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ON THE SMOOTHNESS OF GENERALIZED SOLUTION FOR PARABOLIC SYSTEM IN DOMAINS WITH CONIC POINTS ON BOUNDARY

ПРО ГЛАДКІСТЬ УЗАГАЛЬНЕНОГО РОЗВ'ЯЗКУ ПАРАБОЛІЧНОЇ СИСТЕМИ В ОБЛАСТЯХ З КОНІЧНИМИ ТОЧКАМИ НА МЕЖІ

We consider the first initial boundary-value problems for parabolic systems in domains with conic points on a boundary. We study the smoothness of their solutions with respect to spatial variables.

Досліджуються перші початково-крайові задачі для параболічних систем в областях з конічними точками на межі. Вивчається гладкість їх розв'язків відносно часових змінних.

1. Introduction. At present, the boundary-value problems for elliptic equations in domains with nonsmooth boundary are completely studied. The general problems in multimeasured domains with a finite number of conic points on boundary were considered in detail in [1], where the Noether theorems were proved.

In domains with smooth boundary, the nonstationary problems also were considered by many authors. In [2, 3], the unique solvability of boundary-value problems for parabolic equations was established and was shown that if the right-hand side, the initial and the boundary functions are infinitely differentiable, then the solution is also infinitely differentiable.

In this paper, we consider the first initial boundary-value problems for parabolic systems in domains with conic points on a boundary. We study the smoothness of the solutions with respect to spatial variables.

2. Notations. Let Ω be a bounded domain in the space $\mathbb{R}^n$. The boundary $\partial\Omega$ of Ω is assumed to be an infinitely differentiable surface everywhere except the coordinate origin, in the neighborhood of which $\partial\Omega$ coincides with the cone

$$K = \left\{x : \frac{x}{|x|} \in G\right\},$$

where G is a smooth domain on a unit sphere. Let Ω_T denote a cylinder $\Omega \times [0, T)$, $0 \le T \le \infty$, $H^{l,k}(\Omega_T)$ denote a space consisting of $u = (u_1, \ldots, u_s)$ from $L_2(\Omega_T)$, which have the generalized derivatives up to order l with respect to x and up to order k with respect to t belonging to $L_2(\Omega_T)$. The norm in this space is defined as follows:

$$\|u\|_{H^{l,k}(\Omega_T)} = \left(\int\limits_{\Omega_T} \sum_{|\alpha|=0}^{l} \left|D^\alpha u\right|^2 dx\,dt + \int\limits_{\Omega_T} \sum_{j=1}^{k} \left|u_{t^j}\right|^2 dx\,dt \right)^{1/2}.$$

Let $\overset{\circ}{H}{}^{l,k}(\Omega_T)$ be the closure in $H^{l,k}(\Omega_T)$ of the set consisting of all functions which are infinitely differentiable in Ω_T and vanish near $S_T = \partial\Omega \times [0, T)$.

Let $H^{l,k}(e^{-\gamma t}, \Omega_\infty)$ be the space consisting all functions $u(x, t)$ satisfying

$$\|u\|_{H^{l,k}(e^{-\gamma t}, \Omega_\infty)} = \left[\int\limits_{\Omega_\infty} \left(\sum_{|\alpha|=0}^{l} \left|D^\alpha u\right|^2 + \sum_{j=1}^{k} \left|u_{t^j}\right|^2 \right) e^{-2\gamma t}\, dx\,dt \right]^{1/2}.$$

In the same way as above, define $\overset{\circ}{H}{}^{l,k}(e^{-\gamma t}, \Omega_\infty)$. Denote by $H^{l,k}_\beta(\Omega_T)$ the space consisting of all functions $u(x, t)$ satisfying

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$$\|u\|^2_{H^{l,k}_\beta(\Omega_T)} = \left[\int\limits_{\Omega_T}\left(\sum_{|\alpha|=0}^{l} r^{2(\beta+|\alpha|-l)}\left|D^\alpha u\right|^2 + \sum_{j=0}^{k}\left|u_{t^j}\right|^2\right)dx\,dt\right]^{1/2}.$$

Denote by $H^l_\beta(\Omega_T)$ the space consisting of all functions $u(x,t)$ satisfying

$$\|u\|_{H^l_\beta(\Omega_T)} = \left[\int\limits_{\Omega_T}\sum_{|\alpha|+j=0}^{l} r^{2(\beta+|\alpha|+j-l)}\left|D^\alpha u_{t^j}\right|^2 dx\,dt\right]^{1/2}.$$

We define the spaces $H^{l,k}_\beta(e^{-\gamma t},\Omega_\infty)$, $H^l_\beta(e^{-\gamma t},\Omega_\infty)$ by analogy with the definition of $H^{l,k}(e^{-\gamma t},\Omega_\infty)$.

In Ω_∞, we consider the first initial boundary-value problem for the following system:

$$(-1)^m\left[\sum_{|p|,|q|=1}^{m} D^p a_{pq}(x,t)D^q u + \sum_{|p|=1}^{m} a_p(x,t)D^p u + a(x,t)u\right] - u_t = f(x,t). \quad (2.1)$$

Here, a_{pq}, a_p, a are infinitely differentiable and bounded complex-value matrices $s\times s$, $a_{pq}=(-1)^{|p|+|q|}a^*_{pq}$, a_{pq}, $|p|=|q|=m$, are uniformly continuous in $\overline{\Omega}_\infty = \overline{\Omega}\times[0,\infty)$.

We assume that the considered system (2.1) are strongly parabolic, i.e., for each $\xi\in\mathbb{R}^n\setminus\{0\}$ and $\eta\in\mathbb{C}^s\setminus\{0\}$,

$$\sum_{|p|=|q|=m} a_{pq}(x,t)\xi^p\xi^q\eta\overline{\eta} \ge \mu_0|\xi|^{2m}|\eta|^2, \qquad (x,t)\in\overline{\Omega}_\infty, \quad (2.2)$$

where $\xi^p=\xi_1^{p_1}\ldots\xi_n^{p_n}$, μ_0 is a positive constant.

The function $u(x,t)$ is called a generalized solution of the first initial boundary-value problem for system (2.1) in the space $\overset{\circ}{H}{}^{m,1}(e^{-\gamma t},\Omega_\infty)$ if $u(x,0)=0$ and we have

$$\int\limits_{\Omega_T}\Bigg[-u_t\overline{\eta} + \sum_{|p|,|q|=1}^{m}(-1)^{m-1+|p|}a_{pq}D^q u\overline{D^p\eta} +$$

$$+\sum_{|p|=1}^{m}(-1)^{m-1}a_pD^pu\overline{\eta} + (-1)^{m-1}au\overline{\eta}\Bigg]dx\,dt = \int\limits_{\Omega_T} f\overline{\eta}\,dx\,dt \quad (2.3)$$

for each $T>0$ and all functions $\eta\in\overset{\circ}{H}{}^{m,1}(\Omega_T)$ satisfying $\eta(x,T)=0$.

3. Main results. The following result can be obtained on the basis of Galerkin's method (see analogous proof for the hyperbolic case with finite cylinders in [4]).

Theorem 3.1 (smoothness with respect to temporary variable). *Suppose that*

$$\left|\frac{\partial^k a_{pq}}{\partial t^k},\frac{\partial^{k-1}a_p}{\partial t^{k-1}},\frac{\partial^{k-1}a}{\partial t^{k-1}}\right| \le \mu,$$

where $1\le|p|,|q|\le m$, $k\le h+1$, $\mu=\text{const}>0$, $(x,T)\in\overline{\Omega}_\infty$. *Then there exist the positive constants* γ_k, $0\le k\le h$, *such that if* $f_{t^k}\in L^\infty(0,\infty;L_2(\Omega))$, $f_{t^k}(x,0)=0$, $0\le k\le h$, *the generalized solution* $u(x,t)$ *of the first boundary-value problem for the system* (2.1) *exists in the space* $\overset{\circ}{H}{}^{m,1}(e^{-\gamma t},\Omega_\infty)$, *where* $\gamma=\gamma_0$, *and possesses the derivatives with respect to* t *of all orders* $k\le h$. *Moreover, the following inequality is true*:

$$\left\| u_{t^h} \right\|^2_{H^{m,1}(e^{-\gamma_h t}, \Omega_\infty)} \le C \sum_{k=0}^{j} \left\| f_{t^k} \right\|_{L^\infty(0,\infty; L_2(\Omega))},$$

where the positive constant does not depend on u *and* f.

We now consider some results on the smoothness wit respect to spatial variables

Lemma 3.1. *If* $f,\ f_t \in L^\infty(0, \infty; L_2(\Omega))$, $f(x, 0) = 0$, $f_t(x, 0) = 0$ $u(x, t) \in H^{m,1}(e^{-\gamma t}, \Omega_\infty)$ *is a generalized solution of problem* (2.1) *such that* u *whenever* $|x| > R = \text{const}$, *then* $u \in H_m^{2m,1}(e^{-\gamma t}, K_\infty)$ *and*

$$\| u \|^2_{H_m^{2m,1}(e^{-\gamma t}, K_\infty)} \le C\left[\| f \|_{L^\infty(0,\infty; L_2(K))} + \| f_t \|^2_{L^\infty(0,\infty; L_2(K))} \right],$$

where $C = \text{const}$ *is independent of* u, f.

Proof. Rewrite system (2.1) in the form

$$\sum_{|\alpha| \le 2m} b_\alpha(x, t) D^\alpha u = F, \qquad (\ldots$$

where $u_t + f = F$. It follows from Theorem 3.1 that $F \in L^\infty(0, \infty; L_2(K))$.

We consider the sequence of domains Ω^k

$$\Omega^k = \{ x \mid x \in K,\ 2^{-k} \le |x| \le 2^{-k+1} \}, \quad k = 1, 2, \ldots .$$

From theorems on smoothness of a solution of elliptic systems inside the domain an the neighbourhood of a smooth piece of boundary [5], we obtain

$$\int_{\Omega^2} \left| D^\alpha u(x, t) \right|^2 dx \le C \int_{\Omega^1 \cup \Omega^2 \cup \Omega^3} \left(|F(x, t)|^2 + |u(x, t)|^2 \right) dx, \qquad (\ldots$$

$$|\alpha| \le 2m, \quad C = \text{const}.$$

In (3.1), we perform the substitution $x = \left(\frac{4}{2^{k_1}} \right) x'$ for $k_1 > 2$ and then apply estim (3.2). As a result, we have

$$\int_{\Omega^2} \left| D^\alpha_{x'} u(x', t) \right|^2 dx' \le$$

$$\le C_1 \int_{\Omega^1 \cup \Omega^2 \cup \Omega^3} \left(|F(x', t)|^2 \left(\frac{4}{2^{k_1}} \right)^{4m} + |u(x', t)|^2 \right) dx', \quad C_1 = \text{const}.$$

By returning to variables $x_1, \ldots, x_n$, we obtain

$$\int_{\Omega^{k_1}} \left| D^\alpha u(x, t) \right|^2 r^{2(|\alpha| - m)} dx \le$$

$$\le C_2 \int_{\Omega^{k_1 - 1} \cup \Omega^{k_1} \cup \Omega^{k_1 + 1}} \left(|F(x, t)|^2 r^{2m} + r^{-2m} |u(x, t)|^2 \right) dx, \quad C_2 = \text{const}.$$

After summing up these inequalities with respect to all $k_1 > 2$, we arrive at inequality

$$\int_{\sum_{k>2} \Omega^k} \left| D^\alpha u(x, t) \right|^2 r^{2(|\alpha| - m)} dx \le$$

$$\le C_3 \int_{\sum_{k>1} \Omega^k} \left(|F(x, t)|^2 r^{2m} + r^{-2m} |u(x, t)|^2 \right) dx, \quad C_3 = \text{const}. \qquad (3\ldots$$

Since the solution is smooth outside the neighbourhood of a conic point, inequality (3.3) implies

$$\int_K \left|D^\alpha u(x,t)\right|^2 r^{2(|\alpha|-m)}\,dx \le$$

$$\le C_4 \int_K \left(|F(x,t)|^2 r^{2m} + r^{-2m}|u(x,t)|^2\right) dx, \qquad C_4 = \text{const.} \tag{3.4}$$

The conditions $D_\nu^j u\big|_{S_\infty} = 0,\ j = 0, 1, \ldots, m-1,$ yield the estimate

$$\int_K r^{-2m}|u(x,t)|^2\,dx \le C_5 \sum_{|\beta|=m} \int_K \left|D^\beta u\right|^2 dx, \qquad C_5 = \text{const.}$$

This and (3.4) imply

$$\int_K r^{2(|\alpha|-m)}\left|D^\alpha u(x,t)\right|^2 dx \le$$

$$\le C_6 \int_K \left(|f|^2 + |u_t|^2 + \sum_{|\beta|=m}\left|D^\beta u\right|^2\right) dx, \qquad C_6 = \text{const.}$$

By integrating this inequality by t from 0 to ∞ after multiplying its both sides with respect to $e^{-2\gamma t}$, we obtain

$$\int_{K_\infty} r^{2(|\alpha|-m)}\left|D^\alpha u(x,t)\right|^2 e^{-2\gamma t}\,dx\,dt \le$$

$$\le C_6 \int_{K_\infty} \left(|f|^2 + |u_t|^2 + \sum_{|\beta|=m}\left|D^\beta u\right|^2\right) e^{-2\gamma t}\,dx\,dt. \tag{3.5}$$

By virtue of Theorem 3.1, inequality (3.5) implies the statement of Lemma 3.1. This lemma is proved.

Let ω be a local coordinate system on S^{n-1}. We can represent the main part of operator L at origin 0 in the form

$$L_0(0, t, D) = \tau^{-2m} Q(\omega, t, \tau D_\tau, D_\omega), \qquad D_\tau = \frac{i\partial}{\partial\tau}.$$

Hence, the following spectral problem is of great importance:

$$Q(\omega, t, \lambda, D_\omega)\,\upsilon(\omega) = 0, \qquad \omega \in G, \tag{3.6}$$

$$D_\omega^j \upsilon(\omega) = 0, \qquad \omega \in \partial G, \qquad j = 0, 1, \ldots, m-1. \tag{3.7}$$

We prove the following theorem.

Theorem 3.2. *Let* $u(x,t)$ *be the generalized solution of problem* (2.1) *such that* $u \equiv 0$ *for* $|x| > R = \text{const}$ *and let* $f_{t^k} \in L^\infty(0, \infty; L_2(K))$, $f_{t^k}(x, 0) = 0$ *for* $k \le 2m$. *In addition, suppose that the stripe*

$$m - \frac{n}{2} \le \operatorname{Im}\lambda \le 2m - \frac{n}{2}$$

does not contain points of spectrum of problem (3.6), (3.7) *for every* $t \in [0, \infty)$. *Then* $u(x,t) \in H_0^{2m}(e^{-\gamma_{2m} t}, K_\infty)$ *and the following inequality is true:*

$$\|u\|^2_{H_0^{2m}(e^{-\gamma_{2m}t},\,K_\infty)} \le C \sum_{k=0}^{2m} \left\|f_{t^k}\right\|_{L^\infty(0,\infty;L_2(K))}, \qquad C = \text{const.}$$

Proof. First, we prove the inequality

$$\left\| u_{t^s} \right\|^2_{H_0^{2m,0}(e^{-\gamma t}, K_\infty)} \le C \sum_{k=0}^{2m} \left\| f_{t^k} \right\|^2_{L^\infty(0,\infty;L_2(K))}, \quad C = \text{const}, \quad s \le 2m. \tag{3.8}$$

It follows from (3.1) that

$$(-1)^{m-1} L_0(0, t, D) u = F(x, t), \tag{3.9}$$

where $F(x, t) = u_t + f + (-1)^{m-1}[L_0(0, t, D) - L(x, t, D)]u$. Theorem 3.1 and Lemma 3.1 imply that $F(x, t) \in H^{0,0}_{m-1}(e^{-\gamma t}, K_\infty)$. Therefore, $F(x, t) \in H^0_{m-1}(e^{-\gamma t}, K_\infty)$ for almost every $t \in [0, \infty)$. On the other hand, in the stripe $m - \frac{n}{2} \le \operatorname{Im} \lambda \le m + 1 - \frac{n}{2}$, there are no points of spectrum of problem (3.6), (3.7) for every $t \in [0, \infty)$. So, the results of elliptic problem [6] imply that, for almost every $t \in [0, \infty)$, $u \in H^{2m}_{m-1}(e^{-\gamma_{2m} t}, K_\infty)$ and

$$\| u \|^2_{H^{2m}_{m-1}(e^{-\gamma_{2m} t}, K_\infty)} \le C\left[\| f \|^2_{L^\infty(0,\infty;L_2(K))} + \| u_t \|^2_{L^\infty(0,\infty;L_2(K))} \right], \qquad C = \text{const}.$$

The repeating arguments conducted above lead to the inequality

$$\| u \|^2_{H_0^{2m}(e^{-\gamma_{2m} t}, K_\infty)} \le C\left[\| f \|^2_{L^\infty(0,\infty;L_2(K))} + \| u_t \|^2_{L^\infty(0,\infty;L_2(K))} \right]$$

and, moreover, to the inequality

$$\| u \|^2_{H_0^{2m,0}(e^{-\gamma t}, K_\infty)} \le C\left[\| f \|^2_{L^\infty(0,\infty;L_2(K))} + \| u_t \|^2_{L^\infty(0,\infty;L_2(K))} \right].$$

This and Theorem 3.1 imply

$$\| u \|^2_{H_0^{2m,0}(e^{-\gamma t}, K_\infty)} \le C\left[\| f \|^2_{L^\infty(0,\infty;L_2(K))} + \| f_t \|^2_{L^\infty(0,\infty;L_2(K))} \right],$$

i.e., (3.8) is proved for $s = 0$.

Assume that (3.8) is true for $s - 1$. We differentiate system (2.1) s times with respect to t and put $v = u_{t^s}$.

As a result, we obtain

$$(-1)^{m-1} L v = v_t + f_{t^s} + (-1)^m \sum_{k=1}^{s} \binom{s}{k} L_{t^k} u_{t^{s-k}}, \tag{3.10}$$

where

$$L_{t^k} = \sum_{|p|,|q|=1}^{m} D^p \frac{\partial^k a_{pq}}{\partial t^k} D^q + \sum_{|p|=1}^{m} \frac{\partial^k a_p}{\partial t^k} D^p + \frac{\partial^k a}{\partial t^k}.$$

From induction assumption and from arguments that are analogous to the case $s = 0$, we obtain (3.8).

Since

$$\| u \|^2_{H_0^{2m}(e^{-2\gamma_m t}, K_\infty)} \le \sum_{s=0}^{2m-1} \left\| u_{t^s} \right\|^2_{H_0^{2m,0}(e^{-\gamma t}, K_\infty)} + \left\| u_{t^{2m}} \right\|^2_{H_0^{0,0}(e^{-\gamma t}, K_\infty)},$$

the confirmation of theorem follows from (3.8) and Theorem 3.1. Theorem 3.2 is proved.

We now consider the Dirichlet problem for the system

$$(-1)^{m-1} L_0(0, t, D) u = F(x, t), \qquad x \in K. \tag{3.11}$$

Lemma 3.2. *Let $u(x, t)$ be the generalized solution of Dirichlet problem for* (3.11) *for almost every $t \in [0, \infty)$ such that $u \equiv 0$ in the case where $|x| > R =$*

$= \text{const}$, *and let* $u(x,t) \in H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t}, K_\infty)$. *Let, besides that,* $F \in H^{l,0}_{\beta}(e^{-\gamma t}, K_\infty)$. *Then* $u(x,t) \in H^{2m+l,0}_{\beta}(e^{-\gamma t}, K_\infty)$ *and*

$$\|u\|^2_{H^{2m+l,0}_{\beta}(e^{-\gamma t},\, K_\infty)} \le C\Big[\|F\|^2_{H^{l,0}_{\beta}(e^{-\gamma t},\, K_\infty)} + \|u\|^2_{H^{2m+l-1,0}_{\beta}(e^{-\gamma t},\, K_\infty)}\Big],$$

where $C = \text{const}$.

Proof. It is well-known [5] that

$$\int_{\Omega^2} \big|D^\mu u(x,t)\big|^2 dx \le C \int_{\Omega^1\cup\Omega^2\cup\Omega^3} \Big(\sum_{|\alpha|\le l} \big|D^\alpha F(x,t)\big|^2 + |u(x,t)|^2\Big) dx,$$

$$|\mu| \le 2m + l,$$

where Ω^1, Ω^2, Ω^3 are the same as in Lemma 3.1, $C = \text{const}$. From this inequality and from arguments analogous to the proof of inequality (3.3), we obtain

$$\int_K r^{2\beta}\big|D^\mu u(x,t)\big|^2 dx \le$$

$$\le C\int_K \Big(\sum_{|\alpha|\le l} r^{2(\beta+|\alpha|-l)}\big|D^\alpha F(x,t)\big|^2 + r^{2(\beta-2m-l)}|u(x,t)|^2\Big) dx.$$

After integrating this inequality with respect to t from 0 to ∞, we obtain

$$\int_{K_\infty} e^{-2\gamma t} r^{2\beta}\big|D^\mu u(x,t)\big|^2 dx\,dt \le$$

$$\le C\Big[\|F\|^2_{H^{l,0}_{\beta}(e^{-\gamma t},\, K_\infty)} + \int_{K_\infty} e^{-2\gamma t} r^{2(\beta-2m-l)}|u(x,t)|^2 dx\,dt\Big] \le$$

$$\le C\Big[\|F\|^2_{H^{l,0}_{\beta}(e^{-\gamma t},\, K_\infty)} + \|u\|^2_{H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t},\, K_\infty)}\Big]. \tag{3.12}$$

We have

$$\|u\|^2_{H^{2m+l,0}_{\beta}(e^{-\gamma t},\, K_\infty)} =$$

$$= \sum_{|\mu|=2m+l} \int_{K_\infty} e^{-2\gamma t} r^{2\beta}\big|D^\mu u(x,t)\big|^2 dx\,dt + \|u\|^2_{H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t},\, K_\infty)}.$$

This and (3.12) yield the statement of lemma. Lemma 3.2 is proved.

Theorem 3.3. *Let* $f_{t^k} \in L^\infty(0,\infty; L_2(K))$, $f_{t^k}(x,0) = 0$ *for* $k \le l + 2m$, *and let* $u(x,t)$ *be the generalized solution of problem* (2.1) *such that* $u \equiv 0$ *for* $|x| > R = \text{const}$. *In addition, we suppose that, in the stripe*

$$m - \frac{n}{2} \le \operatorname{Im}\lambda \le 2m + l - \frac{n}{2},$$

there are no points of spectrum of problem (3.6), (3.7) *for every* $t \in [0,\infty)$. *Then* $u(x,t) \in H^{2m+l}_0(e^{-\gamma_{2m+l}t}, K_\infty)$ *and the following inequality holds*:

$$\|u\|^2_{H^{2m+l}_0(e^{-\gamma_{2m+l}t},\, K_\infty)} \le C\sum_{k=0}^{2m+l}\big\|f_{t^k}\big\|^2_{L^\infty(0,\infty; L_2(K))}, \quad C = \text{const}.$$

Proof. The theorem will be proved by induction on l. For $l = 0$, the statement of theorem follows from Theorem 3.2. Let this statement be true with substitution of l by $l - 1$.

We prove the inequality

$$\left\| u_{t^s} \right\|^2_{H_0^{2m+l-s}(e^{-\gamma_{2m+l-s}t},\, K_\infty)} \le C \sum_{k=0}^{2m+l} \left\| f_{t^k} \right\|_{L^\infty(0,\infty; L_2(K))} \tag{3.13}$$

for $s = l, l-1, \ldots, 0$, where $C = \text{const}$.

Since $f_{t^k} \in L^\infty(0, \infty; L_2(K))$, $f_{t^k}(x, 0) = 0$ for $k \le l + 2m - 1$, we obtain from Theorem 3.1 that $u_{t^{l+1}} \in H_0^{0,0}(e^{-\gamma t}, K_\infty)$. This and arguments analogous to the proof of Theorem 3.1 yield inequality (3.13) for $s = l$.

Assume that (3.13) is true for $s = l, l-1, \ldots, j+1$. Put $v = u_{t^j}$. From (3.10) it follows that

$$(-1)^{m-1} L v = F_j, \tag{3.14}$$

where

$$F_j = v_t + f_{t^j} + (-1)^m \sum_{k=1}^{j} \binom{j}{k} L_{t^k} u_{t^{j-k}}.$$

By virtue of induction hypothesis with respect to l, we obtain

$$\sum_{k=1}^{j} \binom{j}{k} L_{t^k} u_{t^{j-k}} \in H_0^{l-j}(e^{-\gamma_{l-j}t}, K_\infty).$$

On the other hand, in view of induction assumption with respect to s, we have $v_t \in H_0^{l-j}(e^{-\gamma_{l-j}t}, K_\infty)$. Therefore, $F_j \in H_0^{l-j}(e^{-\gamma_{l-j}t}, K_\infty)$. Since

$$H_0^{l-j}(e^{-\gamma_{l-j}t}, K_\infty) \subseteq H_{-1}^{l-j-1,0}(e^{-\gamma t}, K_\infty),$$

we have $F_j \in H_{-1}^{l-j-1,0}(e^{-\gamma t}, K_\infty)$.

By repeating the arguments analogous to the proof of Theorem 3.1, we obtain that

$$v \in H_{-1}^{2m+l-j-1,0}(e^{-\gamma t}, K_\infty).$$

This and Lemma 3.2 imply

$$u_{t^j} = v \in H_0^{2m+l-j,0}(e^{-\gamma t}, K_\infty),$$

$$\left\| u_{t^j} \right\|^2_{H_0^{2m+l-j,0}(e^{-\gamma t},\, K_\infty)} \le C \sum_{k=0}^{2m+l} \left\| F_{t^k} \right\|^2_{L^\infty(0,\infty; L_2(K))}, \qquad C = \text{const}. \tag{3.15}$$

We have

$$\left\| u_{t^j} \right\|^2_{H_0^{2m+l-j}(e^{-\gamma_{2m+l-j}t},\, K_\infty)} \le \left\| u_{t^{j+1}} \right\|^2_{H_0^{2m+l-j-1}(e^{-\gamma_{2m+l-j+1}t},\, K_\infty)} + \left\| u_{t^j} \right\|^2_{H_0^{2m+l-j,0}(e^{-\gamma t},\, K_\infty)}. \tag{3.16}$$

By virtue of induction hypothesis with respect to s, it follows from (3.13) that

$$\left\| u_{t^{j+1}} \right\|^2_{H_0^{2m+l-j-1}(e^{-\gamma_{2m+l-j-1}t},\, K_\infty)} \le C \sum_{k=0}^{2m+l} \left\| f_{t^k} \right\|^2_{L^\infty(0,\infty; L_2(K))}, \qquad C = \text{const}.$$

This and (3.15), (3.16) yield the inequality

$$\left\| u_{t^j} \right\|^2_{H_0^{2m+l-j}(e^{-\gamma_{2m+l-j}t},\, K_\infty)} \le C \sum_{k=0}^{2m+l} \left\| f_{t^k} \right\|^2_{L^\infty(0,\infty; L_2(K))}, \qquad C = \text{const}.$$

We obtain the statement of theorem for $j = 0$.

Consider the theorem on the smoothness of generalized solution in domains with a conic point on boundary.

Theorem 3.4. *Let* $u(x, t)$ *be the generalized solution of problem* (2.1) *and let* $f_{t^k} \in L^\infty(0, \infty; L_2(\Omega))$, $f_{t^k}(x, 0) = 0$ *for* $k \le l + 2m$. *In addition, we assume that*

$$m - \frac{n}{2} \le \operatorname{Im}\lambda \le l + 2m - \frac{n}{2}$$

does not contain the points of spectrum of problem (3.6), (3.7) *for every* $t \in [0, \infty)$. *Then*

$$u(x, t) \in H_0^{2m+l}(e^{-\gamma_{2m+l}t}, \Omega_\infty)$$

and the following inequality holds:

$$\|u\|^2_{H_0^{2m+l}(e^{-\gamma_{2m+l}t}, \Omega_\infty)} \le C \sum_{k=0}^{2m+l} \|f_{t^k}\|^2_{L^\infty(0,\infty; L_2(\Omega))}, \qquad C = \text{const}.$$

Proof. We surround the point 0 by the neighbourhood U_0 with so small diameter that the intersection of Ω and U_0 coincides with K.

Consider the function $u_0 = \varphi_0 u$, where $\varphi_0 \in \overset{\circ}{C}{}^\infty(U_0)$ and $\varphi_0 \equiv 1$ in some neighbourhood of 0.

The function u_0 satisfies the system

$$(-1)^{m-1} L(x, t, D) u_0 - (u_0)_t = \varphi_0 f + L'(x, t, D) u,$$

where L' is a linear differential operator of order less than $2m$.

The coefficients of this operator depend on the choice of function φ_0 and equal 0 outside U_0. This and arguments analogous to the proof of Theorem 3.3 imply

$$\|\varphi_0 u\|^2_{H_0^{2m+l}(e^{-\gamma_{2m+l}t}, \Omega_\infty)} \le C \sum_{k=0}^{2m+l} \|f_{t^k}\|^2_{L^\infty(0,\infty; L_2(\Omega))}. \tag{3.17}$$

The function $\varphi_1 u = (1 - \varphi_0) u$ equals 0 in the neighbourhood of the conic point. We can apply to this function the theorem on the smoothness of solution of elliptic problem in a smooth domain, from which it follows that $\varphi_1 u \in H_0^{2m+l}(e^{-\gamma_{2m+l}t}, \Omega)$. This and Theorem 3.1 yield the relations $\varphi_1 u \in H_0^{2m+l}(e^{-\gamma_{2m+l}t}, \Omega_\infty)$ and

$$\|\varphi_1 u\|^2_{H_0^{2m+l}(e^{-\gamma_{2m+l}t}, \Omega_\infty)} \le C \sum_{k=0}^{2m+l} \|f_{t^k}\|^2_{L^\infty(0,\infty; L_2(\Omega))}. \tag{3.18}$$

Since $u = \varphi_0 u + \varphi_1 u$, it follows from (3.17), (3.18) that

$$\|u\|^2_{H_0^{2m+l}(e^{-\gamma_{2m+l}t}, \Omega_\infty)} \le C \sum_{k=0}^{2m+l} \|f_{t^k}\|^2_{L^\infty(0,\infty; L_2(\Omega))}.$$

The theorem is proved.

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