UDC 517.5 + 517.43

V. N. Pivovarchik (South-Ukr. Ped. Univ., Odessa)

ON SPECTRA OF A CERTAIN CLASS OF QUADRATIC OPERATOR PENCILS WITH ONE-DIMENSIONAL LINEAR PART*

ПРО СПЕКТРИ ПЕВНОГО КЛАСУ КВАДРАТИЧНИХ ОПЕРАТОРНИХ В'ЯЗОК З ОДНОВИМІРНОЮ ЛІНІЙНОЮ ЧАСТИНОЮ

We consider a class of quadratic operator pencils that occur in many problems of physics. The part of such a pencil linear with respect to the spectral parameter describes the viscous friction in problems of small vibrations of strings and beams. Patterns in location of eigenvalues of such pencils are established. If the viscous friction (damping) is pointwise, then the operator in the linear part of the pencil is one-dimensional. For this case, rules in the location of the purely imaginary eigenvalues are found.

Розглянуто певний клас квадратичних операторних в'язок, що виникають у багатьох задачах фізики. Лінійна за спектральним параметром частина в'язки описує в'язке тертя в задачах про малі коливання струн та стержнів. Встановлено закономірності в розташуванні власних значень таких в'язок. Якщо в'язке тертя зосереджене в одній точці, то оператор у лінійній за параметром частині в'язки є одновимірним. Для цього випадку знайдено порядок розташування суто уявних власних значень.

1. Introduction. Pioneering results on direct and inverse problems of small transversal vibrations of an inhomogeneous string with pointwise damping were obtained by M. G. Krein and A. A. Nudelman [1, 2]. In these papers conditions were obtained necessary and sufficient for a sequence of complex numbers to be the spectrum of a string whose density belongs to the class of so-called S-strings. It should be mentioned that in implicit form the necessary and sufficient conditions for a certain subclass were obtained in [3]. Later vibrating systems with point-wise damping were considered in many publications [4 – 12]. One of the general approaches to abstract versions of such problems is to use the theory of entire functions see [3, 13]. The spectra of strings are considered there as the sets of zeros of function of Hermite – Biehler class (see [13, p. 307] for the definition) or generalized Hermite – Biehler class. For compressed beam vibrations (see [12]) one needs to use so-called shifted Hermite – Biehler functions (see [14]). Another approach is to use the theory of quadratic operator pencils. Here an important step was done in the famous paper by M. G. Krein and H. Langer [15]. This approach was used in [4, 5, 8, 16] and in many other papers. In present paper some abstract results on quadratic operator pencils are obtained and applied to boundary problems which have eigenvalues in both upper and lower half-planes. In Section 2 we describe general results on location of spectra of quadratic operator pencils of the form $L(\lambda) = \lambda^2 M - i\lambda K - A$ with $M \geq 0$, $K \geq 0$, $A = A^* \geq \beta I$, $-\infty < \beta < 0$. In Section 3 we consider the case of one-dimensional operator K. In Section 4 the results of Sections 2 and 3 are applied to spectral problems which occur in physics.

2. Abstract results. Let us denote by $B(H)$ the set of bounded operators acting on a separable Hilbert space H. We deal here with the following quadratic operator pencil

$$L(\lambda) = \lambda^2 M - i\lambda K - A,$$

where $M \in B(H)$, $K \in B(H)$ and A is a closed operator on H with the domain $D(A)$ dense in H. The domain of the pencil is chosen as usually: $D(L(\lambda)) = D(M) \cap D(K) \cap \cap D(A) = D(A)$. Thus, it is independent of λ.

* This work is supported by grant UK2-2811-OD-06 of Ukrainian Ministry of Education and Science and US Civil Research and Development Foundation (CRDF).

Definition 2.1. *The pencil $L(\lambda)$ is said to be monic if $M = I$, where I is the identity operator.*

In what follows we suppose the following conditions to be satisfied:

Conditions I. *$M \geq 0$ and $K \geq 0$ are bounded operators, $A = A^* \geq -\beta I$ (β is a positive number); for some $\beta_1 > \beta$ there exists $(A + \beta_1 I)^{-1} \in S_\infty$, here by S_∞ we denote the set of compact operators on H; $\ker A \cap \ker K \cap \ker M = \{0\}$.*

Definition 2.2. *The set of values $\lambda \in \mathbf{C}$ such that $L^{-1}(\lambda)$ exists in $B(H)$ is said to be the resolvent set $\rho(L(\lambda))$ of $L(\lambda)$. The spectrum of the pencil $L(\lambda)$ is denoted by $\sigma(L(\lambda))$, i.e., $\sigma(L(\lambda)) = \mathbf{C}\backslash\rho(L(\lambda))$.*

Definition 2.3. *A number $\lambda_0 \in \mathbf{C}$ is said to be an eigenvalue of the pencil $L(\lambda)$ if there exists a vector $y_0 \in D(A)$ (called an eigenvector of $L(\lambda)$) such that $y_0 \neq 0$ and $L(\lambda_0)y_0 = 0$. Vectors $y_1, \ldots , y_{m-1}$ are called associated to y_0 if*

$$\sum_{s=0}^{k} \frac{1}{s!} \left. \frac{d^s L(\lambda)}{d\lambda^s} \right|_{\lambda=\lambda_0} y_{k-s} = 0, \quad k = \overline{1, m-1}.$$

The number m is said to be the length of the chain composed of the eigen- and associated vectors. The geometric multiplicity of an eigenvalue is defined to be the number of the corresponding linearly independent eigenvectors. The algebraic multiplicity of an eigenvalue is defined to be the greatest value of the sum of the lengths of chains corresponding to linearly independent eigenvectors. An eigenvalue is said to be isolated if it has some deleted neighborhood contained in the resolvent set. An isolated eigenvalue λ_0 of finite algebraic multiplicity is said to be normal if the image $\operatorname{Im}\, L(\lambda_0)$ is closed.

In the case of linear monic operator pencils $(\lambda I - A)$ with bounded operator A this definition of a normal eigenvalue coincides with the corresponding definition in [17] (Chapter I, Paragraph 2) for operators. Under Conditions I the theorem about analytic Fredholm operator valued functions which can be found in [18] (Chapter XI, Corollary 8.4) implies that the spectrum of $L(\lambda)$ consists of normal eigenvalues only.

We denote by $\mathbf{C}^+$ ($\mathbf{C}^-$) the open upper (lower) half-plane.

The following lemma is a generalization of statement 2.4^0 in [15].

Lemma 2.1. 1. *If $A \geq 0$, then the spectrum of $L(\lambda)$ (if not empty) is located in the closed upper half-plane.*

2. *If $A >> 0$, i.e., $A \geq \epsilon I$, $\epsilon > 0$ and $K > 0$, then the spectrum of $L(\lambda)$ (if not empty) is located in the open upper half-plan.*

3. *If $A >> 0$ and $\lambda^2 y - Ay \neq 0$ for all real λ and all nonzero $y \in \ker K$, then the spectrum of $L(\lambda)$ (if not empty) is located in the open upper half-plane.*

Proof. Let y_0 be an eigenvector corresponding to the eigenvalue λ_0. Then

$$(L(\lambda_0)y_0, y_0) = 0, \quad y_0 \neq 0,$$

and consequently,

$$((\operatorname{Re}\lambda_0)^2 - (\operatorname{Im}\lambda_0)^2)(My_0, y_0) + \operatorname{Im}\lambda_0(Ky_0, y_0) - (Ay, y_0) = 0 \tag{2.1}$$

and

$$\operatorname{Re}\lambda_0(2\operatorname{Im}\lambda_0(My_0, y_0) - (Ky_0, y_0)) = 0. \tag{2.2}$$

If $\operatorname{Re}\lambda_0 \neq 0$, then the inequality $\operatorname{Im}\lambda_0 \geq 0$ follows from (2.2). If $\operatorname{Re}\lambda_0 = 0$, then (2.1) implies $\operatorname{Im}\lambda_0 \geq 0$. The first assertion is proved.

Now let $A >> 0$ and $K > 0$. Let us set $\operatorname{Im}\lambda_0 = 0$ into (2.1) and (2.2). Then for $\operatorname{Re}\lambda_0 = 0$ (2.1) implies $(Ay_0, y_0) = 0$ what contradicts the inequality $A >> 0$. If $\operatorname{Re}\lambda_0 \neq 0$, $\operatorname{Im}\lambda_0 = 0$, then (2.2) implies $(Ky_0, y_0) = 0$, a contradiction. The second assertion is proved.

Let $y_0 \in \ker K$ be an eigenvector corresponding to a real eigenvalue λ_0. Then $L(\lambda_0)y_0 = \lambda_0^2 y_0 - Ay_0 = 0$, a contradiction.

Now let A be not positive but still bounded below. Then the pencil $L(\lambda)$ has eigenvalues in the open lower half-plane (see below).

Lemma 2.2. 1. *The part of the spectrum of $L(\lambda)$ located in the open lower half-plane lies on the imaginary axis.*

2. *If $K > 0$, then the part of the spectrum of $L(\lambda)$ located in the closed lower half-plane lies on the imaginary axis.*

Proof. Let y_0 be an eigenvector corresponding to an eigenvalue λ_0 with $\operatorname{Im}\lambda_0 < 0$. Then for $\operatorname{Re}\lambda_0 \neq 0$ equation (2.2) implies

$$(My_0, y_0) = (Ky_0, y_0) = 0$$

and consequently, $My_0 = Ky_0 = 0$. Then Conditions I imply $Ay_0 \neq 0$ and $L(\lambda_0)y_0 = = Ay_0 \neq 0$, a contradiction. The first assertion is proved.

If $K > 0$, then for $\operatorname{Im}\lambda_0 \leq 0$ the equality $\operatorname{Re}\lambda_0 = 0$ follows from (2.2).

Lemma 2.2 and the following lemma were proved in [19] for the case of monic operator pencils, i.e., for $M = I$.

Lemma 2.3. 1. *All the eigenvalues of $L(\lambda)$ located in $\overline{\mathbf{C}^-}\backslash\{0\}$ are semisimple, i.e., they do not possess associated vectors.*

2. *If $K > 0$, then all the eigenvalues of $L(\lambda)$ located in the closed lower half-plane are semisimple.*

Proof. Let λ_0 be an eigenvalue of $L(\lambda)$ located in the open lower half-plane (on the imaginary axis according to Lemma 2.2). Let us denote by y_0 (one of) the corresponding eigen- and by y_1 the associated vector. By Definition 2.3

$$\lambda_0^2 My_1 - i\lambda_0 Ky_1 - Ay_1 + 2\lambda_0 My_0 - iKy_0 = 0. \tag{2.3}$$

Multiplying (2.3) by y_0 we obtain

$$((\lambda_0^2 - i\lambda_0 K - A)y_1, y_0) + \big((2\lambda_0 M - iK)y_0, y_0\big) = 0. \tag{2.4}$$

Taking into account that λ_0 is pure imaginary we obtain from (2.4):

$$\big(y_1, (\lambda_0^2 M - i\lambda_0 K - A)y_0\big) + \big((2\lambda_0 M - iK)y_0, y_0\big) = 0,$$

what means

$$i\big((2\operatorname{Im}\lambda_0 M - K\big)y_0, y_0) = 0. \tag{2.5}$$

Equality (2.5) is possible for $\operatorname{Im}\lambda_0 < 0$ only if $(My_0, y_0) = (Ky_0, y_0) = 0$, i.e., if $My_0 = Ky_0 = 0$. In this case $L(\lambda_0)y_0 = -Ay_0 = 0$ and, consequently, $y_0 \in \ker M \cap \cap \ker K \cap \ker A$. Then due to Conditions I we have $y_0 = 0$, a contradiction.

Let now an eigenvalue $\lambda_0 \in \mathbf{R}\backslash\{0\}$. Then (2.2) implies $(Ky_0, y_0) = 0$, and, consequently, $Ky_0 = 0$ and $(\lambda_0^2 M - A)y_0 = 0$. Then (2.4) is equivalent to

$$\big(y_1, (\lambda_0^2 M + i\lambda_0 K - A)y_0\big) + 2\lambda_0 (My_0, y_0) = 0,$$

or

$$(My_0, y_0) = 0.$$

That means $My_0 = 0$. Hence, taking into account $Ky_0 = 0$ we obtain $Ay_0 = 0$, what contradicts Conditions I. The first statement is proved.

Let $K > 0$ and let λ_0 be a real eigenvalue of $L(\lambda)$. Let y_0 and y_1 be the corresponding eigen- and associated vectors. Then according to Lemma 2.2 $\lambda_0 = 0$ or, what is the same, $y_0 \in \ker A$ and (2.3) can be written as follows

$$Ay_1 + iKy_0 = 0. \tag{2.6}$$

Multiplying (2.6) by y_0 we obtain

$$(Ay_1, y_0) + i(Ky_0, y_0) = (y_1, Ay_0) + i(Ky_0, y_0) = i(Ky_0, y_0) = 0, \tag{2.7}$$

what contradicts $K > 0$.

Lemma 2.4. *If* $M + K \geq \epsilon I$ $(\epsilon > 0)$, $\dim \ker A > 0$ *and* $\dim(\ker A \cap \ker K) = p \geq 0$, *then the algebraic multiplicity of* $\lambda = 0$ *as an eigenvalue of* $L(\lambda)$ *is equal to* $p + \dim \ker A$.

Proof. Let $0 \neq y_0 \in \ker A$ and let y_1 be an associated to y_0 vector. Then

$$\left.\frac{dL(\lambda)}{d\lambda}\right|_{\lambda=0} y_0 + L(0)y_1 = -iKy_0 - Ay_1 = 0. \tag{2.8}$$

If $y_0 \in \ker K$ then y_1 can be chosen equal to 0. If $y_0 \notin \ker K$, then (2.8) implies

$$-i(Ky_0, y_0) - (Ay_1, y_0) = -i(Ky_0, y_0) - (y_1, Ay_0) = -i(Ky_0, y_0) = 0.$$

Combining the last equality with the condition $K \geq 0$ we obtain $Ky_0 = 0$, a contradiction. It remains to prove that the third vector of the chain does not exist. Suppose it does exist and denote it by y_2. Then

$$-Ay_2 - iKy_1 + My_0 = 0.$$

Consequently,

$$0 = -(Ay_2, y_0) - i(Ky_1, y_0) + (My_0, y_0) = (My_0, y_0)$$

and $(Ky_0, y_0) = (My_0, y_0) = 0$ contradicts the conditions of Lemma 2.4.

Usually it is more convenient to deal with bounded operator pencils. Let us introduce the following auxiliary pencil:

$$\tilde{L}(\lambda) = L(\lambda)(\beta_1 I + A)^{-1}.$$

Since $A \geq -\beta I > -\beta_1 I$, the pencil $L_1(\lambda)$ is bounded and the following lemma follows from Lemma 20.1 in [20].

Lemma 2.5.

$$\sigma(\tilde{L}(\lambda)) = \sigma(L(\lambda)).$$

Let us introduce the following parameter-dependent operator pencil:

$$L(\lambda, \eta) = \lambda^2 M - i\lambda\eta K - A. \tag{2.9}$$

It is clear that $L(\lambda, 1) = L(\lambda)$. Lemma 2.5 enables us to use the results of [21] (see also [22, 23]) established for bounded operator pencils. Adapted for our aims these results can be given in the following form.

Theorem 2.1. *Let the domain Ω contain the only eigenvalue λ_0 of the pencil $L(\lambda, \eta_0)$. Denote by m the algebraic multiplicity of λ_0. Then there exist numbers $\epsilon > 0$ and $m_1 \in \mathbf{N}$, $m_1 \leq m$, such that the following assertions are true in the neighborhood $|\eta - \eta_0| < \epsilon$:*

1. *$L(\lambda, \eta)$ possesses exactly m_1 different eigenvalues inside the domain Ω. Those eigenvalues can be arranged in groups $\lambda_{ij}(\eta)$ $\left(i = \overline{1, l};\ j = \overline{1, p_i};\ \sum_{i=1}^{l} p_i = m_1\right)$, in such a way that the functions of the group, i.e., $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ip_i}$ compose a complete set of a p_i-valued function. In this case those eigenvalues can be presented in the form of the following series*:

$$\lambda_{ij}(\eta) = \lambda_0 + \sum_{k=1}^{\infty} a_{ik} \left(\left((\eta - \eta_0)^{\frac{1}{p_i}} \right)_j \right)^k, \quad j = 1, 2, \ldots, p_i, \tag{2.10}$$

where $\left((\eta - \eta_0)^{\frac{1}{p_i}}\right)j$, $j = \overline{1, p_i}$, is the complete set of branches of the function $(\eta - \eta_0)^{\frac{1}{p_i}}$, p_i is the chain length of the eigenvector and associated vectors corresponding to the eigenvalue λ_0 and to the eigenvector y_{i0} of the pencil $L(\lambda, \eta_0)$.

2. *The basis of the eigen-space corresponding to $\lambda_{ij}(\eta)$ can be presented in the following form*:

$$y_{ij}^{(q)}(\eta) = y_{i0}^{(q)} + \sum_{k=1}^{\infty} y_{ik}^{(q)}(((\eta - \eta_0)^{\frac{1}{p_i}})_j)^k, \quad j = 1, 2, \ldots, p_i, \quad q = 1, 2, \ldots, \alpha_i, \tag{2.11}$$

where α_i is the geometric multiplicity of $\lambda_{ij}(\eta)$, $y_{i0}^{(q)}$ belong to the eigen-subspace of $L(\lambda, \eta_0)$ corresponding to the eigenvalue λ_0.

It should be mentioned that this theorem is a generalization of the Weierstrass theorem on function analytic in two variables [24, p. 476].

Definition 2.4. *The total algebraic multiplicity of the spectrum of $L(\lambda)$ lying in a domain Ω is defined to be the number $\sum_{i=1}^{n} m_i$, where m_i, $i = \overline{1, p}$, are the algebraic multiplicities of all the eigenvalues lying in Ω.*

Theorem 2.2. 1. *Let, in addition to Condition* I, *be $M = I$. Then the total algebraic multiplicity of the spectrum of $L(\lambda)$ located in the open lower half-plane coincides with the total algebraic multiplicity* (*here it is the same as total geometric multiplicity*) *of the negative spectrum of the operator A.*

2. *If, in addition $K > 0$, then the total algebraic multiplicity of the spectrum of $L(\lambda)$ located in the closed lower half-plane coincides with the total algebraic multiplicity of the nonpositive spectrum of A.*

Proof. 1. We are going to prove that the total algebraic multiplicity of the spectrum of the pencil $L(\lambda)$ located in the open lower half-plane coincides with the total algebraic multiplicity of the spectrum of the pencil $\lambda^2 I - A$ located in the open lower half-plane, or what is the same with the total algebraic multiplicity of the negative spectrum of the operator A. We consider the pencil $L(\lambda, \eta)$ as a perturbation of the pencil $\lambda^2 I - A$.

Let $\eta_0 \in [0, 1]$ and let λ_0 ($\operatorname{Re} \lambda_0 = 0$, $\operatorname{Im} \lambda_0 < 0$) be an eigenvalue of $L(\lambda, \eta_0)$. Then due to Lemma 2.3 this eigenvalue is semisimple. Then formulae (2.10), (2.11) for the eigenvalues of $L(\lambda, \eta)$ in the lower half-plane can be simplified:

$$\lambda_i(\eta) = \lambda_0 + \sum_{k=1}^{\infty} a_{ik} (\eta - \eta_0)^k, \tag{2.12}$$

$$y_i^{(q)}(\eta) = y_{i0}^{(q)} + \sum_{k=1}^{\infty} y_{ik}^{(q)}(\eta - \eta_0)^k, \quad q = \overline{1, \alpha_i}. \tag{2.13}$$

Differentiating the equality

$$L(\lambda_i(\eta), \eta) y_i^{(q)}(\eta) = 0$$

with respect to η and multiplying the resulting equation by $y_i^{(q)}$ we obtain for $\eta = \eta_0$:

$$a_{i1} = \frac{i\lambda_0 \big(K y_{i0}^{(q)}, y_{i0}^{(q)}\big)}{2\lambda_0 \big\|y_{i0}^{(q)}\big\|^2 - i\eta_0 \big(K y_{i0}^{(q)}, y_{i0}^{(q)}\big)} = \frac{i \operatorname{Im} \lambda_0 \big(K y_{i0}^{(q)}, y_{i0}^{(q)}\big)}{2 \operatorname{Im} \lambda_0 \big\|y_{i0}^{(q)}\big\|^2 - \eta_0 \big(K y_{i0}^{(q)}, y_{i0}^{(q)}\big)}. \tag{2.14}$$

It is clear that $\operatorname{Re} a_{i1} = 0$ and $\operatorname{Im} a_{i1} > 0$. It means that the eigenvalues of $L(\lambda, \eta)$ located in the open lower half-plane (on the imaginary axis) move upwards along the imaginary axis when η grows from 0 to 1. To show that eigenvalues do not came from $-i\infty$ let us find $\gamma > 0$ such that $(-i\infty, -i\gamma) \in \rho(L(\lambda, \eta))$ for $\eta \in [0, 1]$. Let $-i\tau$, $\tau > 0$, be an eigenvalue of $L(\lambda, \eta_0)$, where $\eta_0 \in [0, 1]$ and let y be one of the corresponding eigenvectors. Then

$$\tau^2 \|y\|^2 + \eta_0 \tau (Ky, y) + (Ay, y) = 0 \tag{2.15}$$

and consequently $\tau \le \sqrt{|\beta|}$. Consequently, the eigenvalues does not come from $-i\infty$. It means that

$$N(0) \ge N(1), \tag{2.16}$$

where by $N(\eta)$ we denote the total algebraic multiplicity of the spectrum of $L(\lambda, \eta)$ located in the open lower half-plane. Lemma 2.4 shows that the algebraic multiplicity of $\lambda = 0$ as of an eigenvalue of $L(\lambda, \eta)$ does not depend on η for $\eta \in (0, 1]$. That means that moving upwards as η changes from 0 to 1 the eigenvalues of $L(\lambda, \eta)$ on the negative imaginary half-axis do not cross the origin.

2. If $K > 0$ then according to Lemma 2.2 the only real eigenvalue can be at $\lambda = 0$. According to Lemma 2.3 this eigenvalue is semisimple. Therefore, the multiplicity of $\lambda = 0$ as an eigenvalue of $L(\lambda)$ is the same as its multiplicity as an eigenvalue of A. Combining this result with statement 1 of Theorem 2.2 we obtain statement 2.

Statement 1 of this theorem remains true for operators A admitting essential spectrum on $[0, \infty)$ under more restrictive conditions as it was shown in [25, 19]. There exist many related theorems [26–29]. This theorem remains true in the case of not symmetric operator K but such that $\operatorname{Re} \mathrm{K} >> 0$ (under some additional restrictions) see [30, 31].

Theorem 2.3. *Let us assume that, in addition to Condition* I, $M + K \ge \epsilon I$, $\epsilon > 0$. *Then Statement* 1 *of Theorem* 2.2 *is true.*

Proof. Due to Statement 1 of Theorem 2.2 the assertion of Theorem 2.3 is true for the operator pencil

$$\tilde{L}(\lambda, 1) = \lambda^2 I - i\lambda K - A.$$

Let us prove that it is true for each $\eta \in [0, 1]$ for the pencil

$$\tilde{L}(\lambda, \eta) = \lambda^2((1 - \eta)I + \eta M) - i\lambda K - A.$$

If $i\tau$, $\tau < 0$, is an eigenvalue of $\tilde{L}(\lambda, \eta)$ and y is the corresponding eigenvector, then

$$\tau = \frac{(Ky,y) - \sqrt{(Ky,y)^2 - 4((1-\eta)I + \eta M)y,y)(Ay,y)}}{2((1-\eta)I + \eta M)y,y)} =$$

$$= \frac{-2(Ay,y)}{(Ky,y) + \sqrt{(Ky,y)^2 - 4(((1-\eta)I + \eta M)y,y)(Ay,y)}}.$$

Due to inequalities $K \geq 0$, $M \geq 0$, $M + K \geq \epsilon > 0$ and $A \geq \beta I$, $\beta > -\infty$, we obtain that $\tau(\eta) \geq -\infty$ for each $\eta \in [0,1]$. It means that eigenvalues do not come from and do not leave at $-i\infty$. On the other, hand they do not cross the origin what can be proved in the same way as in proof of Theorem 2.2.

In what follows we consider the spectrum of $L(\lambda)$ in the upper half-plane.

Lemma 2.6. 1. *Let* $M \geq \epsilon I$, $\epsilon > 0$. *Then if the eigenvalue* λ_k *is not pure imaginary or if it is not semisimple then* $\operatorname{Im}\lambda_k \in [0, m_1]$, *where*

$$m_1 = \frac{1}{2} \sup_{0 \neq y \in D(A)} \frac{(Ky,y)}{(My,y)}.$$

2. *Let* $M >> 0$ *and* $K >> 0$ *on* $D(A)$. *Then if the eigenvalue* λ_k *is not pure imaginary or if it is not semisimple then* $\operatorname{Im}\lambda_k \in [m_2, m_1]$, *where*

$$m_2 = \frac{1}{2} \inf_{0 \neq y \in D(A)} \frac{(Ky,y)}{(My,y)}.$$

Proof. Let λ_k be a not pure imaginary eigenvalue of $L(\lambda)$. Then due to Lemma 2.2 it lies in the closed upper half-plane. Assertion 1 of Lemma 2.6 follows from (2.2). If λ_k is a pure imaginary eigenvalue in the closed upper half-plane having a chain of length 2 then Assertion 1 of Lemma 2.6 follows from (2.5). The proof of Assertion 2 is analogous.

We can consider $\theta = \eta^{-1}$ as the spectral parameter instead of λ when it is convenient.

Lemma 2.7. *Let all eigenvalues of* $L(\lambda, \theta^{-1})$ *be of geometric multiplicity* 1 *for all* θ *and let* $K \geq \nu I$, $\nu > 0$. *Let* $\theta_0 \in \mathbf{R}\backslash\{0\}$ *be an eigenvalue of the operator-function*

$$Q(\lambda, \theta) = I - \theta\big(i\lambda^{-1}K^{-\frac{1}{2}}AK^{-\frac{1}{2}} - i\lambda K^{-\frac{1}{2}}MK^{-\frac{1}{2}}\big)$$

for $\lambda_0 \in \mathbf{R}\backslash\{0\}$. *Then this eigenvalue is holomorphic as a function of* λ *in some real neighborhood* $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $\epsilon > 0$:

$$\theta(\lambda) = \theta_0 + \sum_{k=p}^{\infty} b_k (\lambda - \lambda_0)^k, \tag{2.17}$$

where $p \in \mathbf{N}$, $b_p \in \mathbf{R}\backslash\{0\}$.

Proof. The spectrum of $Q(\lambda_0, \theta)$ in the domain $\mathbf{C}\backslash\{0\}$ consists of normal eigenvalues only what follows from the above mentioned theorem about analytic operator-function [18] (Chapter XI, Corollary 8.4). The geometric multiplicity of each eigenvalue of $Q(\lambda_0, \theta)$ is equal to 1 because it coincides with that of the corresponding eigenvalue of $L(\lambda_0, \theta^{-1}) = -i\theta^{-1}\lambda_0 K^{\frac{1}{2}} Q(\lambda_0, \theta) K^{\frac{1}{2}}$. Now it is possible to apply the Rellich – Nagy theorem [32] to finish the proof.

Lemma 2.8. *Let the conditions of Lemma* 2.7 *be satisfied. Let* $\lambda_0 = i\tau_0$, $\tau_0 > 0$, *and* $\eta_0 > 0$, *then in some neighborhood of* (λ_0, η_0), *i.e., in* $\big\{(\lambda, \eta)\colon |\lambda - \lambda_0| < \epsilon,\ |\eta - \eta_0| < \delta, \epsilon > 0, \delta > 0\big\}$ *all the eigenvalues are given by the following formula*:

$$\lambda_j(\eta) = \lambda_0 + \sum_{k=1}^{\infty} \beta_k \left((\eta - \eta_0)_j^{\frac{1}{r}}\right)^k, \quad j = 1, 2, \ldots, r, \tag{2.18}$$

where $\beta_1 \neq 0$ *is a real or pure imaginary,* $(\eta - \eta_0)_j^{\frac{1}{r}}$, $j = 1, 2, \ldots, r$, *means the complete set of branches of the root.*

Proof. We obtain this result immediately after inverting (2.17).

Theorem 2.4. *Let, in addition to Condition* I, *be* $M = I$, *then for every* (*pure imaginary*) *eigenvalue* λ_{-k} *of* $L(\lambda)$ *from the closed lower half-plane there exists a pure imaginary eigenvalue* (*denote it* λ_k) *such that*

$$\mathrm{Im}(\lambda_k + \lambda_{-k}) \geq 0. \tag{2.19}$$

Proof. The eigenvalues of $L(\lambda, \eta)$ are piecewise analytic functions of η. They may loose analyticity only when they coincide. This follows from the results above. The eigenvalues located on $(-i\infty, 0)$ are analytic functions of $\eta > 0$ (see (2.12)) and move upwards along the imaginary axis when η increases. We identify $\lambda_j(\eta)$ as the eigenvalue satisfying the conditions $\lambda_{-j}(0) = -\lambda_j(0)$, where $\mathrm{Im}\ \lambda_{-j}(0) < 0$ and $\mathrm{Re}\ \lambda_{-j}(0) = 0$. For sufficiently small $\eta > 0$ we have $\mathrm{Im}\ \lambda_j(\eta) > 0$, $\mathrm{Re}\ \lambda_j(\eta) = 0$ due to the symmetry of the problem, Lemma 2.3 and the fact that all the normal eigenvalues of the pencil are semisimple when $\eta = 0$. It is easy to derive (see (2.14)) the following formula for the derivative:

$$\lambda_j'(\eta) = \frac{i\lambda_j(\eta)\,(Ky_j(\eta), y_j(\eta))}{2\lambda_j\,(\eta)\,\|y_j\,(\eta)\,\|^2 - i\eta\,(Ky_j\,(\eta)\,, y_j\,(\eta))}. \tag{2.20}$$

This formula implies $\mathrm{Im}\ \lambda_j'(\eta) \geq 0$ and $\mathrm{Re}\ \lambda_j'(\eta) = 0$ for $\eta \geq 0$ small enough. Hence, our theorem is true for $\eta \geq 0$ small enough. While $\eta > 0$ increases, $\lambda_j'(\eta)$ can change its sign only when the denominator in the right-hand side of (2.20) vanishes, i.e., when eigenvalues coalesce. If such a coalescence takes place on the interval $(0, i\infty)$, then the eigenvalues involved behave according to formula (2.18). Such a coalescence on the interval $(0, i\infty)$ is of one of the following three types. The first one has r odd in (2.18). In this case we identify the eigenvalue moving upwards along the imaginary axis after the coalescence as the one which moved upwards along the imaginary axis before the coalescence. By a coalescence of the second type we mean one which has r even and β_1 purely imaginary ($\beta_1 \neq 0$) in (2.18). After such a coalescence two new purely imaginary eigenvalues appear which are moving in opposite directions along the imaginary axis, and such a coalescence cannot violate Theorem 2.2. The third type of coalescence has even r and real $\beta_1 \neq 0$. Let $\lambda_j(\eta)$ take part in such a coalescence at $\eta = \eta_0 \in (0, 1]$. Then a coalescence of the second type indeed occurred at some $\eta \in (0, \eta_0)$ in some point $\lambda_\times \in (0, \lambda_j(\eta_0))$ on the imaginary axis. In this case the eigenvalue that has arisen after this coalescence and is moving upwards is identified as $\lambda_j(\eta)$.

To finish the proof we will show that for all $\eta \in [0, 1]$ the pure imaginary eigenvalues lie on some interval $[-i\gamma, i\gamma]$ ($\gamma \in (0, \infty)$). Let $i\tau$ ($\tau \in \mathbf{R}$) be an eigenvalue of $L(\lambda, \eta)$ and y be the corresponding eigenvector. Then

$$\tau^2\|y\|^2 - \tau\eta(Ky, y) + (Ay, y) = 0$$

and, consequently,

$$|\tau| \leq \frac{1}{2}\left(\eta\|K\| + \sqrt{\eta^2\|K\|^2 + 4\beta}\right). \tag{2.21}$$

***Remark* 2.1.** In [33] it was proved that under more restrictive condition $K >> 0$ the inequality $\mathrm{Im}(\lambda_k + \lambda_{-k}) > 0$ is valid.

3. Pencils with one-dimensional linear part. Let us consider the quadratic operator pencil $L(\lambda, \eta)$ with operators M, K, A acting in the Hilbert space $H \oplus C$ and satisfying Condition I, moreover, let the following condition be valid:

Condition II.

$$K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problems related to such pencils occur in the theory of vibrations of strings and elastic beams with pointwise damping [1–6, 11, 12, 26, 34–36]. The so-called Regge problem (see [36–41] and others) can be also reduced to the spectral problems for such pencils.

We consider the order in location of pure imaginary eigenvalues of $L(\lambda)$. The order was found for different cases in [12, 35, 42–45]. In [14] it was shown that the spectra of the problems in all these papers are the sets of zeros of the so-called shifted Hermite–Biehler functions or, as in case of [12], of shifted generalized Hermite–Biehler functions. These functions possess the mentioned order in location of pure imaginary zeros. But in some cases it is not so easy to prove that the spectrum of the problem coincides with the set of zeros of a shifted Hermite–Biehler function. Another approach to describe the spectra of such problems is to use the methods of operator theory.

Lemma 3.1. 1. *Let Conditions* I *and* II *be valid. Let* $\lambda = -i\tau$ *and* $\lambda = i\tau$ *with* $\tau \in \mathbf{R}\backslash\{0\}$ *be eigenvalues of the operator pencil* $L(\lambda, \eta_0)$, *where* $\eta_0 \in (0, 1]$. *Then* $\lambda = -i\tau$ *and* $\lambda = i\tau$ *are eigenvalues of* $L(\lambda, \eta)$ *for each* $\eta \in [0, 1]$.

2. *Let* $\lambda \in \mathbf{R}\backslash\{0\}$ *be an eigenvalue of the operator pencil* $L(\lambda, \eta_0)$, *where* $\eta_0 \in$ $\in (0, 1]$. *Then* λ *and* $-\lambda$ *are eigenvalues of* $L(\lambda, \eta)$ *for each* $\eta \in [0, 1]$.

Proof. 1. Suppose $\lambda = -i\tau$ and $\lambda = i\tau$ with $\tau \in \mathbf{R}\backslash\{0\}$ are eigenvalues of the operator pencil $L(\lambda, \eta_0)$. Then denote by $Y_1 = \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix}$, the eigenvector of $L(\lambda, \eta_0)$ corresponding to $\lambda = i\tau$ and by $Y_2 = \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}$ the eigenvector corresponding to $\lambda = -i\tau$. Then

$$(-\tau^2 M + \tau\eta_0 K - A)Y_1 = 0,$$

$$(-\tau^2 M - \tau\eta_0 K - A)Y_2 = 0$$

and consequently

$$-\tau^2(Y_2, MY_1) + \tau\eta_0(Y_2, KY_1) - (Y_2, AY_1) = 0,$$

$$-\tau^2(MY_2, Y_1) - \tau\eta_0(KY_2, Y_1) - (AY_2, Y_1) = 0.$$

Taking into account the symmetry of the operators we obtain by subtracting:

$$(KY_2, Y_1) = y_{12}y_{22} = 0.$$

If $y_{12} = 0$, then Y_1 is the eigenvector corresponding to the both eigenvalues $i\tau$ and $-i\tau$. These eigenvalues are independent of $\eta \geq 0$ and they are located symmetrically with respect to the real axis.

2. Let $\lambda \neq 0$ be a real eigenvalue and Y be the corresponding eigenvector, then (see the proof of Assertion 1 of Lemma 2.3) $KY = 0$ and, consequently, $-\lambda$ is also an

eigenvalue. If $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, then $KY = 0$ means $y_2 = 0$ and

$$L(\lambda, \eta)Y = \lambda^2 MY - i\lambda\eta KY - AY = \lambda^2 MY - AY = 0.$$

This means λ and $-\lambda$ are eigenvalues for all $\eta > 0$.

Definition 3.1. *Let Conditions* I *and* II *be valid.*

1. *An eigenvalue* $\lambda \neq 0$ *of* $L(\lambda, \eta_0)$, $\eta_0 \in (0, 1]$, *is said to be of type I if* λ^2 *is real and* $-\lambda$ *is also an eigenvalue of* $L(\lambda, \eta_0)$.

2. *Let* $\lambda = 0$ *be an eigenvalue of* A *of geometric multiplicity* $n \geq 1$, *then either* $\dim\ker A \cap \dim\ker K = n$ *or* $\dim\ker A \cap \dim\ker K = n - 1$. *In the first case we say that the pencil* $L(\lambda, \eta)$ *has* $2n$ *eigenvalues of type* I *and no eigenvalues of type* II *at* $\lambda = 0$ *for each* $\eta \in (0, 1]$. *In the second case we say that that there are* $2m - 2$ *eigenvalues of type* I *and one eigenvalue of type* II *at* $\lambda = 0$ *for each* $\eta \in (0, 1]$.

3. *All the other eigenvalues of* $L(\lambda, \eta_0)$, $\eta_0 \in (0, 1]$, *are said to be of type* II (*we consider an eigenvalue of multiplicity* p *as* p *coinciding eigenvalues which can be of different types*).

***Corollary* 3.1.** *Let, in addition to Conditions* I *and* II, *be* $M + K \geq \epsilon I$, $\epsilon > 0$, *then Lemma* 3.1 *shows that there can exist a sequence* (*finite or infinite*) *of eigenvalues of type* I *and these eigenvalues are independent of* η. *This sequence consists of pure imaginary eigenvalues* (*of finite number counting with multiplicities*) *and of real eigenvalues* (*of finite with account of multiplicities or infinite number*). *This sequence is symmetric with respect to the real and imaginary axes. Eigenvalues of type* I *possess no associated vectors* (*except of possible eigenvalue at* $\lambda = 0$).

Proof. Let ζ_0 be an eigenvalue of the linear operator pencil $\zeta M - A$ of multiplicity $s \geq 1$, then the basis of the corresponding subspace can be chosen in such a way that either 1) all s linearly independent eigenvectors are of the form $Y_j = (y_1, 0)^T$, $j = 1, \ldots, s$, or 2) $s - 1$ eigenvectors are of the form $Y_j = (y_1, 0)^T$ and one eigenvector is of the form $Y_s = (0, y_2)^T$ with $y_2 \neq 0$. Then it is clear, that $\pm\sqrt{\zeta_0}$ are eigenvalues of type I of the pencil $L(\lambda, \eta)$ for each $\eta \in [0, 1]$ and the corresponding eigenvectors are Y_j ($j = 1, \ldots, s$ in case 1) and $j = 1, \ldots, s-1$ in case 2). The condition $M+K \geq \epsilon I$ guarantee finiteness of the set of pure imaginary eigenvalues of type I.

Let us describe the location of the rest of eigenvalues which we relate to the subsequence of type II. It is clear that their geometric multiplicity is 1.

Theorem 3.1. *Let, in addition to Conditions* I *and* II, $M + K \geq \epsilon I$, $\epsilon > 0$, *then the eigenvalues of type* II *of the operator pencil* $L(\lambda, \eta)$, $\eta \in (0, 1]$, *possess the following properties:*

1. *All but* κ_2 *terms of the sequence lie in the open upper half-plane.*

2. *All terms in the closed lower half-plane are purely imaginary and occur only once. If* $\kappa_2 \geq 1$, *we denote them as* $\lambda_{-j} = -i|\lambda_{-j}|$, $j = 1, \ldots, \kappa_2$. *We assume that* $|\lambda_{-j}| < < |\lambda_{-(j+1)}|$, $j = 1, \ldots, \kappa_2 - 1$.

3. *If* $\kappa_2 \geq 1$, *the numbers* $i|\lambda_{-j}|$, $j = 1, \ldots, \kappa_2$ (*with the exception of* λ_{-1} *if it equals zero*), *are not terms of the sequence.*

4. *If* $\kappa_2 \geq 2$, *then the number of terms in the intervals* $(i|\lambda_{-j}|,\ i|\lambda_{-(j+1)}|)$, $j = = 1, \ldots, \kappa_2 - 1$, *is odd.*

5. *If* $|\lambda_{-1}| > 0$, *then the interval* $(0, i|\lambda_{-1}|)$ *contains no terms at all or an even number of terms.*

Proof. To prove this theorem it is enough to use the arguments in the proof of Theorem 2.3 keeping in mind that by Definition 3.1 and Lemma 3.1 if $\lambda = -i\tau$, $\tau > 0$, is an eigenvalue of type II, then $\lambda = i\tau$ is not.

Theorem 3.2. *Let, in addition to Conditions* I *and* II, *be* $M >> 0$. *Then*:

1) *if* $\kappa_2 \geq 1$, *then the interval* $(i|\lambda_{-\kappa_2}|, i\infty)$ *contains an odd number of terms*;

2) *if* $\kappa_2 = 0$, *then the sequence has an odd number of positive imaginary terms.*

Proof. It remains to show that for all $\eta \in [0, 1]$ the pure imaginary eigenvalues lie on some interval $[-i\gamma, i\gamma]$, $\gamma < \infty$. But it has been proved already (see (2.21)).

4. Examples. 1. Regge problem [36–41].

This problem occurs in the scattering theory when the potential is supposed to have finite support

$$-y'' + q(x)y = \lambda^2 y, \tag{4.1}$$

$$y(0) = 0, \tag{4.2}$$

$$y'(a) + i\lambda y(a) = 0. \tag{4.3}$$

Here λ is the spectral parameter and the potential q is real-valued and belongs to $L_2(0, a)$.

Let us introduce the operators A, K and M acting in the Hilbert space $H = L_2(0, a) \cup \cup \mathbf{C}$ according to the formulae

$$A \begin{pmatrix} v(x) \\ c \end{pmatrix} = \begin{pmatrix} -v''(x) + q(x)\, v(x) \\ v'(a) \end{pmatrix}, \tag{4.4}$$

$$D(A) = \left\{ \begin{pmatrix} v(x) \\ c \end{pmatrix} : \ v(x) \in W_2^2(0, a), \quad v(0) = 0, \ c = v(a) \right\}, \tag{4.5}$$

$$M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to show that A is selfadjoint and bounded below and that there exists $-\beta_1 < < -\beta \leq \|y\|^{-2}(Ay, y)$ such that $(A + \beta_1 I)^{-1}$ is a compact operator.

Let us consider the nonmonic quadratic operator pencil of the form

$$L(\lambda) = \lambda^2 M - i\lambda K - A.$$

We identify the spectrum of problem (4.1)–(4.3) with the spectrum of the pencil $L(\lambda)$. It is clear that $P \geq 0$, and $K \geq 0$. The spectrum of the pencil consists of normal eigenvalues (see Section 2).

Let us prove that all of these eigenvalues are of type II. Suppose a real $\lambda \neq 0$ is an egenvalue (being real nonzero it must be of type I). Then $-\lambda$ is also an eigenvalue and according to the proof of Lemma 3.1 $c = v(a) = 0$ in (4.4). Therefore, the second component of the equation $L(\lambda_0)Y = 0$ gives $v'(a) = 0$ what contradicts $v(a) = 0$. In the same way, one can prove that there are no symmetrically located pure imaginary eigenvalues and that the possible eigenvalue at the origin is simple.

Thus, the conditions of Theorem 3.1 are satisfied and all the eigenvalues are of type II and therefore statements 1–5 of Theorem 3.1 are valid.

2. The problem of small vibrations of a damped smooth inhomogeneous string in a particular case of point mass at the right end can be reduced to the following problem:

$$y''(\lambda,x)+(\lambda^2-i\,\lambda p-q(x))y(\lambda,x)=0, \tag{4.6}$$

$$y(\lambda,0)=0, \tag{4.7}$$

$$y'(\lambda,a)+(-m\,\lambda^2+i\,\alpha\,\lambda+\beta)\,y(\lambda,a)=0, \tag{4.8}$$

where $p>0$, $m>0$, $\alpha>0$ are constants, $q(x)\in L_2(0,a)$ is a real-valued function. It follows from the physical meaning of this problem that all the eigenvalues of this problem lie in the open upper half-plane [35].

Transformation of the spectral parameter $z=\lambda-i\dfrac{p}{2}$ leads to the following problem:

$$y''(z,x)+\left(z^2+\frac{p^2}{4}-q(x)\right)y(z,x)=0, \tag{4.9}$$

$$y(z,0)=0, \tag{4.10}$$

$$y'(z,a)+(-mz^2+i(\alpha-mp)z+\beta_1)y(z,a)=0, \tag{4.11}$$

where $\beta_1=\beta+\dfrac{p^2m}{4}-\dfrac{\alpha p}{2}$.

The spectrum of problem (4.9) – (4.11) coincides with the spectrum of the following operator pencil:

$$\mathcal{L}(z)=z^2M_1-izK_1-A_1 \tag{4.12}$$

acting in $L_2(0,a)\oplus\mathbf{C}$, where

$$D(\mathcal{L})=D(A_1)=\left\{\begin{pmatrix}y(x)\\ y(a)\end{pmatrix}:\ y(x)\in W_2^2(0,a),\quad y(0)=0\right\},$$
$$A_1\begin{pmatrix}y(x)\\ y(a)\end{pmatrix}=\begin{pmatrix}-y''+q(x)y-\dfrac{p^2}{2}y\\ y'(a)+\beta_1y(a)\end{pmatrix} \tag{4.13}$$

and

$$M_1=\begin{pmatrix}I & 0\\ 0 & m^2\end{pmatrix},\qquad K_1=\begin{pmatrix}0 & 0\\ 0 & (\alpha-pm)I\end{pmatrix}. \tag{4.14}$$

Let $\alpha>mp$. As in previous problem we conclude that the pencil $\mathcal{L}$ possesses only normal eigenvalues of type II. It allows to apply again Theorem 3.1 to the pencil $\mathcal{L}$ and after inverse transformation $\lambda=z+i\dfrac{p}{2}$ to obtain the result obtained in [14] by another method:

1) all (if any) eigenvalues in the closed half-plane $\mathrm{Im}\lambda\le\dfrac{p}{2}$ are pure imaginary and simple $\left(\text{we denote them } \{\lambda_{-j}\},\ j=1,2,\ldots,\kappa, \text{ in the following order: } \left|\dfrac{p}{2}-\lambda_{-j}\right|<\right.$ $\left.<\left|\dfrac{p}{2}-\lambda_{-j-1}\right|\right)$;

2) all the points $i(p-|\lambda_{-j}|)$, $j=1,2,\ldots,\kappa$, do not belong to the spectrum of (4.6) – (4.8) except of $i(p-|\lambda_{-1}|)$ if $\lambda_{-1}=i\dfrac{p}{2}$;

3) each interval $\big(i(p-|\lambda_{-j}|),\, i(p-|\lambda_{-j-1}|\big)$, $j=1,2,\ldots,\kappa-1$, contains odd number (with account of multiplicities) of the eigenvalues;

4) if $\lambda_{-1}\neq i\dfrac{p}{2}$, then the interval $\left(i\dfrac{p}{2},\, i(p-|\lambda_{-1}|)\right)$ contains even number (with account of multiplicities) of the eigenvalues.

The case of $\alpha<mp$ can be considered in the same way.

3. Forth order problem. In [12] the following spectral problem was considered:

$$y^{(4)}-(g(x)y')'=\lambda^2 y, \tag{4.15}$$

$$y(0)=y''(0)=0, \tag{4.16}$$

$$y(a)=0, \tag{4.17}$$

$$y''(a)+i\alpha\lambda y'(a)=0 \tag{4.18}$$

describing small transversal vibrations of an elastic beam. Here $g(x)$ is a continuously differentiable real function describing the distributed stretching or compressing force. The left end of the beam is hinge connected and the right end is hinge connected with damping.

We associate with problem (4.15) – (4.18) the following operator pencil:

$$L_2(\lambda)=\lambda^2 M_2-i\lambda K_2-A_2,$$

where

$$D(L_2)=D(A_2)=\left\{\begin{pmatrix} y(x)\\ y'(a)\end{pmatrix}:\ y(x)\in W_2^4(0,a),\quad y(0)=y''(0)=y(a)=0\right\},$$

$$A_2\begin{pmatrix} y(x)\\ y'(a)\end{pmatrix}=\begin{pmatrix} y^{(4)}-(g(x)y')'\\ y''(a)\end{pmatrix}$$

and

$$M_2=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\qquad K_2=\begin{pmatrix} 0 & 0\\ 0 & \alpha\end{pmatrix}.$$

It is clear that geometric multiplicity of an eigenvalue of this pencil can be 1 or 2 because of conditions $y(0)=y''(0)=0$. Therefore, $L_2(\lambda)$ can have eigenvalues of the both types I and II. Thus, we have deduced one of the results of [12] (see Theorem 3.1 there).

1. *Krein M. G., Nudelman A. A.* On direct and inverse problems for frequencies of boundary dissipation of inhomogeneous string // Dokl AN SSSR. – 1979. – **247**, № 5. – P. 1046 – 1049 (in Russian).
2. *Krein M. G., Nudelman A. A.* On some spectral properties of an inhomogeneous string with dissipative boundary condition // J. Operator Theory. – 1989. – **22**. – P. 369 – 395 (in Russian).
3. *Arov D. Z.* Realization of a canonical system with a dissipative boundary condition at one end of the segment in terms of the coefficient of dynamical compliance // Sibirsk. Math. J. – 1975. – **16**. – P. 440 – 463 (in Russian).
4. *Veselic K.* On linear vibrational systems with one-dimensional damping // Appl. Anal. – 1988. – **29**. – P. 1 – 18.
5. *Veselic K.* On linear vibrational systems with one-dimensional damping. II // Integral Equat. Oper. Theory. – 1990. – **13**. – P. 1 – 18.
6. *Cox S., Zuazua E.* The rate at which energy decays in a string damped at one end // Communs Part. Different. Equat. – 1994. – **19**, № 1/2. – P. 213 – 243.

7. *Shubov M. A.* Asymptotics of resonances and geometry of resonance states in the problem of scattering of acoustic waves by a spherically symmetric inhomogeneity of the density // Different. and Integral Equat. – 1995. – **8**, № 5. – P.1073 – 1115.
8. *Pivovarchik V. N.* Inverse problem for a smooth string with damping at the end // J. Operator Theory. – 1997. – **38**. – P. 243 – 263.
9. *Pivovarchik P. N.* Direct and inverse problems for a damped string // Ibid. – 1999. – **42**. – P. 189 – 220.
10. *Mennicken R., Pivovarchik V. N.* An inverse problem for an inhomogeneous string with an interval of zero density // Math. Nachr. – 2003. – **259**. – P. 1 – 15.
11. *Pivovarchik V. N.* Direct and inverse three-point Sturm – Liouville problem with parameter-dependent boundary conditions // Asympt. Anal. – 2001. – **26**, № 3, 4. – P. 219 – 238.
12. *Möller M., Pivovarchik V. N.* Spectral properties of a forth order differential equation // Z. Anal. und Anwendungen. – 2006. – **25**, № 3. – S. 341 – 366.
13. *Levin B. Ja.* Distribution of zeros of entire functions // Transl. Math. Monogr. – Providence R.I.: Amer. Math. Soc., 1980. – **5**. – 523 p.
14. *Pivovarchik V., Woracek H.* Shifted Hermite – Biehler functions and their applications // Integral Equat. Operator Theory. – 2007. – **57**. – P. 101 – 126.
15. *Krein M. G., Langer H.* On some mathematical principles in the linear theory of damped oscillations of continua. I, II // Ibid. – 1978. – **1**. – P. 364 – 399, 539 – 566.
16. *Freitas P., Grinfeld M., Knight P.* Stability for finite-dimensional systems with indefinite damping // Adv. Math. Sci. and Appl. – 1997. – **7**. – P. 435 – 460.
17. *Gohberg I. C., Krein M. G.* Introduction to the theory of linear nonselfadjoint operators // Transl. Math. Monogr. – Providence R.I.: Amer. Math. Soc., 1969. – **18**. – 342 p.
18. *Gohberg I. C., Goldberg S., Kaashoek M. A.* Classes of linear operators. – Basel: Birkhäuser, 1990. – Vol. 1. – 431 p.
19. *Pivovarchik V. N.* On positive spectra of one class of polynomial operator pencils // Integral Equat. Operator Theory. – 1994. – **19**. – P. 314 – 326.
20. *Markus A. S.* Introduction to the theory of polynomial operator pencils // Transl. Math. Monogr. – Providence R.I.: Amer. Math. Soc., 1988. – **71**. – 279 p.
21. *Gohberg I. C., Sigal E. I.* An operator generalization of the logarithmic residue theorem and Rouché s theorem // Math. Sb. – 1971. – **13**, № 1. – P. 603 – 625 (in Russian).
22. *Eni M. V.* Stability of the root-number of an analytic operator-function and on perturbations of its characteristic numbers and eigenvectors // Sov. Math. Dokl. – 1967. – **8**. – P. 542 – 545.
23. *Kostyuchenko A. G., Orazov M. B.* The problem of oscillations of an elastic half-cylinder and related self-adjoint quadratic pencils // Trudy Sem. Petrovskogo. – 1981. – **6**. – P. 97 – 147 (in Russian).
24. *Markushevich A. I.* Theory of analytic functions. – Moscow: Nauka, 1968. – Vol 1. – 478 p.
25. *Pivovarchik V. N.* On eigenvalues of a quadratic operator pencil // Funkts. Anal. i Pril. – 1989. – **25**. – P. 80 – 81 (in Russian).
26. *Shkalikov A. A.* Operator pencils arising in elasticity and hydrodynamics: the instability index formula // Operator Theory: Adv. Appl. – 1996. – **87**. – P. 358 – 385.
27. *Adamyan V. M., Pivovarchik V. N.* On the spectra of some classes of quadratic operator pencils // Ibid. – 1998. – **106**. – P. 23 – 36.
28. *Adamyan V. M., Mennicken R., Pivovarchik V. N.* On the spectral theory of degenerate operator pencils // Ibid. – 2001. – **124**. – P. 1 – 19.
29. *Adamyan V. M., Langer H., Möller M.* Compact perturbations of spectra of self-adjoint quadratic operator pencils of definite type // Integral Equat. Operator Theory. – 2001. – **39**. – P. 127 – 152.
30. *Pivovarchik V. N.* On Spectra of quadratic operator pencils in the right half-plane // Mat. Zametki. – 1989. – **45**, № 6. – P. 101 – 103 (in Russian).
31. *Pivovarchik V. N.* On the total algebraic multiplicity of the spectrum in the right half-plane for a class of quadratic operator pencils // St.Peterburg Math. J. – 1992. – **3**, № 2. – P. 447 – 454 (in Russian).
32. *Riesz F., Sz-Nagy B.* Lecons d'analyse fonctionelle. – Budapest: Acad. Kiadv., 1952. – 386 p.
33. *Van der Mee C., Pivovarchik V. N.* Some properties of the eigenvalues of a Schrödinger equation with energy-dependent potential // Contemp. Math. – 2002. – **307**. – P. 305 – 310.
34. *Gomilko A. M., Pivovarchik V. N.* On bases of eigenfunctions of boundary problem associated with small vibrations of damped nonsmooth inhomogeneous string // Asympt. Anal. – 1999. – **20**, № 3-4. – P. 301 – 315.

35. *Van der Mee C., Pivovarchik V. N.* The Sturm – Liouville inverse spectral problem with boundary conditions depending on the spectral parameter // Opusc. Math. – 2005. – **25**, № 2. – P. 243 – 260.
36. *Hrushev S.* The Regge problem for a string, unconditionally convergent eigenfunction expansions, and unconditional bases of exponentials in $L_2(-T,T)$ // J. Operator Theory. – 1985. – **14**. – P. 67 – 85.
37. *Regge T.* Construction of potential from resonances // Nuovo cim. – 1958. – **9**, № 3. – P. 491 – 503.
38. *Regge T.* Construction of potential from resonances // Ibid. – № 5. – P. 671 – 679.
39. *Kravitskii A. O.* On double expansion into series of eigenfunctions of a nonself-adjoint boundary problem // Differents. Uravneniya. – 1968. – **4**, № 1. – P. 165 – 177 (in Russian).
40. *Kogan V. M.* On double completeness of the set of eigen- and associated functions of the Regge problem // Funkts. Anal. i Pril. – 1971. – **5**, № 3. – P. 70 – 74 (in Russian).
41. *Sergeev A. G.* Asymptotics of jost-functions and eigenvalues of the Regge problem // Differents. Uravneniya. – 1972. – **8**, № 5. – P. 925 – 927 (in Russian).
42. *Van der Mee C., V. Pivovarchik V. N.* A Sturm – Liouville inverse problem with boundary conditions depending on the spectral parameter // Funct. Anal. and Appl. – 2002. – **36**, № 4. – P. 315 – 317 (in Russian).
43. *Simon B.* Resonances in one-dimension and Fredholm determinants // J. Funct. Anal. – 2000. – **178**, № 2. – P. 396 – 420.
44. *Korotyaev E.* Stability for inverse resonance problem // Res. Notes. – 2004. – **73**. – P. 3927 – 3936.
45. *Pivovarchik V. N., van der Mee C.* The Inverse generalized Regge problem // Inverse Problems. – 2001. – **17**. – P. 1831 – 1845.

Received 15.11.2006