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ON 2-SYMMETRIC WORDS IN NILPOTENT GROUPS

ПРО 2-СИМЕТРИЧНІ СЛОВА В НІЛЬПОТЕНТНИХ ГРУПАХ

We find the nilpotency class of a group of 2-symmetric words for free nilpotent groups, free nilpotent metabelian groups, and free (nilpotent of class c)-by-abelian groups.

Знайдено клас нільпотентності групи 2-симетричних слів для вільних нільпотентних груп, вільних нільпотентних метабелевих груп та вільних груп у класі всіх розширень нільпотентних ступеня c груп за допомогою абелевих груп.

1. Introduction. Symmetric words for a group G are closely related to the fixed points of the automorphisms permuting generators in their corresponding relatively free groups. The problem of characterizing the group of 2-symmetric words for a given group G was initiated by Płonka [1, 2] who, among other things, gave a complete description for nilpotent groups of class ≤ 3. Descriptions of 2-symmetric words are known for the free metabelian groups and free soluble groups of derived length 3 (Macedońska and Solitar [3]) and free soluble groups of arbitrary derived length (Tomaszewski [4]).

In all cases above the group of 2-symmetric words is abelian and 2-symmetric words have usually the form vv^{σ}, where σ is the automorphism which permute generators in the 2-generator free group. Since if v and v^{σ} commute in G then vv^{σ} is obviously 2-symmetric word for G, such description is sometimes called "*trivial*".

Unexpectedly, the situation for nilpotent groups of higher classes is quite different. The results of [5 – 7] show that for free nilpotent groups of class 4 and 5 the group of 2-symmetric words in nonabelian. The same is also true for free metabelian of arbitrary nilpotency class group [5], free (nilpotent of class 2)-by-abelian groups and free centre-by-metabelian groups [8]. Moreover, even in the case of free nilpotent group of class 3 there exist 2-symmetric words not of the form vv^{σ}. In this paper we discuss this phenomena and find the nilpotency class of the group of 2-symmetric words for some new relatively free groups.

Let F be a free group generated by x, y. A binary word $w(x, y) \in F$ is called 2-*symmetric word* for a group G if $w(g, h) = w(h, g)$ for all g, h in G.

The 2-symmetric words for a group G are the same as these for the group $F(G)$, the two generator relatively free group of the variety generated by G ($F(G) \simeq F/V$ for some verbal subgroup V of F). Let σ be the automorphism of F induced by the mapping $x \to y$, $y \to x$. Clearly, w is a 2-symmetric word for G if and only if $w \equiv$ $\equiv w^{\sigma} \mod V$ (i.e. $w = w^{\sigma} \cdot w_0$, where $w_0 \in V$). So, we can carry out all our calculations in F while working modulo V. There is a natural homomorphism ϕ : $F \to F(G)$ ($x \to \bar{x}$, $y \to \bar{y}$) which induces an automorphism $\bar{\sigma}$ of $F(G)$, when V is σ-invariant. Denote by

$$S^{(2)}(G) = \{\bar{w} \in F(G): \bar{w} = \bar{w}^{\bar{\sigma}}\} = \mathrm{Fix}(\bar{\sigma})$$

a group of fixed points of $\bar{\sigma}$. Since $\phi^{-1}(S^{(2)}(G))$ consists of all 2-symmetric words for G we called $S^{(2)}(G)$ a group of 2-symmetric words for G.

In this paper we consider $S^{(2)}(G)$ in the case of G-free nilpotent group of class c, which we denote by $S^{(2)}(\mathfrak{N}_c)$, free nilpotent metabelian group $S^{(2)}(\mathfrak{N}_c \wedge \mathfrak{A}^2)$ and free (nilpotent of class c)-by-abelian group $S^{(2)}(\mathfrak{N}_c \cdot \mathfrak{A})$ (where $\mathfrak{N}_c$, $\mathfrak{A}$, $\mathfrak{A}^2$ are varieties of nilpotent groups of class c, abelian and metabelian groups correspondingly). Our main results here are.

Theorem 1. *The groups* $S^{(2)}(\mathfrak{N}_c)$ *and* $S^{(2)}(\mathfrak{N}_c \wedge \mathfrak{A}^2)$ *are finitely generated nilpotent groups of class* $c-2$.

Theorem 2. *The group* $S^{(2)}(\mathfrak{N}_c \cdot \mathfrak{A})$ *is infinitely generated nilpotent group of class* c.

In our proofs, the words ww^{σ} play the significant role in finding the nilpotency class. However, not all 2-symmetric words have such form. For example, the word $x^2y^2[y,x]^2[y,x,x]^2$ from $S^{(2)}(\mathfrak{N}_3)$ [5, 7] is not equal to ww^{σ} for any $w \in F$ (here $[x,y] = x^{-1}y^{-1}xy$ is a commutator of elements x, y, commutators of higher weight are defined as left-normed, for other definitions we refer to [9]).

2. Proofs of main results. The following lemma is fundamental in our considerations.

Lemma 1. *If* $R \le S \le F$, $w \in S$ *and* $w \equiv w^{\sigma} \mod R$ *then* $(w \cdot w^{\sigma})^{\sigma} \equiv w \cdot w^{\sigma} \mod [R,S]$.

Proof. If $w \equiv w^{\sigma} \mod R$ then $w^{\sigma}w^{-1} \in R$ and $[w^{\sigma}w^{-1}, w] \in [R,S]$. But we have

$$[w^{\sigma}w^{-1}, w] = (w^{\sigma}w^{-1})^{-1}w^{-1}w^{\sigma}w^{-1}w = w(w^{\sigma})^{-1}w^{-1}w^{\sigma} = [w^{-1}, w^{\sigma}].$$

It means that

$$w^{\sigma}w^{-1} \equiv w^{-1}w^{\sigma} \mod [R,S]$$

and

$$w^{\sigma}w \equiv ww^{\sigma} \mod [R,S].$$

Now

$$(ww^{\sigma})^{\sigma} \equiv w^{\sigma}w \equiv ww^{\sigma} \mod [R,S].$$

Denote $d_{ij} = [x,y]^{x^i y^j}$ ($i, j \in \mathbb{Z}$ — integers).

Corollary 1. *For any natural number* c, $c \ge 2$, *there exist 2-symmetric word* $w \mod \gamma_{c+1}(F)$, *such that* $w \in F - \gamma_2(F)$, *and 2-symmetric word* $v \mod \gamma_{c+1}(F')$, *such that* $v \in F' - \gamma_2(F')$.

Proof. It follows from [1] that the word $w = x^2y^2[y,x]^2$ is 2-symmetric $\mod \gamma_3(F)$. By Lemma 1

$$ww^{\sigma} = x^4y^4 \cdot w_0, \qquad w_0 \in \gamma_2(F),$$

is 2-symmetric $\mod \gamma_4(F)$ and by induction we have nontrivial word

$$u = x^{2^{c-1}}y^{2^{c-1}} \cdot u_0, \qquad u_0 \in \gamma_2(F),$$

which is 2-symmetric $\mod \gamma_{c+1}(F)$. The same procedure applied to the word $d_{ij}^2(d_{ij}^{\sigma})^2[d_{ij}^{\sigma}, d_{ij}]$ which is 2-symmetric $\mod \gamma_3(F')$ by [8] give us 2-symmetric word $\mod \gamma_{c+1}(F')$ of the form $d_{ij}^{2^{c-1}}(d_{ij}^{\sigma})^{2^{c-1}} \cdot w_1$, $w_1 \in \gamma_2(F')$.

Example. The words

$$[\underbrace{y,x,x,\ldots,x}_{c}][\underbrace{y,x,y,\ldots,y}_{c}]^{-1}, \qquad [\underbrace{y,x,x,\ldots,x}_{c-1}][\underbrace{y,x,y,\ldots,y}_{c-1}]^{-1}, \qquad c \ge 3,$$

are 2-symmetric word $\mod \gamma_{c+1}(F)$. In general, for $k \ge [c/2]$ the product of two commutators of length k

$$[y, \underbrace{x, \ldots, x}_{l}, y, \ldots, y][x, \underbrace{y, \ldots, y}_{l}, x, \ldots, x]$$

is 2-symmetric word $\mod \gamma_{c+1}(F)$ but, this is not always the case, for example the product

$$[y, x, x, y][x, y, y, x] \equiv 1 \mod \gamma_5(F).$$

It explains the advantages of writing the words as a product of basic commutators and using the uniqueness of such product $\mod \gamma_{c+1}(F)$.

Corollary 2. *1. For all k, $3 \le k \le c$, there exist 2-symmetric word w $\mod \gamma_{c+1}(F)$ such that $w \in \gamma_k(F) - \gamma_{k+1}(F)$.*

2. For all $k \le c$ there exist 2-symmetric word $\mod \gamma_{c+1}(F')$ such that $w \in \gamma_k(F') - \gamma_{k+1}(F')$.

Proof. 1. The word

$$w = [\underbrace{y, x, x, \ldots, x}_{k}][\underbrace{y, x, y, \ldots, y}_{k}]^{-1}, \quad k \ge 3,$$

is 2-symmetric $\mod \gamma_{k+1}(F)$. Applying to this word procedure from Lemma 1 we obtain the word

$$u = [\underbrace{y, x, x, \ldots, x}_{k}]^s \cdot [\underbrace{y, x, y, \ldots, y}_{k}]^{-s} \cdot u_0, \quad u_0 \in \gamma_{k+1}(F),$$

which is 2-symmetric $\mod \gamma_{c+1}(F)$.

2. The corollary follows from the fact that the words $d_{ij}^2 (d_{ij}^{\sigma})^2 [d_{ij}^{\sigma}, d_{ij}]^2$ and $[d_{kl}, d_{mn}][d_{kl}^{\sigma}, d_{mn}^{\sigma}]$ are 2-symmetric $\mod \gamma_3(F')$ and

$$[\underbrace{d_{ij}^{\sigma}, d_{ij}, \ldots, d_{ij}}_{k}] \cdot [\underbrace{d_{ij}^{\sigma}, d_{ij}, d_{ij}^{\sigma}, \ldots, d_{ij}^{\sigma}}_{k}]^{-1}$$

is 2-symmetric word $\mod \gamma_{k+1}(F')$.

Lemma 2. *In any nilpotent group G of class c $(c \ge 3)$ we have for any natural number a*

$$[y^a, x^a] \equiv [y, x]^{a^2} [y, x, x]^{a\binom{a}{2}} [y, x, y]^{a\binom{a}{2}} \mod \gamma_4(F).$$

Proof. Modulo $\gamma_4(G)$

$$[y^n, x] \equiv [y, x]^n [y, x, y]^{\binom{n}{2}}, \quad [y, x^m] \equiv [y, x]^m [y, x, x]^{\binom{m}{2}}$$

and

$$[y^n, x^m] \equiv [y^n, x]^m [y^n, x, x]^{\binom{m}{2}}.$$

These give us

$$[y^n, x^m] \equiv [y, x]^{nm} [y, x, x]^{n\binom{m}{2}} [y, x, y]^{m\binom{n}{2}}.$$

Lemma 3. *If $w \equiv w^{\sigma} \mod \gamma_{k+1}(F)$, $c \ge 3$, then*

$$w \equiv x^a y^a [y, x]^b [y, x, x]^{c_1} [y, x, y]^{c_2} \mod \gamma_4(F),$$

where $a^2 = 2b$ and $c_1 + c_2 = a\binom{a}{2}$.

Proof. If $w \equiv w^{\sigma} \mod \gamma_{c+1}(F)$, then $w \equiv w^{\sigma} \mod \gamma_4(F)$ and by [2] has a form

$$w = x^a y^a [y, x]^b [y, x, x]^{c_1} [y, x, y]^{c_2} \cdot w_0, \quad w_0 \in \gamma_4(F).$$

We have

$$w^{\sigma} = y^a x^a [y,x]^b [x,y,y]^{c_1} [x,y,x]^{c_2} \cdot w_0^{\sigma} =$$

$$= x^a y^a [y,x]^{a^2-b} [y,x,x]^{a\binom{a}{2}-c_2} [y,x,y]^{a\binom{a}{2}-c_1} \cdot w_1, \qquad w_1 \in \gamma_4(F),$$

and comparing $w = w^{\sigma}$ we obtain $a^2 = 2b$ and $c_1 + c_2 = a\binom{a}{2}$.

Lemma 4. *If* w_1, w_2 *are 2-symmetric words* $\operatorname{mod}\gamma_{c+1}(F)$, *then* $[w_1, w_2] \in \gamma_4(F)$. *There exist 2-symmetric words* w_3, w_4 $\operatorname{mod}\gamma_{c+1}(F)$ *such that* $[w_3, w_4] \notin \gamma_5(F)$.

Proof. If w_1, w_2 are 2-symmetric words $\operatorname{mod}\gamma_{c+1}(F)$ then by Lemma 3 have the form

$$w_1 = x^a y^a [y,x]^b [y,x,x]^{c_1} [y,x,y]^{c_2} \cdot w_{01},$$

$$w_2 = x^{\bar{a}} y^{\bar{a}} [y,x]^{\bar{b}} [y,x,x]^{\bar{c}_1} [y,x,y]^{\bar{c}_2} \cdot w_{02},$$

where $w_{01}, w_{02} \in \gamma_4(F)$ and $a^2 = 2b$ and $c_1 + c_2 = a\binom{a}{2}, \bar{a}^2 = 2\bar{b}$, $\bar{c}_1 + \bar{c}_2 = \bar{a}\binom{\bar{a}}{2}$.

We have

$$[w_1, w_2] = [y,x]^{\phi_1} [y,x,x]^{\phi_2} [y,x,y]^{\phi_3} [y,x,x,x]^{\phi_4} [y,x,x,y]^{\phi_5} [y,x,y,y]^{\phi_6} \cdot w_0,$$

where $w_0 \in \gamma_5(F)$ and

$$\phi_1 = 0,$$

$$\phi_2 = -\bar{a}\binom{a}{2} + a\binom{\bar{a}}{2} - \bar{b}a + b\bar{a} = 0,$$

$$\phi_3 = -a\binom{\bar{a}}{2} + \bar{a}\binom{a}{2} - \bar{b}a + b\bar{a} - \bar{a}a^2 + a(\bar{a})^2 = 0,$$

$$\phi_4 = -\bar{a}\binom{a}{3} + a\binom{\bar{a}}{3} - \bar{b}\binom{a}{2} + b\binom{\bar{a}}{2} + c_1\bar{a} - \bar{c}_1 a,$$

$$\phi_5 = -\binom{a}{2}\binom{\bar{a}}{2} + \binom{\bar{a}}{2}\binom{a}{2} - a\bar{a}\binom{a}{2} + a\bar{a}\binom{\bar{a}}{2}(c_1 + c_2)\bar{a} - (\bar{c}_1 + \bar{c}_2)a = 0,$$

$$\phi_6 = -a\binom{\bar{a}}{3} + \bar{a}\binom{a}{3} - \bar{b}\binom{a}{2} + b\binom{\bar{a}}{2} -$$

$$- \bar{a}a\binom{a}{2} - a^2\binom{\bar{a}}{2} + a\bar{a}\binom{\bar{a}}{2} + (\bar{a})^2\binom{a}{2} + c_2\bar{a} - \bar{c}_2 a,$$

which show that $[w_1, w_2] \in \gamma_4(F)$.

If $[w_1, w_2] \in \gamma_5(F)$ then $\phi_4 = 0$, but ϕ_4 depends linearly on c_1 so multiplying w_1 by 2-symmetric word of the form

$$u = [y,x,x]^{\phi} [y,x,y]^{-\phi} \cdot u_1$$

($\phi \neq 0$, $u_1 \in \gamma_4(F)$), which exist by Corollary 2, we obtain $[w_1 u, w_2] \notin \gamma_5(F)$.

Proof of Theorem **1.** Let $N = S^{(2)}(\mathfrak{N}_c)$. Since N is a subgroup of $F/\gamma_{c+1}(F)$ the nilpotency class of N is at most c and N is finitely generated.

From Lemma 4 it follows that for any w_1, w_2, 2-symmetric words $\operatorname{mod}\gamma_{c+1}(F)$ their commutator $[w_1, w_2] \in \gamma_4(F)$. It means that the nilpotency class of N is at most $c - 2$. There exist also two 2-symmetric words w_1, w_2 such that $[w_1, w_2] \notin \gamma_5(F)$. It means that

$$[w_1, w_2] = [y,x,x,x]^{\psi} \cdot [y,x,y,y]^{-\psi} \cdot w_0,$$

where $w_0 \in \gamma_5(F)$, $\psi \neq 0$. The commutator

$$[w_1, w_2, w_2] = [y, x, x, x, x]^{\bar{\alpha}\psi} \cdot [y, x, y, y, y]^{-\bar{\alpha}\psi} \cdot \overline{w}_0, \qquad \overline{w}_0 \in \gamma_6(F),$$

which shows that $[w_1, w_2, w_2]$ belongs to $\gamma_5(F) - \gamma_6(F)$. Continuing this process we see that N has nilpotency class at least $c-2$.

Since 2-generator nilpotent group of class 4 is metabelian, all calculations preceded are valid in the case of free nilpotent metabelian group.

***Proof of Theorem* 2.** Let $R = S^{(2)}(\mathfrak{N}_c \cdot \mathfrak{A}^2)$. If $w \equiv w^\sigma \bmod \gamma_{c+1}(F')$ then $w \equiv w^\sigma \bmod F''$ and by Proposition 1 of [8] we have $w \in F'$. It means that R is a subgroup of $F'/\gamma_{c+1}(F')$ which is nilpotent of class c, so R has nilpotency class at most c. Now we show that nilpotency class of R is equal to c. From [8] it follows that we have infinite number (for different pairs (i, j)) of words of the form $d_{ij}^2(d_{ij}^\sigma)^2[d_{ij}^\sigma, d_{ij}]^2$ which are 2-symmetric $\bmod \gamma_3(F')$. Starting from these words we obtain, using procedure from Corollary 1, the words

$$w_{ij} = d_{ij}^{2^{c-1}}(d_{ij}^\sigma)^{2^{c-1}} \cdot \overline{w}_{ij}, \qquad \overline{w}_{ij} \in \gamma_2(F'),$$

which are 2-symmetric $\bmod \gamma_{c+1}(F')$. Let $(ij) \neq (kl)$. Then

$$[w_{ij}, w_{kl}] = [d_{ij}, d_{kl}]^s[d_{ij}^\sigma, d_{kl}]^s[d_{ij}, d_{kl}^\sigma]^s[d_{ij}^\sigma, d_{kl}^\sigma]^s \cdot w_0, \qquad w_0 \in \gamma_3(F'),$$

is not contained in $\gamma_3(F')$ since $s = 2^{2c-2} \neq 0$. Similarly, the commutator $[w_{ij}, w_{kl}, w_{kl}]$ is not contained in $\gamma_4(F')$ since it has a subword $[d_{ij}, d_{kl}, d_{kl}]$ is some nontrivial power. Continuing this process we see that the nilpotency class of R is c. R is infinitely generated since the set $\{w_{ij} : (ij) \in \mathbb{Z} \times \mathbb{Z}\}$ cannot lie in finitely generated subgroup of R.

1. *Płonka E.* Symmetric operations in groups // Collog. math. – 1970. – **21**. – P. 186.
2. *Płonka E.* Symmetric words in nilpotent groups of class ≤ 3 // Fund. math. – 1977. – **97**. – P. 95 – 103.
3. *Macedońska O., Solitar D.* On binary σ-invariants words in a group // Contemp. Math. – 1994. – **169**. – P. 431 – 449.
4. *Tomaszewski W.* Automorphisms permuting generators in groups and their fixed points // Ph. D. Thesis. – Siles. Univ., 1999.
5. *Hołubowski W.* Symmetric words in metabelian groups // Communs Algebra. – 1995. – **23**. – P. 5161 – 5167.
6. *Hołubowski W.* Symmetric words in nilpotent groups of class 5 // Groups St Andrews 1997 in Bath, I (London Math. Soc. Lect. Note Ser.). – 1999. – **260**. – P. 363 – 367.
7. *Hołubowski W.* Symmetric words in free nilpotent groups of class 4 // Publ. Math. (Debrecen). – 2000. – **57**. – P. 411 – 419.
8. *Gupta C. K., Hołubowski W.* On 2-symmetric words in groups // Arch. Math. – 1999. – **73**. – P. 327 – 331.
9. *Neumann H.* Varieties of groups. – Berlin: Springer, 1967. – 192 p.

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