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T. K. Yuldashev (Kyrgyz Law Acad., Osh),
J. A. Artykova (Osh Univ., Kyrgyzstan)

VOLTERRA FUNCTIONAL INTEGRAL EQUATION OF THE FIRST KIND WITH NONLINEAR RIGHT-HAND SIDE AND VARIABLE LIMITS OF INTEGRATION

ФУНКЦІОНАЛЬНО-ІНТЕГРАЛЬНЕ РІВНЯННЯ ВОЛЬТЕРРИ ПЕРШОГО РОДУ З НЕЛІНІЙНОЮ ПРАВОЮ ЧАСТИНОЮ І ЗМІННИМИ МЕЖАМИ ІНТЕГРУВАННЯ

We prove a theorem on the existence and uniqueness of a solution on a Volterra functional integral equation of the first kind with nonlinear right-hand side and nonlinear deviation. We use the method of successive approximations in combination with the method of compressing mapping.

Доведено теорему про існування та єдиність розв'язку функціонально-інтегрального рівняння Вольтерри першого роду з нелінійним відхиленням. При цьому використано метод послідовних наближень у поєднанні з методом стискаючих відображень.

In this paper, we consider a Volterra functional integral equation of the form

$$\int\limits_{\alpha(t)}^{\beta(t)} K(t,s)u\big[u(s)\big]ds = f\Big(t, u\big[\delta(t,u(t))\big]\Big), \quad t \in T_1, \tag{1}$$

with initial value condition

$$u(t) = g(t), \quad t \in E_0 \equiv [t_0; t_1], \tag{2}$$

where $K(t,s) \in C(T_0^2)$, $0 \le K(t) \equiv K(t,t)$, $f(t,u) \in C(T_1 \times X)$, $T_0^2 \equiv T_0 \times T_0$, $T_1 \equiv [t_1; T]$, $T_0 \equiv [t_0; T]$, $0 < t_0 < T < \infty$, $t_0 < t_1$, X is a bounded closed set in $R \equiv (-\infty; \infty)$, $t_0 \le \alpha(t) < \beta(t) \le T$, $\alpha(t)$, $\beta(t) \in C(T_0)$, $\delta(t,u) \in C(T_1 \times X)$, $t_0 \le \delta(t,u) \le t$, $g(t) \in C(E_0)$.

We study the existence and uniqueness of a solution of the Volterra functional integral equation (1) with initial value condition (2) on the segment T_1. Here, we use the method of successive approximations in combination with the method of compressing mapping.

We note that the Volterra integral equation of the first kind, in which the right-hand side is presented by $f(t)$-known continuous function studied by many authors (see bibliography in [1]).

The Volterra functional integral equations with such right-hand side were considered in our works [2–4]. The present paper is the further development of these works.

We rewrite the Volterra functional integral equation (1) in the following form:

$$\int\limits_{\alpha(t)}^{\beta(t)} K(t,s)u(s)ds =$$

$$= \int\limits_{\alpha(t)}^{\beta(t)} K(t,s)\big[u(s) - u[u(s)]\big]ds + f\big(t, u\big[\delta(t,u(t))\big]\big), \quad t \in T_1,$$

or

$$u(t) + \int\limits_{t_0}^{t} K(t,s)u(s)ds =$$

$$= u(t) + \int\limits_{\alpha(t)}^{\beta(t)} K(t,s)\big[u(s) - u[u(s)]\big]ds + \int\limits_{t_0}^{\alpha(t)} K(t,s)u(s)ds+$$

$$+ \int\limits_{\beta(t)}^{t} K(t,s)u(s)ds + f\big(t, u\big[\delta(t,u(t))\big]\big), \quad t \in T_1. \tag{3}$$

We change equation (3) as follows:

$$u(t) + \int\limits_{t_0}^{t} K(s)u(s)ds =$$

$$= -\int\limits_{t_0}^{t} \big[K(t,s) - K(s)\big]u(s)ds + f_0(t,u), \quad t \in T_1,$$

where we denote the right-hand side of (2) by $f_0(t,u)$, i.e.,

$$f_0(t,u) = u(t) + \int\limits_{\alpha(t)}^{\beta(t)} K(t,s)\big[u(s) - u[u(s)]\big]ds+$$

$$+ \int\limits_{t_0}^{\alpha(t)} K(t,s)u(s)ds + \int\limits_{\beta(t)}^{t} K(t,s)u(s)ds + f\big(t, u\big[\delta(t,u(t))\big]\big), \quad t \in T_1. \tag{4}$$

Hence, using the resolvent method for $\big[-K(s)\big]$, we obtain

$$u(t) = -\int\limits_{t_0}^{t} \big[K(t,s) - K(s)\big]u(s)ds + f_0(t,u)+$$

$$+ \int\limits_{t_0}^{t} K(s)\exp\big\{-\varphi(t,s)\big\}\Big\{-f_0(s,u) + \int\limits_{t_0}^{s} \big[K(s,\tau) - K(\tau)\big]u(\tau)d\tau\Big\}ds, \quad t \in T_1, \tag{5}$$

where $\varphi(t,s) = \int_s^t K(\tau)d\tau$, $\varphi(t,t_0) = \varphi(t)$, $\varphi(t_1) \neq 0$.

Applying Direchlet's formulation to (5), we derive

$$u(t) = \int\limits_{t_0}^{t} H(t,s)u(s)ds + f_0\big((t,u(t)\big)\exp\big\{-\varphi(t)\big\}+$$

$$+ \int\limits_{t_0}^{t} K(s)\exp\big\{-\varphi(t,s)\big\}\Big[f_0\big(t,u(t)\big) - f_0\big(s,u(s)\big)\Big]ds, \quad t \in T_1, \tag{6}$$

where

$$H(t,s) \equiv -\big[K(t,s) - K(s)\big] \exp\big\{-\varphi(t,s)\big\} -$$

$$-\int\limits_s^t K(\tau) \exp\big\{-\varphi(t,\tau)\big\}\big[K(t,s) - K(\tau,s)\big] d\tau$$

and $f_0(t,u)$ is defined from (4).

The Volterra functional integral equations (1) and (6) are equivalent.

Theorem. *Assume that the following conditions are satisfied*:

1) $\big|K(\tau,s) - K(\eta,t)\big| \le L_1(s)\varphi(\tau,\eta)$, $0 \le L_1(s)$;

2) $f(t,u) \in \mathrm{Bnd}(M) \cap \mathrm{Lip}(L_{3|u})$, $0 \le M$, $L_3 = \mathrm{const}$;

3) $\big|\varphi(t,s)\big| \le L_4|t-s|$, $0 \le L_4 = \mathrm{const}$;

4) $\delta(t,u) \in \mathrm{Lip}(L_{5|u})$, $0 \ge L_5 = \mathrm{const}$;

5) *for all* $t \in T_1$ *there holds* $\rho = \rho(t) < 1$,

$$\rho = \int\limits_{t_0}^t \big\|L_1(s)\big\| ds + \int\limits_{t_0}^t \big\|L_1(s)\big\| ds +$$

$$+\Big(1 + \Delta_1 + L_3 + (2 + L_2 L_4)\Delta_2 + L_2 L_3 L_4 L_5\Big) \times$$

$$\times \left(\exp\big\{-\varphi(t)\big\} + 2\int\limits_{t_0}^t \big\|K(s)\big\| \exp\big\{-\varphi(t,s)\big\} ds \right),$$

$$\Delta_1 = \int\limits_{t_0}^{\alpha(t)} \big\|K(t,s)\big\| ds + \int\limits_{\beta(t)}^t \big\|K(t,s)\big\| ds,$$

$$\Delta_2 = \int\limits_{\alpha(t)}^{\beta(t)} \big\|K(t,s)\big\| ds, \quad 0 \le L_2 = \mathrm{const}.$$

Then the Volterra functional integral equation (1) *with initial value condition* (2) *has the unique solution on* T_1.

The theorem is proved by the method of successive approximations, which is defined by the following relations:

$$u_0(t) = \begin{cases} g(t), & t \in E_0, \\ f(t,0)\exp\big\{-\varphi(t)\big\} + & \\ + \displaystyle\int\limits_{t_0}^t K(s) \exp\big\{-\varphi(t,s)\big\}\big[f(t,0) - f(s,0)\big] ds, & t \in T_1, \end{cases}$$

$$u_{k+1}(t) = \begin{cases} g(t), & t \in E_0, \\ \displaystyle\int_{t_0}^{t} H(t,s)u_k(s)ds + f_0\bigl(t, u_k(t)\bigr)\exp\bigl\{-\varphi(t)\bigr\} + \\ \displaystyle + \int_{t_0}^{t} K(s)\exp\bigl\{-\varphi(t,s)\bigr\}\Bigl[f_0\bigl(t,u_k(t)\bigr) - f_0\bigl(s,u_k(s)\bigr)\Bigr]ds, \\ & k = 0,1,\ldots, t \in T_1. \end{cases}$$

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