquad (35)$$

Substituting by $t = \pi - \arccos\xi$ and $c_i = \cos(i\pi/n)$, in (35), we get directly II of Theorem 1.

Theorem 2. *The fundamental Hermite polynomial* $\overline{h}_{2n+1,i}(\xi) = \overline{h}^{\bullet}_{2n+1,i}(\xi)$ *constructed on the closed interval* $[-1, 1]$ *by the nodes* $\xi^{\bullet}_j = -\cos(j\pi/n)$, $j = \overline{0, n}$, *can be written in the form:*

I. *For* $i = 1, 2, \ldots, n-1$:

$$\overline{h}^{\bullet}_{2n+1,i}(\xi) = \frac{1}{2n^2}\left\{c_i - c_{2i}T_1(\xi) + \sum_{k=2}^{2n-2}(-1)^k\left[c_{i(k+1)} - c_{i(k-1)}\right]T_k(\xi) + \right.$$

$$\left. + \left[c_{2i} - \frac{1}{2}\right]T_{2n-1}(\xi) - c_i T_{2n}(\xi) + \frac{1}{2}T_{2n+1}(\xi)\right\}, \qquad c_i = \cos\frac{i\pi}{n}.$$

II. *For* $i = 0$, $i = n$:

$$\overline{h}^{\bullet}_{2n+1,i}(\xi) = \left(\frac{1}{8n^2}\right)\left[c_i - T_1(\xi) + \frac{1}{2}T_{2n-1}(\xi) - c_i T_{2n}(\xi) + \frac{1}{2}T_{2n+1}(\xi)\right].$$

Proof. I. From (7), (5), (18) and (21) we get

$$\bar{h}^{\bullet}_{2n+1,i}(\xi) = (\xi-\xi_i)\frac{(1-\xi^2)\sin^2(n\arccos\xi)}{n^2(\xi-\xi_i)^2}. \tag{36}$$

Let $\xi=-\cos t$ and $\xi_i=-\cos t_i$, where $t_i=i\pi/n$. Then (36) becomes:

$$\bar{h}^{\bullet}_{2n+1,i}(\xi) = \frac{1}{n^2}\left\{\frac{\sin^2 t}{\cos t_i-\cos t}\right\}\sin^2 nt =$$

$$= \frac{1}{n^2}\left\{\cos t_i+\cos t + \sin t_i\left[\frac{\sin t_i}{\cos t_i-\cos t}\right]\right\}\sin^2 nt. \tag{37}$$

We have

$$\frac{\sin t_i}{\cos t_i-\cos t} = -\frac{1}{2}\left[\cot\frac{t+t_i}{2} - \cot\frac{t-t_i}{2}\right]. \tag{38}$$

From (37) and (38) we have

$$\bar{h}^{\bullet}_{2n+1,i}(\xi) = \frac{1}{n^2}\left\{\cos t_i+\cos t - \frac{1}{2}\sin t_i\left[\cot\frac{t+t_i}{2} - \cot\frac{t-t_i}{2}\right]\right\}\sin^2 nt. \tag{39}$$

From (20), (31) and (39) we get

$$\bar{h}^{\bullet}_{2n+1,i}(\xi) =$$

$$= \frac{1}{2n^2}\left\{(1-\cos 2nt)(\cos t+\cos t_i) - \sin t_i\left[Q_{2n}(t+t_i)-Q_{2n}(t-t_i)\right]\right\} =$$

$$= \frac{1}{2n^2}\left\{\cos t+\cos t_i - \frac{1}{2}\cos(2n+1)t - \right.$$

$$\left. - \frac{1}{2}\cos(2n+1)t - \cos t_i\cos 2nt - 2\sin t_i\sum_{k=1}^{2n-1}\sin kt_i\cos kt\right\} =$$

$$= \frac{1}{2n^2}\left\{\cos t_i+\cos(2t_i)\cos t + \sum_{k=1}^{2n-1}[\cos(k+1)t_i - \cos(k-1)t_i]\cos kt - \right.$$

$$\left. - \left[\cos 2t_i - \frac{1}{2}\right]\cos(2n-1)t - \cos t_i\cos 2nt - \frac{1}{2}\cos(2n+1)t\right\}. \tag{40}$$

Substituting $t=\pi-\arccos\xi$ and $c_i=\cos(i\pi/n)$ in (40) we get directly I of theorem 2.

II. For $i=0$ or $i=n$:

$$\bar{h}^{\bullet}_{2n+1,i}(\xi) = \frac{1}{4n^2}\left\{\frac{\sin^2 t}{\cos t_i-\cos t}\right\}\sin^2 nt = \frac{1}{4n^2}(\cos t_i+\cos t)\sin^2 nt =$$

$$= \frac{1}{8n^2}[\cos t_i+\cos t][1-\cos 2nt] =$$

$$= \frac{1}{8n^2}\left\{\cos t_i+\cos t - \cos t_i\cos 2nt - \frac{1}{2}[\cos(2n-1)t + \cos(2n+1)t]\right\}. \tag{41}$$

Substituting $t=\pi-\arccos\xi$ and $c_i=\cos(i\pi/n)$ in (40) we get directly II of Theorem 2.

3. Fundamental Hermitian interpolating polynomials with respect to the zeros of the $(n+1)$ degree Chebyshev polynomials. In this Section we give useful formulae for the fundamental Hermite polynomials $h_{2n+1}(\xi)$ and $\overline{h}_{2n+1}(\xi)$ respectively, on the closed interval $[-1, 1]$ and by taking the nodes:

$$x_i = \xi_j^\circ = -\cos\frac{(2j+1)\pi}{2n+2}. \tag{42}$$

The nodes (42) are the zeros of the polynomials $T_{n+1}(\xi)$ and we have

$$\omega_{n+1}^\circ(\xi) = (\xi-\xi_0^\circ)(\xi-\xi_1^\circ)\ldots(\xi-\xi_n^\circ) = C_n^\circ T_{n+1}(\xi) = C_n^\circ \cos((n+1)\arccos\xi),$$
$$C_n^\circ = 2^{-n}. \tag{43}$$

By differentiation (43) two times we get

$$\omega_{n+1}^{\circ\prime}(\xi) = C_n^\circ \frac{(n+1)\sin((n+1)\arccos\xi)}{\sqrt{1-\xi^2}} \tag{44}$$

and

$$\omega_{n+1}^{\circ\prime\prime}(\xi) = C_n^\circ \frac{n+1}{1-\xi^2}\left[\frac{\xi\sin((n+1)\arccos\xi)}{\sqrt{1-\xi^2}} - (n+1)\cos((n+1)\arccos\xi)\right]. \tag{45}$$

So, for $j = 0, 2, \ldots, n$ we get

$$\omega_{n+1}^{\circ\prime}(\xi_j) = C_n^\circ \frac{(n+1)(-1)^{n+j}}{\sqrt{1-\xi_j^2}} \quad \text{and} \quad \omega_{n+1}^{\circ\prime\prime}(\xi_j) = C_n^\circ \frac{(n+1)(-1)^{n+j}\xi_j}{\sqrt{\left(1-\xi_j^2\right)^3}}. \tag{46}$$

Theorem 3. *The fundamental Hermite polynomial* $h_{2n+1,i}(\xi) = h_{2n+1,i}^\circ(\xi)$ *constructed on the closed interval* $[-1, 1]$ *by the nodes* $\xi_j^\circ = -\cos\dfrac{(2j+1)\pi}{2n+2}$, $j = \overline{0, n}$, *can be written in the form:*

$$h_{2n+1,i}^\circ(\xi) = \frac{1}{n+1}\left\{1 + 2\sum_{k=1}^{2n+1}(-1)^k\left[1-\frac{k}{2n+2}\right]S_{i,k}T_k(\xi)\right\},$$
$$S_{i,k} = \cos\frac{(2i+1)k\pi}{2n+2}, \qquad i = \overline{0, n}.$$

Proof. From (7), (27) we get

$$h_{2n+1,i}^\circ(\xi) = \left[1 - \frac{\omega_{n+1}''(\xi_i)}{\omega_{n+1}'(\xi_i)}(\xi-\xi_i)\right] l_{n,i}^2(\xi). \tag{47}$$

And, from (5), (7), (43), (44) and (45) we get

$$h_{2n+1,i}^\circ(\xi) = [1-\xi\xi_i]\frac{T_{n+1}^2(\xi)}{(n+1)^2(\xi-\xi_i)^2}. \tag{48}$$

Put $\xi = -\cos t$ and $\xi_i = -\cos t_i$, $t_i = \dfrac{(2i+1)\pi}{2n+2}$.

Therefore (48) becomes:

$$h_{2n+1,i}^\circ(\xi) = \frac{1}{(n+1)^2}\frac{1-\cos t\cos t_i}{(\cos t - \cos t_i)^2}\cos^2(n+1)t. \tag{49}$$

Now we have

$$\frac{1-\cos t \cos t_i}{(\cos t - \cos t_i)^2} = \frac{1}{4}\left[\frac{1}{\sin^2((t+t_i)/2)} + \frac{1}{\sin^2((t-t_i)/2)}\right],$$
$$\cos^2(n+1)t = \sin^2[(n+1)(t \pm t_i)]. \tag{50}$$

From (9), (50) and (49) we get

$$h^{\circ}_{2n+1,i}(\xi) = \frac{1}{n+1}\{F_{2n+2}(t+t_i) + F_{2n+2}(t-t_i)\} =$$
$$= \frac{1}{n+1}\left\{1 + 2\sum_{k=1}^{2n+1}\left[1-\frac{k}{2n+2}\right]\cos\frac{k(2i+1)\pi}{2n+2}\cos k(t)\right\}. \tag{51}$$

Substituting $t = \pi - \arccos\xi$ in (51). Theorem 3 is this proven.

Theorem 4. *The fundamental Hermite polynomial* $\overline{h}_{2n+1,i}(\xi) = \overline{h}^{\circ}_{2n+1,i}(\xi)$ *constructed on the closed interval* $[-1, 1]$ *by the nodes* $\xi^{\circ}_j = -\cos\frac{(2j+1)\pi}{2n+2}$, $j = \overline{0, n}$, *can be written in the form:*

$$\overline{h}^{\circ}_{2n+1,i}(\xi) = \frac{1}{2(n+1)^2}\sum_{k=1}^{2n+1}(-1)^k\left[S_{i,k+1} - S_{i,k-1}\right]T_k(\xi), \qquad S_{i,k} = \cos\frac{(2i+1)k\pi}{2n+2}.$$

Proof. I. From (5), (7), (43) and (44) we get

$$\overline{h}^{\circ}_{2n+1,i}(\xi) = (\xi-\xi_i)l^2_{n,i}(\xi) = \frac{1}{(n+1)^2}\frac{(1-\xi_i^2)T^2_{n+1}(\xi)}{\xi-\xi_i}. \tag{52}$$

Let $\xi = -\cos t$ and $\xi_i = -\cos t_i$, $t_i = \frac{(2i+1)\pi}{2n+2}$. Hence:

$$\overline{h}^{\circ}_{2n+1,i}(\xi) = \frac{1}{(n+1)^2}\frac{\sin^2 t_i}{\cos t_i - \cos t}\cos^2(n+1)t. \tag{53}$$

Consequently we have

$$\frac{\sin t_i}{\cos t_i - \cos t} = -\frac{1}{2}\left[\cot\frac{t+t_i}{2} - \cot\frac{t-t_i}{2}\right]. \tag{54}$$

Using (20), (50) and (54) in (53):

$$\overline{h}^{\circ}_{2n+1,i}(\xi) = -\frac{1}{2(n+1)^2}[Q_{2n+2}(t+t_i) - Q_{2n+2}(t-t_i)]\sin t_i =$$
$$= -\frac{\sin t_i}{(n+1)^2}\sum_{k=2}^{2n-1}\sin kt_i \cos kt =$$
$$= \frac{1}{2(n+1)^2}\sum_{k=2}^{2n-1}\left[S_{i,k+1} - S_{i,k-1}\right]\cos kt_i. \tag{55}$$

Substituting $t = \pi - \arccos\xi$ in (55). Theorem 4 is thus proven.

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