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DIFFUSION APPROXIMATION WITH EQUILIBRIUM OF EVOLUTIONARY SYSTEMS SWITCHED BY SEMI-MARKOV PROCESSES

ДИФУЗІЙНА АПРОКСИМАЦІЯ З РІВНОВАГОЮ ЕВОЛЮЦІЙНИХ СИСТЕМ, ЩО ПЕРЕМИКАЮТЬСЯ НАПІВМАРКОВСЬКИМИ ПРОЦЕСАМИ

We consider an evolutionary system switched by a semi-Markov process. For this system we obtain a nonhomogeneous diffusion approximation results where the initial process is compensated by the averaging function in the average approximation scheme.

Для систем, що перемикаються напівмарковськими процесами, одержано результати про неоднорідну дифузійну апроксимацію, де висхідний процес компенсується усередненою функцією в апроксимаційній схемі усереднення.

1. Introduction. Dynamic systems described by evolutionary equation is a classical topic in stochastic modelling. Asymptotic analysis of such systems is studied by several authors (see, e.g., [1 – 5]).

The usual asymptotic approach, in the diffusion approximation scheme, consist into normalize the process about an equilibrium point obtained by a balance condition with respect to the equilibrium distribution. Another diffusion approximation can be obtained by considering fluctuation with respect to the average process. In a previous work we have studied evolutionary systems with Markov switching in two cases [6]. The first case when the average process is a deterministic function and the second case when the average was a stochastic process.

In the present paper, we compensate the initial process by an averaging deterministic function instead of an equilibrium point (see, e.g., [6]) and we obtain a nonhomogeneous diffusion approximation result.

In Section 2 we describe processes implied in our analysis. In Section 3 we present result (Theorem 1) and in Section 4 the proof of this theorem.

2. Preliminaries. Let E be a Polish space and $\mathscr{E}$ its Borel σ-algebra. We call the measurable space $(E, \mathscr{E})$ a standard state space.

The semi-Markov continuous stochastic system is considered in the series scheme with small series parameter $\varepsilon > 0$, $\varepsilon \to 0$, described by a solution of the evolutionary equation in $\mathbb{R}^d$

$$\frac{d}{dt}U^\varepsilon(t) = a_\varepsilon\left(U^\varepsilon(t); x\left(\frac{t}{\varepsilon^2}\right)\right). \tag{1}$$

The velocity function admit the following representation:

$$a_\varepsilon(u; x) = a(u; x) + \varepsilon a_1(u; x), \tag{2}$$

where $u \in \mathbb{R}^d$ and $x \in E$.

The semi-Markov switching process $x(t)$, $t \geq 0$, on the standard state space $(E, \mathscr{E})$, is given by the semi-Markov kernel

$$Q(x, B, t) = P(x, B) F_x(t), \tag{3}$$

where $x \in E$, $B \in \mathscr{E}$, and $t \geq 0$, and supposed to be supposed to be uniformly ergodic with the stationary distribution $\pi(B)$, $B \in \mathscr{E}$, satisfying the relation

$$\pi(dx) = \rho(dx)\frac{m(x)}{m}, \tag{4}$$

where $\rho(B)$, $B \in \mathcal{E}$, is the stationary distribution of the embedded Markov chain x_n, $n \ge 0$, given by the stochastic kernel

$$P(x, B) := \mathbb{P}\left(x_{n+1} \in B \,\middle|\, x_n = x\right). \tag{5}$$

In addition

$$m(x) := \int_0^\infty \overline{F}_x(t)\,dt, \qquad \overline{F}_x(t) := 1 - F_x(t), \quad m := \int_E \rho(dx)m(x). \tag{6}$$

It is well-known (see, e.g., [3]) that under some additional conditions the stochastic system $U^\varepsilon(t)$, $t \ge 0$, converges weakly to the deterministic average process $\hat{U}(t)$, $t \ge$ ≥ 0, defined by a solution of the average evolutionary equation

$$\frac{d}{dt}\hat{U}(t) = \hat{a}\left(\hat{U}(t)\right), \tag{7}$$

with the average velocity

$$\hat{a}(u) := \int_E \pi(dx)a(u; x). \tag{8}$$

It is natural that the fluctuation of the stochastic system around the average process can be described by the diffusion process (see [6]). The diffusion approximation scheme for the semi-Markov continuous stochastic system (1) here considered for the centered and normalized process

$$\zeta^\varepsilon(t) = \varepsilon^{-1}\left[U^\varepsilon(t) - \hat{U}(t)\right]. \tag{9}$$

3. Main results. The main result is formulated as follows.

Theorem 1. *Let the stochastic evolutionary system* (9) *be defined by relations* (1) – (9) *and the following conditions be fulfilled:*

C1) *the switching semi-Markov process* $x(t)$, $t \ge 0$, *is uniformly ergodic with stationary distribution* $\pi(dx)$ *on the compact phase space* E;

C2) *the following asymptotic expansions take place:*

$$a(v + \varepsilon u; x) = a(v; x) + \varepsilon u a'_v(v; x) + \theta_0^\varepsilon(v, u; x),$$

$$a_1(v + \varepsilon u; x) = a_1(v; x) + \theta_1^\varepsilon(v, u; x),$$

where, for any $R > 0$,

$$\sup_{\substack{|v| \le R \\ |u| \le R \\ x \in E}} \left|\theta_i^\varepsilon(v, u; x)\right| \to 0, \quad \varepsilon \to 0, \quad i = 0, 1.$$

In addition the velocity functions $a(u; x)$ *and* $a_1(u; x)$ *satisfy the global solution of the equation* (1) *and* (7).

Then the weak convergence for $0 \le t \le T$,

$$\zeta^\varepsilon(t) \Rightarrow \zeta^0(t), \quad \varepsilon \to 0, \tag{10}$$

takes place. The limit diffusion process $\zeta^0(t)$, $t \ge 0$, *is determined by the generator of the coupled process* $\zeta^0(t)$, $\hat{U}(t)$, $t \ge 0$,

$$\mathbb{L}\varphi(u; v) = b(u, v)\varphi'_u(u, v) + \frac{1}{2}B(v)\varphi''_u(u, v) + \hat{a}(v)\varphi'_v(u, v). \tag{11}$$

Here

$$b(u; v) = \hat{a}_1(v) + u\hat{a}'(v), \tag{12}$$

$$\hat{a}(v) = \int_E \pi(dx)a(v; x), \qquad \hat{a}_1(v) = \int_E \pi(dx)a_1(v; x),$$

where prime and double prime mean first and second derivatives respectively.

The covariance matrix $B(v)$, $v \in \mathbb{R}^d$, *is determined by the relations*

$$B(v) = B_0(v) + B_1(v), \tag{13}$$

$$B_0(v) = 2\int_E \pi(dx)\tilde{a}(v; x)R_0\tilde{a}(v; x),$$

$$B_1(v) = 2\int_E \pi(dx)\mu(x)\tilde{a}(v; x)\tilde{a}^*(v; x), \tag{14}$$

$$\mu(x) = \frac{m_2(x) - 2m^2(x)}{m(x)}$$

where a^* *means transpose of vector* a.

Remarks. **1.** The particular case $\mu(x) = 0$ correspond to the exponential distribution $F_x(t) = 1 - \exp\{-\lambda(x)t\}$. As a corollary in this case, we get the results given in [6].

2. The limit diffusion process $\zeta^0(t)$, $t \geq 0$, in nonhomogeneous in time and is solution of the following SDE

$$d\zeta^0(t) = \left[a_1\big(\hat{U}(t)\big) + \hat{a}'\big(\hat{U}(t)\big)\zeta^0(t)\right]dt + B^{1/2}\big(\hat{U}(t)\big)dW(t),$$

where $W(t)$, $t \geq 0$, is the standard Wiener process in $\mathbb{R}^d$.

3. The stationary regime for the average process $\hat{U}(t)$, $t \geq 0$, is realized when he average velocity $\hat{a}(v)$ has an equilibrium point $\rho : \hat{a}(\rho) = 0$. Then the limit diffusion process $\hat{\xi}(t)$, $t \geq 0$, is of the Ornstein – Uhlenbeck process with the following generator:

$$\hat{\mathbb{L}}\varphi(u) = b(u)\varphi'(u) + \frac{1}{2}B\varphi''(u),$$

where

$$b(u) = b_1 + ub_0, \qquad b_1 = \hat{a}_1(\rho), \qquad b_0 = \hat{a}'(\rho), \qquad B = B(\rho).$$

4. Proof. The proof of Theorem 1 is divided on several steps. At first, the extended Markov chain

$$\zeta_n^\varepsilon = \zeta^\varepsilon(\varepsilon^2\tau_n), \qquad \hat{U}_n^\varepsilon = \hat{U}\big(\varepsilon^2, \tau_n\big), \qquad x_n = x(\tau_n), \quad n \geq 0, \tag{15}$$

is considered, where τ_n, $n \geq 0$, is the sequence of the Markov renewal moments (moments of jumps of the semi-Markov process $x(t)$, $t \geq 0$), that is,

$$\tau_{n+1} = \tau_n + \theta_{n+1}, \quad n \geq 0,$$

$$F_x(t) = \mathbb{P}\big(\theta_{n+1} \leq t \,\big|\, x_n = x\big).$$

Let us introduce the following families of semigroups:

$$\Gamma_t^\varepsilon(x)\varphi(u) = \varphi\big(U_x^\varepsilon(t)\big),$$

$$U_x^\varepsilon(0) = u \in \mathbb{R}^d, \tag{16}$$

where $U_x^{\varepsilon}(t)$, $t \geq 0$, is a solution of the evolutionary system

$$\frac{d}{dt} U_x^{\varepsilon}(t) = a_{\varepsilon}\big(U_x^{\varepsilon}(t); x\big), \quad x \in E,$$

and, similarly,

$$\hat{A}_t \varphi(v) = \varphi\big(\hat{U}(t)\big),$$
$$\hat{U}(0) = v \in \mathbb{R}^d, \tag{17}$$

where $\hat{U}(t)$, $t \geq 0$, is a solution of the average evolutionary system (7).

It is worth noticing that the generators of semigroups (16) and (17) are respectively:

$$\mathbf{\Gamma}_{\varepsilon}(x)\varphi(u) = a_{\varepsilon}(u; x)\varphi'(u),$$

$$\hat{\mathbf{A}}\varphi(v) = \hat{a}(v)\varphi'(v).$$

The following generators will be also used:

$$\mathbf{\Gamma}(x)\varphi(u) = a(u; x)\varphi'(x),$$

$$\tilde{\mathbf{\Gamma}}(x)\varphi(u) = \tilde{a}(u; x)\varphi'(x), \qquad \tilde{a}(u; x) := a(u; x) - \hat{a}(u).$$

The main object in asymptotic analysis with semi-Markov processes is the compensating operator of the extended embedded Markov chain (15) which is given ere in the next lemma.

Lemma 1. *The compensating operator of the extended embedded Markov chain* (15) *is determined by the relation*

$$\mathbb{L}^{\varepsilon}\varphi(u, v, x) = \varepsilon^{-2} q(x)\left[\int_0^{\infty} F_x(dt)\Gamma^{\varepsilon}_{\varepsilon^2 t}(x, v)\overline{\Gamma}^{\varepsilon}_{\varepsilon^2 t}(v)\hat{A}_{\varepsilon^2 t} P\varphi(u, v, x) - \varphi(u, v, x)\right], \tag{18}$$

where the semigroup $\Gamma^{\varepsilon}_t(x, v)$, $t \geq 0$, *is defined by the generator*

$$\mathbf{\Gamma}^{\varepsilon}(x, v)\varphi(u) = a^{\varepsilon}(v + \varepsilon u; x)\varphi'(u), \tag{19}$$

$$a^{\varepsilon}(u; x) := \varepsilon^{-1} a_{\varepsilon}(u; x) = \varepsilon^{-1} a_{\varepsilon}(u; x) + a_1(u; x), \tag{20}$$

the semigroup $\overline{\Gamma}^{\varepsilon}_t(v)$, $t \geq 0$, *is defined by the generator*

$$\overline{\mathbf{\Gamma}}^{\varepsilon}(v)\varphi(u) = -\varepsilon^{-1}\hat{\mathbf{A}}(v)\varphi(u) := -\varepsilon^{-1}\hat{a}(v)\varphi'(u). \tag{21}$$

It is worth noticing that the generator $\mathbf{\Gamma}^{\varepsilon}(x, v)$ in (19) can be transformed by using condition C_2 of Theorem 1, as follows:

$$\mathbf{\Gamma}^{\varepsilon}(x, v) = \varepsilon^{-1}\mathbf{\Gamma}_{\varepsilon}(x, v), \tag{22}$$

$$\mathbf{\Gamma}^{\varepsilon}(x, v)\varphi(u) :=$$

$$:= a_{\varepsilon}(v + \varepsilon u; x)\varphi'(u) = a(v; x)\varphi'(u) + \varepsilon b(v, u; x)\varphi'(u) + \theta^{\varepsilon}(v, u; x)\varphi(u),$$

where by definition

$$b(v, u; x) = a_1(v; x) + u a'_v(v; x).$$

***Proof of Lemma* 1.** The proof of this lemma is based on the conditional expectation of the extended embedded Markov chain (15) which is calculated by using (1) – (9)

$$\mathbb{E}\left[\varphi\big(\zeta^{\varepsilon}_{n+1}, \hat{U}^{\varepsilon}_{n+1}, x_{n+1}\big) \,\Big|\, \zeta^{\varepsilon}_n = u,\ \hat{U}^{\varepsilon}_n = v,\ x_n = x\right] =$$

$$= \int_0^\infty F_x(dt)\mathbb{E}\left[\varphi\left(u + \varepsilon^{-1}\left[\int_0^{\varepsilon^2 t} a_\varepsilon\big(U_x^s(s); x\big)ds - \int_0^{\varepsilon^2 t} \hat{a}\big(\hat{U}(s)\big)ds\right],\right.\right.$$

$$\left.\left. v + \int_0^{\varepsilon^2 t} \hat{a}\big(\hat{U}(s)\big)ds, x_{n+1}\right) \middle| U_x^\varepsilon(0) = v + \varepsilon u,\ \hat{U}_n^\varepsilon = v,\ x_n = x\right] =$$

$$= \int_0^\infty F_x(dt)\Gamma^\varepsilon_{\varepsilon^2 t}(x, v)\overline{\Gamma}^\varepsilon_{\varepsilon^2 t}(v)\hat{A}_{\varepsilon^2 t}P\varphi(u, v, x) \ =:\ \mathbb{F}^\varepsilon(x)P\varphi(u, v, x).$$

The next step in the asymptotic analysis is to construct the asymptotic expansion of the compensating operator with respect to ε.

Lemma 2. *The compensating operator* (18) – (21) *admits the following asymptotic representation of test function* $\varphi \in C_0^{3,2}\big(\mathbb{R}^d \times \mathbb{R}^d\big)$:

$$\mathbb{L}^\varepsilon \varphi(u, v, x) = \varepsilon^{-2} Q\varphi(\cdot, \cdot, x) + \varepsilon^{-1}\tilde{\mathbf{\Gamma}}(x, v)P\varphi(u, \cdot, \cdot) +$$

$$+ \left[\mathbb{L}_0(x, v)P\varphi(\cdot, v, \cdot) + \hat{\mathbf{A}}P\varphi(u, \cdot, \cdot)\right] + \theta_l^\varepsilon \varphi(u, v, x), \tag{23}$$

with the negligible term

$$\left\|\sup_{x\in E}\left|\theta_l^\varepsilon \varphi(u, v, x)\right|\right\| \to 0, \quad \varepsilon \to 0.$$

Here, by definition,

$$Q\varphi(x) = q(x)[P - I]\varphi(x), \tag{24}$$

is the generator of the associated Markov process $x_0(t)$, $t \geq 0$, *with the intensity function*

$$q(x) := \frac{1}{m(x)}, \qquad m(x) := \int_0^\infty \overline{F}_x(t)\,dt.$$

The generator $\tilde{\mathbf{\Gamma}}(x, v)$, *and the operator* $\mathbb{L}_0(x, v)$ *are defined as follows:*

$$\tilde{\mathbf{\Gamma}}(x, v)\varphi(u) = \tilde{a}(v; x)\varphi'(u), \tag{25}$$

and

$$\mathbb{L}_0(x, v)\varphi(u) = b(v, u; x)\varphi'(u) + \frac{1}{2}B_1(v; x)\varphi''(u), \tag{26}$$

$$b(v, u; x) := a_1(v; x) + u\,a_v'(v; x), \tag{27}$$

$$B_1(v; x) := \mu_2(x)\tilde{a}(v; x)\tilde{a}^*(v; x), \tag{28}$$

$$\mu_2(x) := \frac{m_2(x)}{m(x)}, \quad m_2(x) := \int_0^\infty t^2 F_x(dt). \tag{29}$$

Proof. At the beginning the compensating operator is transformed as follows:

$$\mathbb{L}^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-2}q(x)\big[\mathbb{F}^\varepsilon(x) - I\big]P. \tag{30}$$

Now, the following algebraic identity is used:

$$abc - 1 = (a-1) + (b-1) + (c-1) + (a-1)(b-1) + (a-1)(c-1) +$$

$$+ (b-1)(c-1) + (a-1)(b-1)(c-1). \tag{31}$$

Setting

$$a := \Gamma^{\varepsilon}_{\varepsilon^2 t}(x, v), \qquad b := \overline{\Gamma}^{\varepsilon}_{\varepsilon^2 t}(v), \qquad c := \hat{A}_{\varepsilon^2 t}, \tag{32}$$

the terms in (30) with (31) and (32) are transformed by using the integral equation for semigroup

$$F^{\varepsilon}_a(x) := \int_0^{\infty} F_x(dt)\Big[\Gamma^{\varepsilon}_{\varepsilon^2 t}(x, v) - I\Big] = \varepsilon^2 \mathbf{\Gamma}^{\varepsilon}(x, v) \int_0^{\infty} \overline{F}_x(t)\Gamma^{\varepsilon}_{\varepsilon^2 t}(x, v)\,dt =$$

$$= \varepsilon^2 m(x)\mathbf{\Gamma}^{\varepsilon}(x, v) + \varepsilon^4\Big[\mathbf{\Gamma}^{\varepsilon}(x, v)\Big]^2 \int_0^{\infty} \overline{F}_x(t)\Gamma^{\varepsilon}_{\varepsilon^2 t}(x, v)\,dt =$$

$$= \varepsilon^2 m(x)\mathbf{\Gamma}^{\varepsilon}(x, v) + \varepsilon^4 \frac{m_2(x)}{2}\Big[\mathbf{\Gamma}^{\varepsilon}(x, v)\Big]^2 + \varepsilon^6\Big[\mathbf{\Gamma}^{\varepsilon}(x, v)\Big]^3 F^{\varepsilon}_{a3}(x),$$

where

$$\overline{F}^{(k+1)}_x(t) := \int_t^{\infty} \overline{F}^{(k)}_x(s)\,ds, \qquad \overline{F}^{(1)}_x(t) := \overline{F}_x(t),$$

$$F^{\varepsilon}_{a3}(x) := \int_0^{\infty} \overline{F}^{3}_x(t)\Gamma^{\varepsilon}_{\varepsilon^2 t}(x, v)\,dt.$$

Taking into account (22) the following expansion is obtained:

$$F^{\varepsilon}_a(x) = \varepsilon m(x)\Gamma_{\varepsilon}(x, v) + \varepsilon^2 \frac{m_2(x)}{2}[\Gamma_{\varepsilon}(x, v)]^2 + \varepsilon^2 \theta^{\varepsilon}_a(x, v), \tag{33}$$

with the negligible term

$$\theta^{\varepsilon}_a(x, v) := \varepsilon\Big[\Gamma^{\varepsilon}(x, v)\Big]^3 F^{\varepsilon}_{a3}(x),$$

on test function $\varphi \in C^3_0\big(\mathbb{R}^d\big)$.

Similarly, the asymptotic expansion can be obtained for the next two terms in (31), (32)

$$F^{\varepsilon}_b(x) := \int_0^{\infty} F_x(dt)\Big[\overline{\Gamma}^{\varepsilon}_{\varepsilon^2 t}(v) - I\Big] = -\varepsilon m(x)\hat{\mathbf{A}}(v) + \varepsilon^3 \frac{m_2(x)}{2}\Big[\hat{\mathbf{A}}(v)\Big]^2 + \varepsilon^3 \theta^{\varepsilon}_b(x, v), \tag{34}$$

with the negligible term

$$\theta^{\varepsilon}_b(x, v) := \varepsilon\Big[\hat{\mathbf{A}}(v)\Big]^3 F^{\varepsilon}_{b3}(x),$$

$$F^{\varepsilon}_{b3}(x) := \int_0^{\infty} \overline{F}^{(3)}_x(t)\overline{\Gamma}^{\varepsilon}_{\varepsilon^2 t}(v)\,dt,$$

on test function $\varphi \in C^3_0\big(\mathbb{R}^d\big)$.

Analogously,

$$F^{\varepsilon}_c(x) := \int_0^{\infty} F_x(dt)\Big[\hat{A}_{\varepsilon^2 t} - I\Big] = \varepsilon^2 m(x)\hat{\mathbf{A}}(v) + \varepsilon^2 \theta^{\varepsilon}_c(x), \tag{35}$$

with the negligible term

$$\theta_c^\varepsilon(x) := \left[\hat{\mathbf{A}}(v)\right]^3 F_{c2}^\varepsilon(x), \qquad F_{c2}^\varepsilon(x) := \int_0^\infty \overline{F}_x^{(2)}(dt)\hat{A}_{\varepsilon^2 t}\, dt,$$

on test function $\varphi \in C_0^2(\mathbb{R}^d)$.

At last we analyze the next term

$$F_{ab}^\varepsilon(x) := \int_0^\infty F_x(dt)\left[\Gamma_{\varepsilon^2 t}^\varepsilon(x,v) - I\right]\left[\overline{\Gamma}_{\varepsilon^2 t}^\varepsilon(v) - I\right] = \Gamma^\varepsilon(x,v)\overline{\Gamma}^\varepsilon(v)F_{ab1}^\varepsilon(x),$$

$$F_{ab1}^\varepsilon(x) := \int_0^\infty F_x(dt)\left[\int_0^{\varepsilon^2 t}\Gamma_s^\varepsilon(x,v)\,ds \int_0^{\varepsilon^2 t}\overline{\Gamma}_s^\varepsilon(v)\,ds\right] =$$

$$= 2\varepsilon^4 \int_0^\infty F_x^{(2)}(dt)\Gamma_{\varepsilon^2 t}^\varepsilon(x,v)\overline{\Gamma}_{\varepsilon^2 t}^\varepsilon(v) + \varepsilon^4\theta_{ab}^\varepsilon(x).$$

Hence, by (21) and (22), we get

$$F_{ab}^\varepsilon(x) := -\varepsilon^2 m_2(x)\mathbf{\Gamma}_\varepsilon(x,\ v)\hat{\mathbf{\Gamma}}(v) + \varepsilon^2\theta_{ab}^\varepsilon(x), \tag{36}$$

with negligible term $\theta_{ab}^\varepsilon(x)$ on the test function $\varphi \in C^3(\mathbb{R}^d)$.

It can be easily verified that the last three terms in (31), (32) are negligible on test functions $\varphi \in C^{3,2}(\mathbb{R}^d \times \mathbb{R}^d)$. As a consequence, gathering the extensions (33) – (36), the asymptotic extension (23) – (28) for the compensating operator is obtained.

In the next step in the proof of Theorem 1, the limit generator (11) is calculated by using a solution of singular perturbation problem for the compensating operator (23) (see, e.g., [4, 6]).

Lemma 3. *A solution of singular perturbation problem for the generator* (23)

$$\mathbb{L}^\varepsilon\varphi^\varepsilon(u,v,x) = \mathbb{L}\varphi(u,v) + \theta_L^\varepsilon(u,v,x), \tag{37}$$

on test function $\varphi^\varepsilon(u,v,x) = \varphi(u,v) + \varepsilon\varphi_1(u,v,x) + \varepsilon^2\varphi_2(u,v,x)$, *and negligible term* $\theta_L^\varepsilon(u,v,x)$, *is realized by the generator* $\mathbb{L}$ *given in Theorem* 1, *formulae* (11) – (13).

Proof. According to [4, p. 51] (Lemma 3.3), the limit generator in (37) is represented as follows:

$$\mathbb{L}\Pi = \Pi\tilde{\mathbf{\Gamma}}(x,v)PR_0\tilde{\mathbf{\Gamma}}(x,v)P\Pi + \Pi\mathbb{L}_0(x,v)P\Pi + \Pi\hat{\mathbf{A}}P\Pi,$$

where the projector Π is defined as follows:

$$\Pi\varphi(x) = \int_E \pi(dx)\varphi(x).$$

Let us calculate

$$\mathbb{L}_1\Pi = \Pi\tilde{\mathbf{\Gamma}}(x,v)PR_0\tilde{\mathbf{\Gamma}}(x,v)P\Pi\varphi(u) = \Pi\tilde{\mathbf{\Gamma}}(x,v)PR_0\tilde{\mathbf{\Gamma}}(x,v)\varphi(u) =$$

$$= \Pi\tilde{\mathbf{\Gamma}}(x,v)PR_0\tilde{a}(v;x)\varphi'(u).$$

By the definition of potential operator R_0 [4], we have

$$QR_0 = R_0Q = \Pi - I,$$

or

$$q(x)[P - I]R_0 = \Pi - I,$$

hence, $PR_0 = R_0 + m(x)[\Pi - I]$.

So, we can write

$$\mathbb{L}_1\Pi = \Pi\tilde{\mathbf{\Gamma}}(x,v)[R_0 - m(x)I]\tilde{a}(v;x)\varphi'(u) =$$

$$= \Pi\tilde{\mathbf{\Gamma}}(x,v)R_0\,\tilde{a}(v;x)\varphi'(u) - \Pi\tilde{a}(v;x)m(x)\tilde{a}^*(v;x)\varphi''(u) =$$

$$= \Pi\tilde{a}(v;x)R_0\,\tilde{a}(v;x)\varphi''(u) - \Pi m(x)\tilde{a}(v;x)\tilde{a}^*(v;x)\varphi''(u).$$

Hence, the first term is

$$\mathbb{L}_1\varphi(u) = \frac{1}{2}B_0(v)\varphi''(u) - \frac{1}{2}A_0(v)\varphi''(u), \tag{38}$$

where

$$B_0(v) := 2\int_E \pi(dx)\tilde{a}(v;x)R_0\,\tilde{a}(v;x),$$

$$A_0(v) := 2\int_E \pi(dx)m(x)\tilde{a}(v;x)\tilde{a}^*(v;x).$$

The next term is

$$\Pi\mathbb{L}_0(x,v)P\Pi\varphi(u) = \Pi b(v,u;x)\varphi'(u) + \frac{1}{2}\Pi B_1(v;x)\varphi''(u) = \mathbb{L}_0(v)\varphi(u),$$

where

$$\mathbb{L}_0(v)\varphi(u) = b(v,u)\varphi'(u) + \frac{1}{2}B_1(v)\varphi''(u), \tag{39}$$

and

$$B_1(v) := \int_E \pi(dx)\mu_2(x)B_1(v;x),$$

functions $B_1(v;x)$, $b(v,u)$, $b(v,u;x)$ and $\mu_2(x)$ are defined respectively in (28), (12), (27) and (29).

Hence, setting together (38) and (39) we obtain the generator $\mathbb{L}$ of Theorem 1.

The last step of the proof concerns the relative compactness of the probability measures of the processes where it can be realized by the standard scheme as it is given in [7] and [8].

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