We estimate the rate of convergence for functions of bounded variation for the Bézier variant of the Szász operators $S_{n,\alpha}(f,x)$. We study the rate of convergence of $S_{n,\alpha}(f,x)$ for the case $0 < \alpha < 1$.

1. Introduction. For the case where $\alpha \geq 1$ or $0 < \alpha < 1$ and a function $f$ is defined on $[0,\infty)$, the Szász – Bézier operator $S_{n,\alpha}$ is defined by

$$S_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right),$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}(x) - J_{n,k+1}(x)$ and $J_{n,k}(x) = \sum_{j=0}^{\infty} s_{n,j}(x)$ with the Szász basis function $s_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}$, $k = 0, 1, 2, \ldots$. It is well known that for $\alpha = 1$, the operators (1) reduce to the well-known Szász – Mirakyan operators

$$S_{n}(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right).$$

The rate of convergence for the Szász – Mirakyan operators on functions of bounded variation was first studied by Cheng [1]. Recently, Zeng [2] and Zeng and Zhao [3] estimated the rates of convergence for the Szász – Bézier operators whenever $\alpha \geq 1$. The rates of convergence for the other case of $\alpha \in (0,1)$ for functions of bounded variation were obtained in [4] and [5], respectively, for the Kantorovich – Bézier operators and Bernstein – Bézier operators. Motivated by this, we extend the results of [1, 3, 6] and study the rate of convergence for the Szász – Bézier operators $S_{n,\alpha}(f,x)$, $0 < \alpha < 1$ for functions of bounded variation.

Our main theorem is stated as follows:

**Theorem.** Let $f$ be a function of bounded variation on every finite subinterval of $[0,\infty)$. Let $f(t) = O(t^r)$ for some $r \in \mathbb{N}$ as $t \to \infty$. Then for $x \in (0,\infty)$, $0 < \alpha < 1$, there exists a positive constant $M(f, \alpha, x, r)$ such that, for $n$ sufficiently large, we have

$$\left|S_{n,\alpha}(f,x) - \frac{1}{2^n} f(x^+) - \left(1 - \frac{1}{2^n}\right) f(x^-) \right| \leq$$

$$\frac{Z(x)}{\sqrt{n x}} \left|f(x^+) - f(x^-)\right| + \frac{1}{\sqrt{2e}n x} \varepsilon_n(x) \left|f(x) - f(x^-)\right| +$$

$$+ \frac{5}{n^2} \sum_{k=1}^{n} \Omega \left(f_x, \frac{x}{\sqrt{k}}\right) + \frac{M(f, \alpha, x, r)}{n^m},$$

where $Z(x) = \min \{0, 0.8 \sqrt{(1+3x)} + 0.5, 1.6x^2 + 1.3\}$,
$$\Omega_x(f, \lambda) = \sup_{t \in [x-\lambda, x+\lambda]} |f(t) - f(x)|, \quad \varepsilon_n(x) = \begin{cases} 1, & \text{if } x = \frac{k'}{n}, \ k' \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_\lambda(t) = \begin{cases} f(t) - f(x-), & 0 \leq t < x, \\ 0, & t = x, \\ f(t) - f(x+), & x < t < \infty. \end{cases}$$

It is clear that:

(i) $\Omega_x(f, h)$ is monotone nondecreasing with respect to $h$;

(ii) $\lim_{h \to \infty} \Omega_x(f, h) = 0$ if $f$ is continuous at the point $x$;

(iii) if $f$ is a function of bounded variation on $[a, b]$ and $V^b_a(f)$ denotes the total variation of $f$ on $[a, b]$, then $\Omega_x(f, h) \leq V^{x-h}_{a}(f)$.

We recall the Lebesgue – Stieltjes integral representation

$$S_{n,\alpha}(f, x) = \int_0^\infty f(t) d_x(K_{n,\alpha}(x, t)),$$

where

$$K_{n,\alpha}(x, t) = \begin{cases} \sum_{k \leq m} Q^{(\alpha)}_{n,k}(x), & 0 \leq t < \infty, \\ 0, & t = 0. \end{cases}$$

We also define

$$H_{n,\alpha}(x, t) = \begin{cases} 1 - K_{n,\alpha}(x, t), & 0 \leq t < \infty, \\ 0, & t = 0. \end{cases}$$

2. Lemmas. In the sequel, we shall need the following lemmas:

**Lemma 1** [6]. For all $x \in (0, \infty)$ and $x \in \mathbb{N}$ we have

$$s_{n,k}(x) < \frac{1}{\sqrt{2e} \sqrt{n} x}.$$

**Lemma 2** [2]. For $x \in (0, \infty)$ we have

$$\left| \sum_{k \geq nx} s_{n,k}(x) - \frac{1}{2} \right| \leq \min \left\{ \frac{0.8 \sqrt{(1+3x)} + 0.5}{1 + \sqrt{n} x}, \frac{1.6x^2 + 1.3}{1 + \sqrt{n} x} \right\}$$

and, for $0 \leq t < x$, we have

$$\sum_{k \leq nt} Q^{(\alpha)}_{n,k}(x) \leq \frac{x}{n(t-x)^2}.$$
Proof. Since \(0 < x < t < \infty\), we have \(\frac{k/n - x}{t - x} \geq 1\) for \(k \geq nt\). Thus,

\[
H_{n,t}(x) = 1 - K_{n,t}(x, t) = 
1 - \sum_{k \geq nt} Q_{n,k}^{(\alpha)}(x) \leq \sum_{k \geq nt} Q_{n,k}^{(\alpha)}(x) \leq \left( \sum_{k \geq nt} \frac{k/n - x}{(t - x)^{2m/\alpha}} s_{n,k}(x) \right)^{\alpha} \leq \frac{1}{(t - x)^{2m/\alpha}} \left( \sum_{k \geq nt} k/n - x \right)^{2m/\alpha} s_{n,k}(x)^{\alpha}.
\]

Put \(l = 2m/\alpha\) and suppose that \([l]\) denotes the integral part of \(l\). Following [4] (Lemma 6), choose the numbers \(p = \frac{2[l]}{2[l] + 2 - l}\), \(q = \frac{2[l]}{l - 2}\).

For each real, put \(\psi(t) = t - x\). Note that \(\frac{2}{p} + \frac{2(l + [l])}{q} = \frac{2[l]}{2[l] + 2 - l} + \frac{l - 2}{l} = l\). The application of Hölder’s inequality yields

\[
\sum_{k = 0}^{\infty} \frac{k/n - x}{(t - x)^{2m/\alpha}} s_{n,k}(x)^{\alpha} \leq \left( \sum_{k = 0}^{\infty} \frac{k/n - x}{(t - x)^{2m/\alpha}} s_{n,k}(x) \right)^{1/p} \left( \sum_{k = 0}^{\infty} \frac{k/n - x}{(t - x)^{2m/\alpha}} s_{n,k}(x) \right)^{1/q} = 
\left( S_{n,1}(\psi^2, x) \right)^{1/p} \left( S_{n,1}(\psi^{2([l]+1)}, x) \right)^{1/q}.
\]

By using the well-known result \(S_{n,1}(\psi^r, x) = O(n^{-r})\) as \(n \to \infty\) \((r = 1, 2, 3, \ldots)\), we obtain

\[
\left( \sum_{k = 0}^{\infty} \frac{k/n - x}{(t - x)^{2m/\alpha}} s_{n,k}(x) \right)^{\alpha} \leq O\left(n^{-\alpha/p - \alpha([l]+1)/q}\right) = O(n^{-m}),
\]

since

\[
-\frac{\alpha}{p} - \frac{\alpha([l]+1)}{q} = -\alpha - \frac{\alpha(l+1)}{2l} = -\alpha - \frac{\alpha l - 2}{l} = -\alpha \frac{l}{2} = -m.
\]

This completes the proof of Lemma 3.

3. Proof of Theorem. We have

\[
f(t) = 2^{-\alpha}f(x+) + (1 - 2^{-\alpha})f(x-) + g(x) + 2^{-\alpha}(f(x+) - f(x-))\text{sign}^{(\alpha)}(t) + 
\left(f(x) - 2^{-\alpha}f(x+) - (1 - 2^{-\alpha})f(x-))\delta_{x}(t),
\]

where

\[
\text{sign}^{(\alpha)}(t - x) := \begin{cases} 
2^{\alpha} - 1 & \text{if } t > x, \\
0 & \text{if } t = x, \text{ and } \delta_{x}(t) = \begin{cases} 
1 & \text{if } x = t, \\
0 & \text{if } x \neq t.
\end{cases}
\end{cases}
\]

Therefore,

\[
\left| S_{n,\alpha}(f, x) - \frac{1}{2^{\alpha}}f(x+) - \left(1 - \frac{1}{2^{\alpha}}\right)f(x-) \right| \leq 
\leq \left| S_{n,\alpha}(f_{x}, x) + \frac{f(x) - f(x-)}{2^{\alpha}}S_{n,\alpha}(\text{sign}^{(\alpha)}(t - x), x) \right|
\]

ISSN 1027-3190. Укр. мат. журн., 2005, т. 57, № 12
We first estimate
\[
S_n(\alpha, x) = 2^n \sum_{k > nx} Q_{n,k}^{(\alpha)}(x) - 1 + \varepsilon_n(x) \|Q_n(\alpha)\|
\]
and
\[
S_n(\alpha, \delta_x, x) = \varepsilon_n(x) \|Q_n(\alpha)\|.
\]
Hence, we have
\[
\frac{f(x) - f(x -)}{2^n} \left[S_n(\alpha, x) + \left[f(x) - \frac{1}{2^n} f(x +) - \left(1 - \frac{1}{2^n}\right)f(x -)\right] S_n(\alpha, \delta_x, x)\right] =
\]
\[
\left[\frac{f(x) - f(x -)}{2^n}\right]^{\alpha}
\]
By mean value theorem, we have
\[
\left[\sum_{j > nx} s_{n,j}(x)\right]^{\alpha} - \frac{1}{2^n} = \alpha \left[\zeta_{n,j}(x)\right]^{\alpha-1} \left[\sum_{j > nx} s_{n,j}(x) - \frac{1}{2^n}\right],
\]
where \(\zeta_{n,j}(x)\) lies between \(\frac{1}{2}\) and \(\sum_{j > nx} s_{n,j}(x)\). In view of Lemma 2, it is observed
that, for \(n\) sufficiently large, the intermediate point \(\zeta_{n,j}\) is arbitrary close to \(\frac{1}{2}\), i.e.,
\[
\zeta_{n,j} = \frac{1}{2 + \varepsilon}
\]
with an arbitrary small \(\varepsilon\). Then we have
\[
\alpha \left[\zeta_{n,j}(x)\right]^{\alpha-1} \leq \alpha (2 + \varepsilon)^{1-\alpha}.
\]
The latter expression is positive and strictly increasing for \(\alpha \in (0, 1)\), since
\[
\frac{\partial}{\partial \alpha} \alpha (2 + \varepsilon)^{1-\alpha} = (2 + \varepsilon)^{1-\alpha}[1 - \alpha \log(2 + \varepsilon)] > 0
\]
for sufficiently small \(\varepsilon\). Thus, it takes maximum value at \(\alpha = 1\). This implies
\[
\alpha \left[\zeta_{n,j}(x)\right]^{\alpha-1} \leq 1.
\]
Hence,
\[
\left[\sum_{j > nx} s_{n,j}(x)\right]^{\alpha} - \frac{1}{2^n} \leq \frac{Z(x)}{1 + \sqrt{nx}}, \quad Z(x) = \min\{0, 8\sqrt{(1 + 3x)} + 0.5, 1, 6x^2 + 1, 3\}.
\]
We also have

\[
ISSN 1027-3190. Украина мат. журн., 2005, т. 57, № 12
\]
\[ Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x) = \alpha(\zeta_{n,k}^{(\alpha)})^{n-1}s_{n,k}(x), \]

where \( J_{n,k+1}(x) < \zeta_{n,k}(x) < J_{n,k}(x) \). Thus, by Lemma 1, we have

\[ Q_{n,k}^{(\alpha)}(x) \leq \frac{1}{\sqrt{2enx}}. \tag{5} \]

Combining the estimates of (3) – (5), we have

\[
\begin{align*}
&Q_x + Q_{\alpha_{x}}(x) + Q_{\alpha_{y}}(x) + Q_{\alpha_{z}}(x) + Q_{\alpha_{w}}(x) + Q_{\alpha_{v}}(x) + Q_{\alpha_{u}}(x) \\
&\leq \frac{Z(x)}{1 + \sqrt[n]{nx}} \left| f(x) - f(x^-) \right| + \frac{1}{\sqrt{2enx}} \left| f(x) - f(x^-) \right|.
\end{align*}
\]

We next estimate \( S_{\alpha}(g_{x}, x) \) as follows:

\[
S_{\alpha}(f_{x}, x) = \int_{0}^{\infty} f_{x}(t) d_{1}(K_{\alpha}(x, t)) =
\]

\[
= \left\{ \int_{t_1} + \int_{t_2} + \int_{t_3} + \int_{t_4} \right\} f_{x}(t) d_{1}(K_{\alpha}(x, t)) = E_1 + E_2 + E_3 + E_4 \quad \text{say,} \tag{6}
\]

where \( t_1 = \left[ 0, x - \frac{x}{\sqrt{n}} \right], \ t_2 = \left[ x - \frac{x}{\sqrt{n}}, x + \frac{x}{\sqrt{n}} \right], \ t_3 = \left[ x + \frac{x}{\sqrt{n}}, 2x \right], \text{ and } t_4 = [2x, \infty). \)

We first estimate \( E_2 \). Noting that \( f_{x}(x) = 0 \), we have

\[
|E_2| \leq \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} \left| g_{x}(t) - g_{x}(x) \right| d_{1}(K_{\alpha}(x, t)) \leq \Omega_{x}(f_{x}, x/\sqrt{n}) \leq \frac{x}{nx} \sum_{k=1}^{n} \Omega_{x} \left( f_{x}, \frac{x}{\sqrt{k}} \right). \tag{7}
\]

We next estimate \( E_1 \). Writing \( y = x - \frac{x}{\sqrt{n}} \) and using Lebesgue – Stieltjes integration by parts, we have

\[
|E_1| = \left| \int_{0}^{y} f_{x}(t) d_{1}(K_{\alpha}(x, t)) \right| \leq \int_{0}^{y} \Omega_{x}(f_{x}, x-t) d_{1}K_{\alpha}(x, t) =
\]

\[
= \Omega_{x}(f_{x}, x-y)K_{\alpha}(x, y) + \int_{0}^{y} \hat{K}_{\alpha}(x, t) d_{1}(-\Omega_{x}(f_{x}, x-t)),
\]

where \( \hat{K}_{\alpha}(x, t) \) is the normalized form of \( K_{\alpha}(x, t) \). Since \( \hat{K}_{\alpha}(x, t) \leq K_{\alpha}(x, t) \) on \((0, \infty)\), by Lemma 2 it follows that

\[
|E_1| \leq \Omega_{x}(f_{x}, x-y) \frac{x}{n(x-y)^2} + \frac{x}{n} \int_{0}^{y} \frac{1}{(x-t)^2} d_{1}(-\Omega_{x}(f_{x}, x-t)).
\]

Integrating by parts the last term, we have

\[
\int_{0}^{y} \frac{1}{(x-t)^2} d_{1}(-\Omega_{x}(f_{x}, x-t)) = \left. -\frac{\Omega_{x}(f_{x}, x-t)}{(x-t)^2} \right|_{0}^{y} + \int_{0}^{y} \Omega_{x}(f_{x}, x-t) \frac{2dt}{(x-t)^2}. \tag{8}
\]
Hence, by replacing the variable $t$ in the last integral by $x - \frac{x}{\sqrt{u}}$, we get

$$|E_1| \leq \frac{2}{nx} \sum_{k=1}^{n} \Omega_k \left( f_x, \frac{x}{\sqrt{k}} \right).$$

(8)

Using the similar method for the estimation of $E_3$, we get

$$|E_3| \leq \frac{2}{nx} \sum_{k=1}^{n} \Omega_k \left( f_x, \frac{x}{\sqrt{k}} \right).$$

(9)

Finally, by assumption, we have the estimate

$$|f_x(t)| \leq M r^{r} \leq M \left( \frac{t-x}{x} \right)^{r} \quad \text{for} \quad t \geq 2x.$$ 

Now

$$|E_4| = \int_{2x}^{\infty} f_x(t) K_{n,\alpha}(x, t) \, dt \leq \int_{2x}^{\infty} |f_x(t)| d_i K_{n,\alpha}(x, t) \leq$$

$$\leq M x^{-r} \int_{0}^{\infty} (t-x)^{r} d_i K_{n,\alpha}(x, t) \leq -M x^{-r} \int_{2x}^{\infty} (t-x)^{r} d_i (1 - K_{n,\alpha}(x, t)) =$$

$$= -M x^{-r} \int_{2x}^{\infty} (t-x)^{r} d_i (H_{n,\alpha}(x, t)) \leq -M x^{-r} \int_{0}^{\infty} (t-x)^{r} d_i (1 - K_{n,\alpha}(x, t)) =$$

$$= M x^{-r} \lim_{R \to \infty} \left( -(t-x)^{r} H_{n,\alpha}(x, t) \int_{2x}^{R} d_i (t-x)^{r} \right) =$$

$$= M x^{-r} \lim_{R \to \infty} \left( -(t-x)^{r} E(\alpha) \int_{2x}^{R} \frac{r E(\alpha)}{n^m (t-x)^m} (t-x)^{r-1} dt \right) =$$

$$= M x^{-r} \lim_{R \to \infty} \left( -(t-x)^{r} \frac{r E(\alpha)}{n^m (t-x)^m} \int_{2x}^{R} (t-x)^{r-1} dt \right) =$$

$$= M \frac{E(\alpha)}{n^m x^m} + \frac{r E(\alpha)}{n^m (m-r)x^{m-r}}, \quad m > r.$$ 

(10)

Combining the estimates of (2) – (10), we obtain the required result.

This completes the proof of the theorem.


Received 10.11.2004