The general Kloosterman sum $K(m, n; k; q)$ over $\mathbb{Z}$ was studied by S. Kanemitsu, Y. Tanigawa, Y. Yuan, Zhang Wenzhou in their research of problem of D. H. Lehmer. In this paper, we obtain the similar estimations of $K(\alpha, \beta; k; q)$ over $\mathbb{Z}[i]$. We also consider the sum $\tilde{K}(\alpha, \beta; h, q; k)$ which has not an analogue in the ring $\mathbb{Z}$ but it can be used for the investigation of the second moment of the Hecke zeta-function of field $\mathbb{Q}(i)$.

The generalizations of the Kloosterman sums (see [9, 10]):}

1. Introduction. The classic Kloosterman sums appeared first in the work of Kloosterman [1] in connection with the representation of natural numbers by binary quadratic forms. The Kloosterman sum is an exponential sum over a reduced residue system modulo $q$:

$$K(a, b; q) := \sum_{x=1}^{q} e^{2\pi i \frac{ax + bx'}{q}}, \quad a, b \in \mathbb{Z}, \quad q > 1 \quad \text{is natural},$$

here and in sequence $x'$ denote the reciprocal to $x$ modulo $q$, i.e., $xx' \equiv 1 \pmod{q}$. By the relation for $q = q_1 q_2$, $(q_1, q_2) = 1$,

$$K(a, b; q) = K(a q_1, b q_1; q_1) K(a q_2, b q_2; q_2)$$

follows that suffices to obtain the estimations $K(a, b; q)$ only for a case $q = p^n$, $p$ be a prime, $n \in \mathbb{N}$.

The greatest difficulty in an estimation of the Kloosterman sums provides the case $q = p$. The estimation $K(a, b; p) \ll p^{\frac{3}{2}}$ under a condition $(a, b, p) = 1$ was obtained in the named work of Kloosterman, and then Davenport [2] improved on it up to $\ll p^2$. A. Weil [3] proved the Riemann hypothesis for algebraic curves of over finite field and obtained the best possible estimation $\ll p^2$.

Davenport [2] studies the general Kloosterman sums over finite field with the multiplicative character $\psi$ of this field

$$K_{\psi}(a, b; p) = \sum_{x \in \mathbb{F}_p^*} \psi(x) e^{2\pi i \frac{ax + bx'}{p}}.$$

The further generalization of the Kloosterman sums concerned with a substitution of a prime field $\mathbb{F}_p$, on it a finite expansion $\mathbb{F}_q$, $q = p^n, n \in \mathbb{N}$. The generalization of the Kloosterman sums concerned with theory of modular forms studies in the works Kuznetsov [4, 5], Bruggeman [6], Deshoillers, Iwaniec [7], Proskurin [8]. In last years in connection with the investigation of the D. H. Lehmer problem was studied others generalizations of the Kloosterman sums (see [9, 10]):

$$K(a, b; q, k, \psi) = \sum_{x \in \mathbb{R}^* (q)} \psi(x) e^{2\pi i \frac{ax + bx'k}{q}}, \quad xx' \equiv 1 \pmod{q},$$

where $\psi$ is a multiplicative character modulo $q$. 

© S. P. VARBANETS, 2007
ISSN 1027-3190. Укр. мат. журн., 2007, т. 59, № 9
The multiple Kloosterman sums introduced Mordell [11]:
\[ K(a_1, \ldots, a_n; q) = \sum_{x_1, \ldots, x_n \in \mathbb{F}_q^*} e^{2\pi i q^{-1} \sum_{i=1}^n a_i x_i}, \]
where \( a_1, \ldots, a_n \in \mathbb{F}_q^*, \quad q = p^n, \quad \sigma_m(c) \) is a trace from \( \mathbb{F}_q \) into \( \mathbb{F}_p \).

The multiple Kloosterman sums are a particular case of the trigonometric sums on an algebraic variety over a finite field. By virtue of the investigations Dwork [12] (which has proved a rationality of the zeta-function of an algebraic variety over finite field), Deligne [13] (which has proved the Riemann hypothesis for an algebraic variety over \( \mathbb{F}_q \)) and Bombieri [14] (which has estimated in terms of a generative polynomial the number of characteristic roots of the zeta-function) was obtained the final estimation (see Deligne [15], Bombieri [14])
\[ K(a_1, \ldots, a_n; q) \leq nq^{n-1}. \]

In this paper we obtain the estimations of general Kloosterman sums over the ring of the Gaussian integers.

**Notations.** We denote \( \mathbb{Z}[i] \) the ring of the Gaussian integers
\[ \mathbb{Z}[i] := \{a + bi|a, b \in \mathbb{Z}, \ i^2 = -1\}. \]

For the designation of the Gaussian integers we shall use the Greek letters \( \alpha, \beta, \gamma, \xi, \eta; \) a Gaussian prime number denote through \( p \) if \( p \notin \mathbb{Z} \). For \( \alpha \in \mathbb{Z}[i] \) we put \( \text{Sp}(\alpha) = \alpha + \bar{\alpha}, \quad N(\alpha) = \alpha \bar{\alpha} \), where \( \bar{\alpha} \) denotes a complex conjugate with \( \alpha; \) \( \text{Sp}(\alpha) \) and \( N(\alpha) \) we name a trace and a norm (accordingly) of \( \alpha \) from \( \mathbb{Q}(i) \) into \( \mathbb{Q} \). \( \mathbb{F}_q \) denotes a field which contain just \( q \) an element, \( q = p^n, \ n \in \mathbb{N} \).

For \( x \in \mathbb{F}_q \) denote through \( \sigma_n(x) \) a trace \( x \) from \( \mathbb{F}_q \) into \( \mathbb{F}_p \), i.e.,
\[ \sigma_n(x) := x + x^p + \ldots + x^{p^{n-1}}, \quad \sigma_1(x) = \sigma(x) = x. \]

The writing \( a \in R(q) \) (accordingly, \( a \in R(q, i) \)) denotes that \( a \in \mathbb{Z} \) (accordingly, \( a \in \mathbb{Z}[i] \)) and \( a \) runs a complete residue system modulo \( q \). Analogous, \( a \in R^*(q) \) (accordingly, \( a \in R^*(q, i) \)) denotes \( a \in \mathbb{Z} \) (accordingly \( a \in \mathbb{Z}[i] \)) and runs a reduced residue system modulo \( q \).

The writing \( \sum_{(U)} \) denotes that the summation runs over the region \( U \) which describe extra. Moreover, \( \exp(z) = e^z, \quad e_q(z) = e^{2\pi i z} \) for \( q \in \mathbb{N} \); the Vinogradov symbol as in \( f(x) \ll g(x) \) means that \( f(x) = O(g(x)) \).

For Gaussian integers \( \alpha, \beta, \gamma \) we define the Kloosterman sum
\[ K(\alpha, \beta; \gamma) = \sum_{\gamma \in R^*(\gamma, i)} \exp\left(\pi i \text{Sp}\frac{\alpha x + \beta x'}{\gamma}\right). \]

Zanbyrbaeva [16] obtained the estimation
\[ K(\alpha, \beta; \gamma) \ll 2^{\nu(\gamma)} N(\gamma) \frac{1}{2} N((\alpha, \beta, \gamma)) \frac{1}{2}, \]
where \( \nu(\gamma) \) is the number distinct prime divisors of \( \gamma; \) \( (\alpha, \beta, \gamma) \) denotes the greatest common divisor of \( \alpha, \beta, \gamma \).

We consider two type of general Kloosterman sums over \( \mathbb{Z}[i] \)
\[ K(\alpha, \beta; k; \gamma, \psi) = \sum_{\gamma \in R^*(\gamma, i)} \psi(x) \exp\left(\pi i \text{Sp}\frac{\alpha x^k + \beta x'^k}{\gamma}\right), \]
where \( \alpha, \beta, \gamma \in \mathbb{Z}[i], \psi \) is multiplicative character modulo \( \gamma \),

ISSN 1027-3190. Укр. мат. журн., 2007, т. 59, № 9
\[ \tilde{K}(\alpha, \beta; h, q; k) = \sum_{\substack{x, y \in \mathbb{Z}[i] \cap N(x, y) \equiv h \mod q}} e_q \left( \frac{1}{2} Sp(\alpha x^j + \beta y^j) \right), \]

where \( \alpha, \beta \in \mathbb{Z}[i], h, q \in \mathbb{N}, (h, q) = 1. \)

We call \( K(\alpha, \beta; k; \gamma, \psi) \) the general power Kloosterman sum and \( \tilde{K}(\alpha, \beta; h, q; k) \) call the norm Kloosterman sum.

Our aim is to obtain non trivial estimations for \( K(\alpha, \beta; k; \gamma, \psi) \) and \( \tilde{K}(\alpha, \beta; h, q; k) \).

2. Auxiliary results. For the proofs of our main results some Lemmas are need.

**Lemma 2.1.** Let \( \mathfrak{p} \) be a Gaussian prime "odd" number, \( \alpha_1, \ldots, \alpha_k \in \mathbb{Z}[i], \quad (\alpha_2, \mathfrak{p}) = \ldots = (\alpha_k, \mathfrak{p}) = 1; \quad \nu_3, \nu_4, \ldots, \nu_k \geq 2, \)

are natural numbers.

Then for every natural \( n \geq 2 \) we have

\[
\left| \sum_{\xi \in \mathbb{Z}[i] \mod \mathfrak{p}^n} \exp \left( 2\pi i \ Sp \left( \frac{\alpha_1 \xi + \alpha_2 \mathfrak{p} \xi^2 + \alpha_3 \mathfrak{p}^\nu_3 \xi^3 + \ldots + \alpha_k \mathfrak{p}^\nu_k \xi^k}{\mathfrak{p}^n} \right) \right) \right| =
\begin{cases}
0 & \text{if } (\alpha_1, \mathfrak{p}) = 1, \\
\frac{1}{N} \left( \frac{\mathfrak{p}}{z} \right)^{\frac{n+1}{2}} & \text{if } \alpha_1 \equiv 0 \mod \mathfrak{p}.
\end{cases}
\]

**Lemma 2.2.** Let \( \mathfrak{p} = 1 + i \) be a Gaussian "even" number and let \( \alpha_j \in \mathbb{Z}[i], \quad j = 1, 2, \ldots, k; \quad (\alpha_2, \mathfrak{p}) = \ldots = (\alpha_k, \mathfrak{p}) = 1. \)

Then for any natural numbers \( \nu_j \geq 2, j = 2, 3, \ldots, k, \) and any \( n \geq 2 \) the following estimate:

\[
\left| \sum_{\xi \in \mathbb{Z}[i] \mod \mathfrak{p}^n} \exp \left( 2\pi i \ Sp \left( \frac{\alpha_1 \xi + \alpha_2 \mathfrak{p} \xi^2 + \alpha_3 \mathfrak{p}^\nu_3 \xi^3 + \ldots + \alpha_k \mathfrak{p}^\nu_k \xi^k}{\mathfrak{p}^n} \right) \right) \right| \leq \delta \cdot 2^{n+1},
\]

holds, where

\[
\delta = \begin{cases}
0 & \text{if } \alpha_1 \not\equiv 0 \mod \mathfrak{p}^2, \\
2 & \text{if } \alpha_1 \equiv 0 \mod \mathfrak{p}^2.
\end{cases}
\]

The assertion of these lemmas are the consequences of the estimates of complete linear sum and Gauss’sum to which we can reduced the primary sums.

**Lemma 2.3.** Let \( \mathfrak{p} \) be a prime number, \( A \in \mathbb{Z}, (A, \mathfrak{p}) = 1, f(x) \in \mathbb{Z}[x], \)

\[
f(x) = a_1 x + a_2 x^2 + p^k a_3 x^3 + \ldots + p^k a_k x^k,
\]

\( (a_i, \mathfrak{p}) = 1, i = 2, 3, \ldots, k; \lambda_j > 0, j = 3, \ldots, k. \)

Then for any \( n \in \mathbb{N} \) the equality

\[
S := \sum_{x \mod \mathfrak{p}^n} e_{\mathfrak{p}^n}(Af(x)) = \varepsilon(n)p^z \varepsilon\mathfrak{p}(AF(a_1, \ldots, a_n))
\]

holds, where \( F(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n], \)

\[
\varepsilon(n) = \begin{cases}
1 & \text{if } n \text{ is even}, \\
\left( \frac{A}{\mathfrak{p}} \right) \cdot (i)^{\frac{n-1}{2}} & \text{if } n \text{ is odd, } \left( \frac{A}{\mathfrak{p}} \right) \text{ is a symbol of Legendry.}
\end{cases}
\]

**Proof.** We set \( x = y + p^{n-1}z, y \mod \mathfrak{p}^{n-1}, z \mod \mathfrak{p}. \) Then we have

\[
\sum_{x \mod \mathfrak{p}^n} e_{\mathfrak{p}^n}(Af(x)) = \sum_{y \mod \mathfrak{p}^{n-1}} \sum_{z \mod \mathfrak{p}} e_{\mathfrak{p}^n}(A(f(y) + p^{n-1}z)f'(y)).
\]
The sum over $z$ gives zero if $f'(y) \not\equiv 0 \pmod{p}$. But we have $f'(y) \equiv a_1 + 2a_2y \pmod{p}$. Let $y_0$ be a root of congruence $a_1 + 2a_2y \equiv 0 \pmod{p}$. Then

$$S = e_p^n (Af(y_0)) \sum_{y \pmod{p^{n-1}}} e_{p^n} \left( A(f(y_0 + py) - f(y_0)) \right) =$$

$$= e_p^n (Af(y_0)) \sum_{y \pmod{p^{n-1}}} e_{p^{n-2}} (Ag(y)) =$$

$$= p e_p^n (Af(y_0)) \sum_{y \pmod{p^{n-2}}} e_{p^{n-2}} (Ag(y)),$$

where

$$g(y) = \frac{f(y_0 + py) - f(y_0)}{p^2} = b_1y + b_2y^2 + p^{\mu_3}b_3y^3 + \ldots + p^{\mu_k}b_ky^k,$$

moreover $b_1, \ldots, b_k$ are linear functions of $a_1, \ldots, a_k$ with the coefficients which depends on $y_0$, and $b_2 \equiv a_2 \pmod{p}$, $(b_j, p) = 1, \mu_j \geq 1, j = 3, \ldots, k$. Thus $g(y)$ is a polynomial such sort as $f(y)$.

These consideration we continue further. Then for $n \equiv 1 \pmod{2}$ we obtain

$$S = p^\frac{n-1}{2} e^{2\pi i A \left[ \frac{f(y_0)}{p} + \frac{g(y)}{p^2} \right]} \sum_{x \pmod{p}} e_p(A(ax^2)) =$$

$$= \left( \frac{A}{p} \right)^{\frac{p-1}{2}} e^{2\pi i A \left[ \frac{f(y_0)}{p} + \frac{g(y)}{p^2} \right] (\frac{p-1}{2}).}$$

The lemma is proved.

**Lemma 2.4.** Let $p$ be a prime number, $p \equiv 3 \pmod{4}$ and let $E_4$ be a set of residue classes $\pmod{p^2}$ of the ring $\mathbb{Z}[i]$, which has norms congruous modulo $p^2$ with $\pm 1$. Then $E_4$ is a cyclical group of order $2(p+1)p^{\ell-1}$.

**Proof.** From an equality $N(\alpha \beta) = N(\alpha)N(\beta)$ follows that $E_4$ is a subgroup of the group of residue classes modulo $p^2$ in $\mathbb{Z}[i]$.

At first let $\ell = 1$. Then the residue classes modulo $p^2$ organizes a field $\mathbb{F}_{p^2}$. Let $g_0$ be a generative element of multiplicative group of this field. We denote

$$g_0^u = x(u) + iy(u),$$

where $x(u), y(u) \in \mathbb{F}_p$ and $i$ is an element of field $\mathbb{F}_{p^2}$ such that $i^2 = -1$. The residue classes $\pmod{p}$ for which norms $\equiv \pm 1 \pmod{p}$ be characterized by a condition

$$x^2 + y^2 \equiv \pm 1 \pmod{p}.$$

Now from (2.2) we have

$$g_0^{pu} = x(u) - iy(u).$$

Hence an element $g_0^u$ has a norm $\equiv \pm 1 \pmod{p}$ iff $g_0^{(p+1)u} = \pm 1$, i.e., iff $\frac{p-1}{2} | u$.

Denote $u = \frac{p-1}{2} t$, $t = 0, 1, \ldots, 2p + 1$, and set $g_0^{u_1} = g$. The classes $g^t$, $t = 0, 1, \ldots, 2p + 1$, are just those and only those which have a norm $\equiv \pm 1 \pmod{p}$. Let $f = g + p\lambda$, $\lambda \in \mathbb{Z}[i]$. Then

$$f^p = g^p + p\lambda_1, \quad \lambda_1 \in \mathbb{Z}[i],$$
Let $g^{2(p+1)} = 1 + pg_1$. We have

$$f^{2(p+1)} - 1 \equiv p(g_1 + 2g^{p+1}1) \pmod{p^2}.$$  

Always we can take $\lambda$ so

$$f^{2(p+1)} = 1 + ph, \quad h \in \mathbb{Z}[i], \quad (h, p) = 1.$$  

Thus we can account that a generative element $g$ of the group $E_\ell$ had selected so $g^{2(p+1)} - 1 \not\equiv 0 \pmod{p^2}$. Now we easily get

$$g^{2(p+1)p^{\ell-1}} \equiv 1 \pmod{p^\ell}, \quad g^k \not\equiv 1 \pmod{p^\ell}, \quad 0 < k < 2(p+1)p^{\ell-1},$$  

for every $\ell = 1, 2, \ldots$. We must show also that for every $\ell = 1, 2, \ldots$ there exists $g_\ell$ such that $g_\ell \equiv g \pmod{p^\ell}$ and $N(g_\ell) \equiv -1 \pmod{p^\ell}$.

For $\ell = 1$ we proved already.

Let $\ell = 2$. If $g = x + iy$ then

$$x^2 + y^2 = -1 + \lambda p,$$

$$g^{2(p+1)} = 1 + p(h_1 + ih_2), \quad (h_1 + ih_2, p) = 1, \quad h_1, h_2 \in \mathbb{Z}.$$  

We have for $k = k_1 + ik_2, k_1, k_2 \in \mathbb{Z}$:

$$N(x + iy + p(k_1 + ik_2)) = -1 + \lambda p + p(2xk_1 + 2yk_2) + p^2(k_1^2 + k_2^2) \equiv$$

$$\equiv -1 + p(\lambda + 2xk_1 + 2yk_2) \pmod{p^2},$$

$$(x + iy + p(k_1 + k_2))^2(p+1)p \equiv$$

$$\equiv (x + iy)^2(p+1)p + 2p^2(x + iy)^2(p+1)p^{\ell-1}(k_1 + ik_2) \pmod{p^3}.$$  

Hence,

$$((x + iy) + p(k_1 + ik_2))^2(p+1)p^2 \lambda + 2xk_1 + 2yk_2 \equiv 0 \pmod{p^2},$$

here $h^{(1)}$ and $\alpha$ are the Gaussian integers and co-prime numbers with $p$.

Next, the congruence $-h^{(1)} \equiv \alpha(k_1 + ik_2) \pmod{p}$ holds only for one assembly of $(k_1^0, k_2^0)$ by modulo $p$. Therefore, if we take $k_1 \not\equiv k_1^0 \pmod{p}$ and define $k_2$ from the congruence

$$\lambda + 2xk_1 + 2yk_2 \equiv 0 \pmod{p^2},$$

then we obtain that $f = x + iy + p(k_1 + ik_2)$ has a norm $\equiv -1 \pmod{p^2}$. Moreover, $f$ belongs to an exponent $2(p+1)p$ by modulo $p^2$ and $f^{2(p+1)p} = 1 + Hp^2, (H, p) = 1$, i.e., $f \in E_2$ and $f$ belongs to an exponent $2(p+1)p^{\ell-1}$ by modulo $p^\ell$ for every $\ell = 2, 3, \ldots$.

Now we note that the $g_3 = f + p^2(m_1 + im_2)$ satisfies by the condition $g_3^{2(p+1)p} = 1 + H_1p^2, (H_1, p) = 1$, for any $m_1, m_2 \in \mathbb{Z}$. We take $m_1, m_1 \in \mathbb{Z},$ such that

$$\lambda_2 + 2f_1m_1 + 2f_2m_2 \equiv 0 \pmod{p},$$

where $\lambda = \frac{N(f) - (-1)}{p^2} = \frac{N(f) + 1}{p^2}, \quad f = f_1 + if_2.$

Then $g_3 = f + p^2(m_1 + im_2)$ is a generative element of the group $E_3$.

Next, by induction. If we defined already $g_{\ell - 1}$ then a generative element of $E_\ell$ will be
where \( m_1, m_2 \) define from a congruence
\[
\lambda_{t-1} + 2g'_{t-1}m_1 + 2g''_{t-1}m_2 \equiv 0 \pmod{p}
\]
(here \( \lambda_{t-1} = \frac{N(g_{t-1}) + 1}{p^{t-1}}, g_{t-1} = g'_{t-1} + ig''_{t-1}, g'_{t-1}, g''_{t-1} \in \mathbb{Z} \)).

The lemma is proved.

**Lemma 2.5.** Let \( p \) be prime number, \( p \equiv 3 \pmod{4}, \ell \in \mathbb{N} \). Then every residue \( x + iy \) a reduced residue system \( \mod{p^\ell} \) of the ring of the Gaussian integers has unique representation in form
\[
x + iy \equiv g^c(u + iv)^d \pmod{p^\ell},
\]
\( c = 0, 1, \ldots, (p - 1)p^{\ell-1} - 1, \quad d = 0, 1, \ldots, (p + 1)p^{\ell-1} - 1, \quad (2.3)
\]
where \( g \) is a primitive root modulo \( p^\ell \) in \( \mathbb{Z}, u + iv \) is a generative element of \( \mathbb{E}_\ell \).

**Proof.** Let \( \varphi(a) \) denote the Euler function on \( \mathbb{Z}[i] \). Then for \( p \equiv 3 \pmod{4} \) we have
\[
\varphi(p^\ell) = N(p^\ell) \left(1 - \frac{1}{N(p)} \right) = p^{2(\ell-1)}(p^2 - 1).
\]

In the relation (2.3) we have \( p^{2(\ell-1)}(p^2 - 1) \) the formally distinguishable expressions of form \( g^c(u + iv)^d \). As for any \( c \) and \( d \) we have \( (g^c(u + iv)^d, p) = 1 \) then for the proof of the assertion of lemma sufficiently to show that the expression (2.3) are pairwise disjoint \( \mod{p^\ell} \) for different assemblies of \( (c, d) \).

Let us assume
\[
g^{c_1}(u + iv)^{d_1} \equiv g^{c_2}(u + iv)^{d_2} \pmod{p^\ell}, \quad c_1 \geq c_2.
\]
Then we have
\[
g^{c_1-c_2} \equiv (u + iv)^{d_2-d_1} \pmod{p^\ell} \quad \text{if} \quad d_2 \geq d_1
\]
or
\[
g^{c_1-c_2} \equiv (u + iv)^{d_2-d_1} \equiv 1 \pmod{p^\ell} \quad \text{if} \quad d_2 < d_1.
\]
And now take account that the sets \( \{g^c\} \) and \( \{(u + iv)^d\} \) has only one common element (it is 1) modulo \( p^\ell \) we obtain all once \( c_1 = c_2, d_1 = d_2 \).

The lemma is proved.

**Corollary.** All reduced classes \( x + iy \) modulo \( p^\ell, p \equiv 3 \pmod{4} \) which has equal norms modulo \( p^\ell \) we can write in form
\[
x + iy \equiv g^c(u + iv)^{2d},
\]
\( d = 0, 1, \ldots, p^{\ell-1}(p + 1) - 1 \quad \text{if} \quad N(x + iy) \equiv g^{2c} \pmod{p^\ell},
\]
\( (x + iy) \equiv g^c(u + iv)^{2d+1},
\]
\( d = 0, 1, \ldots, p^{\ell-1}(p + 1) - 1 \quad \text{if} \quad N(x + iy) \equiv g^{2c} \pmod{p^\ell}
\)
\( \left( \text{here } 0 \leq c \leq \frac{p - 1}{2}p^{\ell-1} - 1 \right). \)

Let \( p \) be a prime number, \( p \equiv 1 \pmod{4} \). Then in the ring \( \mathbb{Z}[i] \) we have \( p = p \cdot \overline{p} \), where \( p \) and \( \overline{p} \) are the complex-conjugate Gaussian prime numbers (\( \overline{p} \neq \pm p, \pm ip \)).

Well-known that \( \{a + bi | a, b = 0, 1, \ldots, p^\ell - 1\} \) is a complete residue system \( \mod{p^\ell} \).

Similarly, for \( p = 2 \) we have \( 2 = -i(1 + i)^2 \) and \( \{a + bi | a, b = 0, 1, \ldots, 2^\ell - 1\} \) is a complete system \( \mod{p^\ell}, p = 1 + i \) in \( \mathbb{Z}[i] \).
3. General Kloosterman sum $K(\alpha, \beta; k; \gamma)$. We consider the sum $K(\alpha, \beta; k; \gamma)$ defining in Introduction for the trivial character $\psi_0$:

$$K(\alpha, \beta; k; \gamma, \psi_0) = K(\alpha, \beta; k; \gamma) = \sum_{(U)} \exp\left(\frac{\pi i Sp \alpha x + \beta x'^k}{\gamma}\right), \quad (3.1)$$

where $U = \{(x, x') \in \mathbb{Z}[i]^2 | x, \ x' (\mod \gamma) \}, xx' \equiv 1 (\mod \gamma)$. Obviously, we have

$$K(\alpha, \beta; k; \gamma) = K(\alpha \gamma_1, \beta \gamma_2; k; \gamma_1)K(\alpha \gamma_1, \beta \gamma_2; k; \gamma_2) \quad \text{if} \quad \gamma = \gamma_1 \gamma_2, \ (\gamma_1, \gamma_2),$$

where $\gamma_1 \gamma_2' \equiv 1 (\mod \gamma_2), \ \gamma_2 \gamma_2' \equiv 1 (\mod \gamma_1)$.

Thus we can therefore assume, without loss of generality, that $\gamma = p^n$, $p$ is a Gaussian prime number.

In part 1 we had obtained a description of a reduced residue system $\mod p^n$, $p = p \equiv 3 (\mod 4)$. For $p \in \mathbb{Z}[i]$, $N(p) = p \equiv 1 (\mod 4)$, a reduced residue system $\mod p^n$ has a form

$$\{a \in \mathbb{Z} | 1 \geq a \geq p^n - 1, \ (a, p) = 1\},$$

and for Gaussian prime “even” number $p = 1 + i$

$$\{a + bi | a, b \in \{0, 1, \ldots, 2^n - 1\}, \ a \equiv 1 (\mod 2), \ b \equiv 0 (\mod 2)\}.$$

**Theorem 3.1.** Let $p$ be Gaussian prime number, $N(p) = p \equiv 1 (\mod 4)$ and let $d = (k, p - 1)$. Then

$$|K(\alpha, \beta; k; p)| \leq 2dN((\alpha, \beta, p))^{\frac{1}{2}}N(p)^{\frac{1}{2}}. \quad (3.2)$$

**Proof.** If $(\alpha, \beta, p) = p$ then our assertion is clear. Let $(\alpha, \beta, p) = 1$. By a description of a reduced residue system $\mod p^n$ we can suppose that $\alpha = a, \ \beta = b, \ a, b \in \mathbb{Z}, \ p = c_1 + ic_2, \ c_1, c_2 \in \mathbb{Z}, \ (c_1, p) = (c_2, p) = 1$.

Thus we have

$$K(\alpha, \beta; k; p) = \sum_{u \in R^+ (p)} e_p \left(\frac{1}{2} Sp (ac_1 - bc_1 u + bc_1 c_2 u^k)\right) = \sum_{u \in R^+ (p)} e_p (ac_1 u^k + bc_1 u^k).$$

The last sum was estimated in [10] but we shall give a calculation in order to make more precise an estimation.

We define

$$\mathcal{I}_k (a) := \# \{x \in \mathbb{Z} | 0 \geq x \geq p - 1, \ x^k \equiv a (\mod p)\},$$

It is clear that

$$\mathcal{I}_k (a) = \begin{cases} d & \text{if} \ d | \text{ind} \ a \\ 0 & \text{otherwise} \end{cases} = \sum_{t=0}^{d-1} e^{2\pi i t \text{ind} \ a \over d}.$$

(Here $\text{ind} \ a$ denote an index of integer $a, \ (a, p) = 1$, by a radix of some primitive root modulo $p$.)

Then we obtain
\[ |K(\alpha, \beta; k; p)| = \left| \sum_{u \in R^*(p)} \mathcal{K}(u)e_p(au + bu') \right| \leq \sum_{t=0}^{d-1} \sum_{u \in R^*(p)} e_d(t \text{ ind } u)e_p(au + bu') \leq 2dp^{\frac{3}{2}}. \tag{3.3} \]

Here we take into account that an inner sum is classical Kloosterman sum weighting by a character and hence estimates as \(2p^{\frac{3}{2}}\) (see Perel’muter [17], Williams [18]).

The theorem is proved.

**Theorem 3.2.** Let \(p \equiv 3 \pmod{4}, k \in \mathbb{N}, d = (k, p^2 - 1).\) Then

\[ |K(\alpha, \beta; k; p)| \leq 2dN((\alpha, \beta, p))^{\frac{3}{2}} N(p)^{\frac{3}{2}}. \tag{3.4} \]

**Proof.** The residue classes \(\text{mod } p\) in the ring \(\mathbb{Z}[i]\) organizes a field \(\mathbb{F}_{p^2}\). Hence, \(\sigma_2(x) = x + x^p\). But we observed that \(x \equiv x^p \pmod{p}\) for \(x \in \mathbb{Z}[i]\). Thus \(S_p(x) = x + \overline{x} \equiv x + x^p \equiv \sigma_2(x) \pmod{p}\).

Hence,

\[ K(\alpha, \beta; k; p) = \sum_{x \in R^*(p)} e_p \left( \frac{1}{2} S_p(\alpha x^k + \beta \overline{x}^k) \right) = \sum_{x \in \mathbb{F}_{p^2}^*} e_p \left( \sigma_2(2' \alpha x^k + 2' \beta x^k) \right), \]

where \(2' \equiv 1 \pmod{p}\).

Let \(g\) denote a primitive element of the field \(\mathbb{F}_{p^2}\), \(\text{ind}_g x = \text{ind } x\) for \(x \in \mathbb{F}_{p^2}\) and let \(\mathcal{I}(u)\) is a number solutions of equation \(x^k = u\) in \(\mathbb{F}_{p^2}\). It follows that

\[ |K(\alpha, \beta; k; p)| = \left| \sum_{t=0}^{d-1} \sum_{x \in \mathbb{F}_{p^2}^*} e_p \left( \sigma_2(2' \alpha x + 2' \beta x') \right) \right| \leq \sum_{x \in \mathbb{F}_{p^2}^*} e_p \left( \sigma_2(2' \alpha x + 2' \beta x') \right) + \sum_{t=1}^{d-1} \sum_{x \in \mathbb{F}_{p^2}^*} \psi_t(x)e_p \left( \sigma_2(2' \alpha x + 2' \beta x') \right), \]

where \(\psi_t(x) = e_d(t \text{ ind } x)\) is a multiplicative character of the field \(\mathbb{F}_{p^2}\).

Again using the estimations of the Kloosterman sums with a character of a finite field we obtain finally

\[ |K(\alpha, \beta; k; p)| \leq 2dN((\alpha, \beta, p))^{\frac{3}{2}} N(p)^{\frac{3}{2}}. \]

The theorem is proved.

For \(p = 1 + i\) we have trivially \(|K(\alpha, \beta; k; p)| = 1\).

Now for \(\gamma = p^n\) we make a substitute \(x \pmod{p^n} = y + p^m z\), where \(y \pmod{p^n}\), \(z \pmod{p^{n-m}}\), \(m = \left[ \frac{n+1}{2} \right] \) and then using the standard technique, we easily obtain the following theorem.

**Theorem 3.3.** Let \(\gamma = (1 + i)^n \prod_{i=1}^{n_0} p_i^{n_i} \prod_{j=1}^{t_1} p_i^{n_j} \pmod{p_i^{(3)}(4)} \). Then

\[ |K(\alpha, \beta; k; \gamma)| \leq 2D\sqrt{N((\alpha, \beta, \gamma))} N(\gamma)^{\frac{3}{2}}, \tag{3.5} \]

where \(D = \prod_{i=1}^{t} (k, p_j - 1) \prod_{j=1}^{t} (p_j^2 - 1)\).
Now we consider a nontrivial multiplicative character \( \psi \) of the field \( \mathbb{F}_q \), \( q = p^r \), \( r \in \mathbb{N} \), \( p \) be a prime number, \( \alpha, \beta \in \mathbb{F}_q \) and \( \alpha \neq 0 \) or \( \beta \neq 0 \). We define the general power Kloosterman sum with a character

\[
K(\alpha, \beta; k; q, \psi) := \sum_{x \in \mathbb{F}_q} \psi(x)e_p\left(\sigma_2(\alpha x^k + \beta x'^k)\right). \tag{3.6}
\]

Let \( d = (k, q - 1) \), \( \psi(x) = e^{2\pi i \frac{\text{ind} x}{q - 1}} \), where \( \text{ind} x \) take in regard to a some primitive element for \( \mathbb{F}_q \). We have two probable cases: \( d \mid h \) and \( d \nmid h \).

We shall prove that \( K(\alpha, \beta; k; q, \psi) = 0 \) in first case. We have for \( \beta \neq 0 \):

\[
\sum_{\alpha \in \mathbb{F}_q} \left| K(\alpha, \beta; k; q, \psi) \right|^2 = \sum_{\alpha \in \mathbb{F}_q} \sum_{x, y \in \mathbb{F}_q^*} \psi(x)\psi(y)e_p\left(\sigma_2(\alpha(x^k - y^k) + \beta(x'^k - y'^k))\right) = \sum_{x \in \mathbb{F}_q^*} \psi(x) \sum_{y \in \mathbb{F}_q^*} e_p\left(\sigma_2(\beta y'^k(x^k - 1))\right) \sum_{\alpha \in \mathbb{F}_q} e_p\left(\sigma_2(\alpha y^k(x^k - 1))\right) = q \sum_{y \in \mathbb{F}_q^*} \psi(y) \sum_{x \in \mathbb{F}_q^*} \left| \sigma_2(\beta y'^k(x^k - 1)) \right| = q(q - 1) \sum_{x \in \mathbb{F}_q^*} \psi(x). \tag{3.7}
\]

In the last sum the summation runs over \( x \in \mathbb{F}_q^* \) for which \( k \text{ ind } x \equiv 0 \pmod{q - 1} \), i.e., \( \text{ind } x = \frac{q - 1}{d} s \), \( s = 0, 1, \ldots, d - 1 \), and thus

\[
\sum_{\alpha} \left| K(\alpha, \beta; k; q, \psi) \right|^2 = q(q - 1) \sum_{s=0}^{d-1} e^{2\pi i \frac{s}{d} h} = 0 \quad \text{if} \quad h \not\equiv 0 \pmod{d}.
\]

If \( d \mid h \) we have \( \psi(x) = e_{q-1}(h_1 \text{ ind } x) = e_{q-1}(h_1 \text{ ind } x^d) \). Hence, setting \( k_1 = \frac{k}{d} \), \( h_1 = \frac{h}{d} \), \( \psi_1^d = \psi \), we obtain

\[
K(\alpha, \beta; k; q, \psi) = \sum_{x \in \mathbb{F}_q} \psi_1(x^d)e_p\left(\sigma_2(\alpha(x^d)^{k_1} + \beta(x'^d)^{k_1})\right) = \sum_{x \in \mathbb{F}_q} \psi_1(x)e_p\left(\sigma_2(\alpha x^{k_1} + \beta x'^{k_1})\right) = \sum_{s=0}^{d-1} \sum_{x \in \mathbb{F}_q^*} e_d(s \text{ ind } x)e_{q-1}(h_1 \text{ ind } x) e_p\left(\sigma_2(\alpha x^{k_1} + \beta x'^{k_1})\right) = \sum_{s=0}^{d-1} \sum_{x \in \mathbb{F}_q^*} \psi_2(x)e_p\left(\sigma_2(\alpha x^{k_1} + \beta x'^{k_1})\right), \tag{3.8}
\]

where \( \psi_2(x) = e_{q-1}(h_2 \text{ ind } x) \), \( h_2 = \frac{s(q - 1) + h}{d} \).

So then we diminished the exponent \( k \) in \( d \)-time if \( d > 1 \). But if \( d = 1 \) then clearly that
\[ K_k(\alpha, \beta; q) = \sum_{x \in \mathbb{F}_q^*} e_{q-1}(hh' \text{ ind } x^k)e_p(\sigma_2(\alpha x^{k_1} + \beta x^{k_1})) = \sum_{x \in \mathbb{F}_q^*} e_{q-1}(hh' \text{ ind } x)e_p(\sigma_2(\alpha x + \beta x')) = K_1(\alpha, \beta; q; \psi_3), \]

where \( hh' \equiv 1 \pmod{(q - 1)}, \psi_3 = \psi^{k'}. \)

The sum \( K_1(\alpha, \beta; q, \psi_3) \) is the Kloosterman sum over \( \mathbb{F}_q \) weighting by a multiplicative character \( \psi_3 \) of the field \( \mathbb{F}_q \) and has an estimation as \( 2q^{1/2} \) if \( \beta \neq 0 \) (see Perel'muter [17]). The relation (3.8) show that if \( (k_1, q - 1) = 1 \) then

\[ |K_k(\alpha, \beta; q, \psi)| \leq 2dq^{1/2}. \]

If \( (k_1, q - 1) = d_1 > 1 \) we again consider two cases

\( (h_1, d_1) = d_1 \) or \( (h_2, d_1) < d_1. \)

But if \( (h_2, d_1) < d_1 \) we have \( K(\alpha, \beta; k; q, \psi) = 0. \)

The case \( h_2;d_1 \) can execute only for those \( s, 0 \leq s \leq d - 1 \), for which \( h_2 \equiv 1 \pmod{d_1} \), i.e., \( s \) must satisfy the congruence

\[ s \frac{q - 1}{d} + h \equiv 0 \pmod{d_1}. \]

But \( \left( d_1, \frac{q - 1}{d} \right) = 1 \) since \( d_1 | k_1 \) and \( \left( k_1, \frac{q - 1}{d} \right) = 1. \)

It follows that we have only one value \( s \) modulo \( d_1 \), and hence, at most \( \left\lfloor \frac{d}{d_1} \right\rfloor + 1 \) the value of \( s \) among \( 0 \leq s \leq d - 1 \), for which \( h_2d_1 \). We apply this reduction and through \( \nu(k) \) steps we obtain the estimation

\[ |K_k(\alpha, \beta; q, \psi)| \leq 2^{\nu(k)+1}kq^{1/2}, \]

where \( \nu(k) \) denote the number a prime divisors of \( k. \)

And so we proved the following theorem.

**Theorem 3.4.** Let \( \alpha, \beta \in \mathbb{F}_q \) and though one of element \( \alpha \) or \( \beta \) is not equal to zero. Then for any multiplicative character \( \psi \) of field \( \mathbb{F}_q \) the estimation

\[ |K(\alpha, \beta; q, \psi)| \leq 2kq^{1/2} \]

holds.

**Corollary.** Let \( p \) be a Gaussian prime number and let \( \chi \) is a multiplicative character of a field of the residue classes \( \text{mod} \, p. \) Then

\[ \left| \sum_{x \pmod{p}} \chi(x) \exp \pi i \text{ Sp} \left( \frac{\alpha x^k + \beta x^k}{p} \right) \right| \leq 2^{\nu(k)+1}kN(p)^{1/2}N((\alpha, \beta, p))^{1/2}. \]

4. General Kloosterman sums over norm. Let \( \alpha, \beta \in \mathbb{Z}[i], \, h \in \mathbb{Z}, \, q \in \mathbb{N}, \, q > 1, \, (h, q) = 1. \) We set

\[ \tilde{K}(\alpha, \beta; h, q) := \sum_{x,y \pmod{q}} e_q \left( \frac{1}{2} \text{ Sp} (\alpha x + \beta y) \right) \quad (4.1) \]

and call the norm Kloosterman sum in \( \mathbb{Z}[i]. \)
For \( q = q_1 q_2 \), \((q_1, q_2) = 1\) we have

\[
\tilde{K}(\alpha, \beta; h, q) = \tilde{K}(\alpha, \beta; h q''_2, q_1) \tilde{K}(\alpha, \beta; h q''_1, q_2) = \\
= \tilde{K}(\alpha q_2, \beta q_2; h, q_1) \tilde{K}(\alpha q_1, \beta q_1; h, q_2).
\]

Thus we shall consider only case \( q = p^n \), \( p \) is prime rational number, \( n \in \mathbb{N} \). We denote \( m_\alpha = \max_{m \geq n} \{ \alpha \equiv 0 \pmod{p^m} \} \).

**Theorem 4.1.** Let \((h, p) = 1\). Then

\[
\tilde{K}(\alpha, \beta; h, p^n) \ll (p^{m_\alpha}, p^{m_\beta}, p^n)^{\frac{1}{2}} p^{\frac{3n}{2}}
\]

with an absolute constant in symbol \( \ll \).

**Proof.** At first let \( n = 1 \). The case \( m_\alpha = m_\beta = 1 \) is a trivial. Thus we shall suppose that \( m_\alpha = 0 \) or \( m_\beta = 0 \). We set \( \alpha = a_1 + i a_2 \), \( \beta = b_1 + i b_2 \) and, hence, \((a_1, a_2, b_1, b_2) = 1\).

For \( p \equiv 1 \pmod{4} \) we have

\[
\tilde{K}(\alpha, \beta; h, p) = \sum_{(U)} e_p(a_1 x_1 - a_2 x_2 + b_1 y_1 - b_2 y_2),
\]

where \( U = \{ x_1, x_2, y_1, y_2 \in \{0, 1, \ldots, p-1\}, (x_1^2 + x_2^2)(y_1^2 + y_2^2) \equiv h \pmod{p} \} \). Let \( \varepsilon_0 \) is a solution of congruence \( x^2 \equiv -1 \pmod{p} \).

We set

\[
u_1 = x_1 + \varepsilon_0 x_2, \quad u_2 = x_1 - \varepsilon_0 x_2, \quad v_1 = y_1 + \varepsilon_0 y_2, \quad v_2 = y_1 - \varepsilon_0 y_2.
\]

Now by (4.3) we obtain

\[
\tilde{K}(\alpha, \beta; h, p) = \sum_{(U)} e_p(A_1 u_1 + A_2 u_2 + B_1 v_1 + B_2 v_2),
\]

where \( U = \{ u_1, u_2, v_1, v_2 \in \{0, 1, \ldots, p-1\}, u_1 v_1 u_2 v_2 \equiv h \pmod{p} \} \).

E. Bombieri [14] proved that the last sum can be estimated as \( \ll p^2 \). If \( p \equiv 3 \pmod{4} \) then the such estimation holds for the sum (4.3) (the proof is analogous).

The case \( p = 2 \) is a trivial.

Now, let \( n \geq 2 \). It is enough to consider only the case \((p^{m_\alpha}, p^{m_\beta}, p^n) = 1\). In this case though one of number \( a_1, a_2, b_1, b_2 \) does not divide on \( p \) (here \( \alpha = a_1 + i a_2 \), \( \beta = b_1 + i b_2 \)). We have

\[
\tilde{K}(\alpha, \beta; h, p^n) = \\
= \sum_{x, y \pmod{p^n}} \frac{1}{p^{n-1}} \sum_{k=0}^{p^n-1} e_{p^n} \left( k(N(x)N(y) - h) + \mathcal{R}(ax) + \mathcal{R}(by) \right) = \\
= \frac{1}{p^n} \sum_{U} e_{p^n} \left( k((x_1^2 + x_2^2)(y_1^2 + y_2^2) - h) + a_1 x_1 - a_2 x_2 + b_1 y_1 - b_2 y_2 \right),
\]

where \( U := \{ k \pmod{p^n}; x_1, x_2 \pmod{p^n}; y_1, y_2 \pmod{p^n} \} \).

Though one out of sums over \( x_1, x_2, y_1, y_2 \) is equal 0 if \( (k, p) = p \) (by a rational analogue of Lemma 2.1).

Thus, supposing \((a_1, a_2, p) = 1\), we have
\[ \tilde{K}(\alpha; \beta; h, p^n) = \]
\[ = \frac{1}{p^n} \sum_U e_p(-kh)e_{p^n} \left( kN(x)(y_1^2 + y_2^2) + \Re(\alpha x) + b_1y_1 - b_2y_2 \right) = \]
\[ = \frac{1}{p^n} \sum_{k \pmod{p^n}} e_{p^n}(-kh) \left( \sum_{x \pmod{p^n}} + \sum_{x \pmod{p^n}} \sum_{N(x) = p} \right) = \sum_1 + \sum_2, \quad (4.5) \]
say, where \( U := \{ k \in R^+(p^n), x \in R(p^n, i), y_1, y_2 \in R(p^n) \} \). Let \( N(x)' \) and \( k' \) are the solutions of the congruences
\[ N(x)u \equiv 1 \pmod{p^n}, \quad ku \equiv 1 \pmod{p^n}, \]
accordingly. Then
\[ \left| \sum_1 \right| = \left| \sum_k, e_{p^n}(-kh) \sum_{x \pmod{p^n}} e_{p^n} \left( 4'N(x)'k'(b_1^2 + b_2^2) + a_1x_1 - a_2x_2 \right) \right|. \quad (4.6) \]
We set
\[ x_1 = x_1^0 + p^mz_1, \quad x_2 = x_2^0 + p^mz_2, \]
\[ 0 \leq x_1^0, x_2^0 \leq p^m - 1, \quad 0 \leq z_1, z_2 \leq p^{n-m} - 1, \quad m = \left\lfloor \frac{n+1}{2} \right\rfloor. \]
It is obvious
\[ N(x)' = (x_1^0 + x_2^0)'(1 - 2p^m(x_1^0 + x_2^0)'(x_1^0z_2 + x_2^0z_1)) \]
and consequently
\[ \left| \sum_1 \right| = \left| \sum_k, e_{p^n}(-kh) \times \right| \times \sum_{x_1^0, x_2^0 \pmod{p^n}} e_{p^n} \left( 4k'(x_1^0 + x_2^0)'(b_1^2 + b_2^2) + a_1x_1^0 - a_2x_2^0 \right) \times \]
\[ \times \sum_{x_1^0, x_2^0 \pmod{p^n}} e_{p^n-m} \left( (A_1 + a_1)z_1 + (A_2 + a_2)z_2 \right), \]
where \( A_1 = 2((x_1^0 + x_2^0)'^2 x_2)^0, \quad A_2 = 2((x_1^0 + x_2^0)'^2 x_1)^0. \)
The summation over \( z_1, z_2 \) gives zero if the congruences
\[ A_1 + a_1 \equiv 0 \pmod{p^{n-m}}, \quad A_2 - a_2 \equiv 0 \pmod{p^{n-m}} \]
or the equivalent congruences
\[ a_2x_1^0 + a_1x_2^0 \equiv 0 \pmod{p^{n-m}}, \quad 2x_2^0 \equiv -a_1(x_1^0 + x_2^0)^2 \pmod{p^{n-m}} \]
are disturbs.

This system of the congruences has at most three solutions modulo \( p^{n-m} \), and therefore at most \( 3p^{m-(n-m)} \) solutions modulo \( p^m \).
Hence,
\[
\left| \sum_1 \right| = \left| p^{2(n-m)} \sum_{(U)} e_p^n (a_1 x_1^0 - a_2 x_1^0) \sum_{k \pmod{p^n}} (kh + k'B) \right| \leq 8 p^{\frac{3}{2} n}, \quad (4.7)
\]
where
\[
U = \left\{ x_1, x_2 \pmod{p^n} \mid a_2 x_1^0 \equiv -a_1 x_2^0 \pmod{p^n-m}, \right. \left. 2x_1 \equiv -a_1 \left( x_1^2 + x_2^2 \right)^2 \pmod{p^n-m} \right\}.
\]
At last, if \( N(x) \equiv 0 \pmod{p} \) then \( \sum_2 = 0 \) by Lemma 2.1.

The theorem is proved.

For natural \( k > 1 \) we set
\[
\tilde{K}(\alpha, \beta; h, q; k) := \sum_{x, y \pmod{q}} \epsilon_q \left( \frac{1}{2} \text{Sp}(\alpha x^k + \beta y^k) \right). \quad (4.8)
\]
It is obvious that \( \tilde{K}(\alpha, \beta; h, q; 1) = K(\alpha, \beta; h, q) \).

The method of investigation of the sum \( \tilde{K}(\alpha, \beta; h, q; k) \) towards suffices to consider the case \( q = p^n \), \( p \) be a prime. At first we shall account that \( p \equiv 3 \pmod{4} \).

**Theorem 4.2.** Let \( p \equiv 3 \pmod{4}, h \in \mathbb{Z}, (h, p) = 1, k \in \mathbb{N}, d = (k, p - 1) \). Then for any Gaussian integers \( \alpha, \beta, (\alpha, \beta, p) = 1 \) the estimation
\[
\left| \tilde{K}(\alpha, \beta; h, p; k) \right| \ll \begin{cases} d^2 p^{\frac{3}{2}} & \text{if } d - 1 \leq \sqrt{p}, \\ dp^2 & \text{if } d \geq \sqrt{p} + 1 \end{cases}
\]
holds.

**Proof.** Let \( k = dk_1, \left( k_1, \frac{p - 1}{d} \right) = 1 \). We have
\[
\sum_{x, y \pmod{p}} e_p \left( \frac{1}{2} \text{Sp}(\alpha x^{k_1} + \beta y^{k_1}) \right) = \\
\sum_{x, y \pmod{p}} e_p \left( \frac{1}{2} \text{Sp}(\alpha x^{k_1} + \beta y^{k_1}) \right) = \\
\sum_{x, y \pmod{p}} e_p \left( \frac{1}{2} \text{Sp}(\alpha x^d + \beta y^d) \right) = \tilde{K}(\alpha, \beta; h, k_1, p; d).
\]

Now, for any multiplicative character \( \chi \) of field \( \mathbb{F}_{p^2} \) we have
\[
\sum_{h \in \mathbb{F}_{p^2}} \chi(h) \tilde{K}(\alpha, \beta; h, p; d) = \\
\sum_{x, y \in \mathbb{F}_{p^2}} \chi(N(x)N(y)) e_p \left( \frac{1}{2} \text{Sp}(\alpha x^d) \right) e_p \left( \frac{1}{2} \text{Sp}(\beta y^d) \right) = \\
\left( \sum_{x \in \mathbb{F}_{p^2}} \chi(N(x)) e_p \left( \frac{1}{2} \text{Sp}(\alpha x^d) \right) \right) \left( \sum_{y \in \mathbb{F}_{p^2}} \chi(N(y)) e_p \left( \frac{1}{2} \text{Sp}(\beta y^d) \right) \right), \quad (4.9)
\]
and moreover the sums on the right of (4.9) can be estimate as \((d - 1)N(p)^\frac{1}{2}\) (see [17]). Thus we obtain

\[
\sum_{h \in \mathbb{F}_p^*} \chi(h) \tilde{K}(\alpha, \beta; h, p; d) \leq (d - 1)^2 p^2.
\]

The application of the Plancherel theorem gives

\[
\sum_{h \in \mathbb{F}_p^*} |K(\alpha, \beta; h, p; d)|^2 \leq (d - 1)^4 p^4.
\]

Now similarly as in the Bombieri work [14] we conclude that the weights of characteristic roots associating with \(\tilde{K}(\alpha, \beta; h, p; d)\) are at most 3 if \((d - 1)^4 < p\). Hence, using the results of Bombieri [14] and Deligne [13] we infer

\[
\tilde{K}(\alpha, \beta; h, p; d) \ll (d - 1)^2 p^2 \ll d^2 p^2 \quad \text{if} \quad d - 1 < \sqrt{p}.
\]

Further, for \(x = x_1 + ix_2, x_1, x_2 \in \mathbb{Z}\), we have \(x_1 - ix_2 \equiv (x_1 + ix_2)p\) (mod \(p\)) and thus \(N(x) \equiv x^{p-1} \pmod{p}\). Hence,

\[
\sum_{x, y \pmod{p}} e_p \left( \frac{1}{2} \sum_{d, \beta \equiv h \pmod{p}} (\chi_d + \beta y^d) \right) = \sum_{x, y \pmod{p}} e_p \left( \frac{1}{2} \sum_{d, \beta \equiv h \pmod{p}} (\chi_d + \beta y^d) \right) = \sum_{\chi \pmod{p}} \sum_{x \pmod{p}} e_p \left( \frac{1}{2} \sum_{d, \beta \equiv h \pmod{p}} (\chi_d + \beta y^d) \right). \tag{4.10}
\]

The congruence \(x^{p-1} \equiv h \pmod{p}\) has exactly \(p + 1\) solutions \(\pmod{p}\). The inner sum in the right in (4.10) estimates as \(\leq 2dp\). This completes the proof of the theorem.

Now, let \(q = p^n, p \equiv 3 \pmod{4}, n \geq 2\). We shall use the description of a reduced residue system \(\pmod{p^n}\) (see Lemma 2.5).

In farther the following assertion are need.

**Lemma 4.1.** Let \(n, k \in \mathbb{N}, p \geq 3\) be a prime, \(u \in \mathbb{Z}, (p, u) = 1\). Then for any natural \(t\) we have

\[
(1 + p^k u)^t \equiv 1 + p^k a_1 t + p^{2k} a_2 t^2 + p^{3k} a_3 t^3 + \ldots + p^{nk} a_n t^n \pmod{p^n},
\]

Moreover \((a_i, p) = 1, i = 1, \ldots, n, \lambda_j > 2k, j = 3, \ldots, n\).

**Proof.** By the relation

\[
\binom{t}{m} = \frac{1}{m!} \left( \frac{1}{2} m m! \right) \frac{1}{2} m m! + \ldots + (-1)^{m-1} m m! (m - 1)! t
\]

and upper estimation of an exponent with which \(p\) enters in \(m!\) we obtain

\[
(1 + p^k u)^t \equiv 1 + p^k a_1 t + p^{2k} a_2 t^2 + p^{3k} a_3 t^3 + \ldots + p^{nk} a_n t^n \pmod{p^n},
\]

where \((a_i, p) = 1, i = 1, \ldots, n, \lambda_j > \left( k - \frac{1}{p - 1} \right) j > 2k\) for \(j = 3, 4, \ldots\).

The lemma is proved.
From the proof of Lemma 2.3 it is obvious that a generative element $u + iv$ of the group $E_1$ can be taken that it is a generative element of the group $E_\ell$ for any fixed $\ell$, $\ell = 2, 3, \ldots$. Let $\ell = \max(5, n)$. We have

$$N((u + iv)^2 \equiv 1 \pmod{p^\ell},$$

Thus

$$N(1 + px_0 + ipy_0) \equiv 1 + 2px_0 + p^2x_0^2 + p^2y_0^2 \equiv 1 \pmod{p^\ell}. $$

Hence, $2px_0 \equiv 0 \pmod{p^\ell}$, $x_0 = px_0^\ell$, $(y_0, p) = 1$. So,

$$(u + iv)^{2(p+1)} \equiv 1 + p^2x_0 + ipy_0, \quad (x_0, p) = (y_0, p) = 1.$$

Now, applying the previous lemma we easily obtain

$$\Re((u + iv)^{2(p+1)t}) \equiv A_0 + A_1t + A_2t^2 + \ldots + A_{n-1}t^{n-1} \pmod{p^n},

\Im((u + iv)^{2(p+1)t}) \equiv B_0 + B_1t + B_2t^2 + \ldots + B_{n-1}t^{n-1} \pmod{p^n},$$

(4.11)

where

$$A_0 \equiv 1 \pmod{p}, \quad B_0 \equiv 0 \pmod{p},$$

$$A_1 \equiv p^2x_0 + 2y_0^2p^2 \pmod{p^3}, \quad \text{i.e.,} \quad A_1 \equiv 0 \pmod{p^3},$$

$$A_2 \equiv -2y_0^2p^2 \pmod{p^3}, \quad \text{i.e.,} \quad A_2 = p^2A_2', \quad (A_2', p) = 1,$$

$$B_1 \equiv py_0 \pmod{p^3}, \quad \text{i.e.,} \quad B_1 \equiv pB_1', \quad (B_1', p) = 1,$$

$$B_2 \equiv A_3 \equiv A_4 \equiv \ldots \equiv A_{n-1} \equiv B_{n-1} \equiv 0 \pmod{p^3}. $$

We set

$$\beta = 2(p+1)t + z, \quad 0 \leq t \leq p^{n-1} - 1, \quad 0 \leq z \leq 2p + 1,$$

and denote

$$(u + iv)^{\beta} = u(z) + iv(z), \quad z = 0, 1, \ldots, 2p + 1.$$}

Then

$$(u + iv)^{\beta} = (u + iv)^{2(p+1)t}(u(z) + iv(z)).$$

And hence, we have

$$\Re\{(u + iv)^{2(p+1)t+z}\} \equiv A_0(z) + A_1(z)t + \ldots + A_{n-1}(z)t^{n-1} \pmod{p^n},$$

(4.12)

where $A_1(z) = A_1u(z) - B_1v(z)$.

We clear up for which values $z$ the congruence $v(z) \equiv 0 \pmod{p}$ holds. Let $v(z) = pv_0(z)$, $v_0(z) \equiv 0 \pmod{p^k}$, $k \geq 0$. Then

$$(u + iv)^{z} = u(z) + ivv_0(z),$$

$$(u + iv)^{z(\pmod{p^n})} \equiv (u(z))^{z(\pmod{p^n})} \pmod{p^n}.$$}

The sequences $\{u + iv)^{2\beta}\}$ and $\{g^n\}$ can have only two common elements modulo $p$: 1 or $-1$. Thus

$$(u(z))^{(p-1)p^{n-1}} \equiv \pm 1 \pmod{p^n}. $$

The congruence $(u(z))^{(p-1)p^{n-1}} \equiv -1 \pmod{p^n}$ is impossible, so the other way we have $(-1)^{p^{n-1}} \equiv (u(z))^{(p+1)p^{n-1}} \equiv 1 \pmod{p^n}$, i.e., $-1 \equiv 1 \pmod{p}$. Hence
\[(u(z))^{(p-1)p^{n-k}} \equiv 1 \pmod{p^n},\]
\[z(p-1)p^{n-k} \equiv 0 \pmod{2(p+1)p^{n-1}}.\]

Since, \((p-1,p+1) = 2\), then \(z \equiv 0 \pmod{(p+1)p^{k-1}}\). Whence it follows that from \(p \nmid v(z)\) we have \(z \equiv p+1\) and from \(p^2 \nmid v(z)\) follows \(z = 0\). So we have
\[p \mid A_1(z), \quad A_1(z) \equiv 0 \pmod{p^2}, \quad i = 2, \ldots, n-1, \quad \text{if} \quad z \neq 0, z \neq p+1,\]
\[A_1(0) = A_1(p+1) \equiv 0 \pmod{p^2}, \quad p^2 \mid A_2(0), \quad p^2 \mid A_2(p+1),\]
\[A_j(0) \equiv A_j(p+1) \equiv 0 \pmod{p^3}, \quad j = 3, 4, \ldots, n-1.\]

We are now in position to prove the following assertion.

**Theorem 4.3.** Let \(p\) be a prime number, \(p \equiv 3 \pmod{4}\), \(h \in \mathbb{Z}\), \((h,p) = 1\), \(k > 1\) is a natural, \(a, b\) are the Gaussian integer, \((a,p) = (b,p) = 1\). Then for \(n \geq 2\)
\[
\left|K(a,b; h, p^n; k)\right| \leq 2p^{\frac{1}{2}n+m} \log p^n, \quad (4.13)
\]
where \(m\) such that \(p^m \parallel k\).

**Proof.** Applying Lemma 2.5, we can write \(a, b\) in the form
\[a = g^{\alpha'}(u + iv)^{\beta'}, \quad b = g^{\alpha''}(u + iv)^{\beta''},\]
where \(g\) is a primitive root \(\pmod{p^n}\) in \(\mathbb{Z}\), \(u + iv\) is a generative element of the group \(E_n\). Then we obtain
\[
K(a,b; h, p^n; k) = \sum_{x,y \pmod{p^n}} e_{p^n} \left(g^{\alpha'} \Re((u + iv)^{\beta'}x^k) + g^{\alpha''} \Re((u + iv)^{\beta''}y^k)\right). \quad (4.14)
\]
Let \(h \equiv g^n \pmod{p^n}\). Then \(h \equiv \pm g^{2\alpha_0} \pmod{p^n}\), where
\[2\alpha_0 = \begin{cases} \alpha & \text{if } \alpha \text{ is even}, \\ \alpha + \frac{p-1}{2}p^{n-1} & \text{if } \alpha \text{ is odd}. \end{cases}\]
The sum over \(x \pmod{p^n}\) in (4.14) we split into two pairs, \(\sum_1 = \sum_1 + \sum_2\).

In sum over \(\sum_1\) we take these \(x \pmod{p^n}\) for which \(N(x) \equiv g^{2\alpha_1} \pmod{p^n}\), and in \(\sum_2\) come upon these \(x \pmod{p^n}\) for which \(N(x) \equiv -g^{2\alpha_1} \pmod{p^n}\). In both cases \(\alpha_1\) runs the values 0, 1, \ldots, \frac{p-1}{2}p^{n-1} - 1. So
\[
K(a,b; h, p^n; k) = \sum_1 + \sum_2. \quad (4.15)
\]
For \(x\) from \(\sum_1\) we have
\[x \equiv g^{\alpha_1}(u + iv)^{2\beta_1} \pmod{p^n},\]
\[\alpha_1 = 0, 1, \ldots, \frac{1}{2}(p-1)p^{n-1} - 1, \quad \beta_1 = 0, 1, \ldots, (p+1)p^{n-1} - 1.\]
Hence,
\[\Re((u + iv)^{\beta_1}x^k) \equiv g^{k\alpha_1} \Re((u + iv)^{2k\beta_1} \pmod{p^n}).\]
From the condition \(N(x)N(y) \equiv h \pmod{p^n}\) it follows

**ISSN 1027-3190. Ukr. mat. журн., 2007, т. 59, № 9**
where $\alpha_2 = \alpha_0 + ((p - 1) p^{n - 1} - 1) \alpha_1$.

And thus we have

$$
N(y) \equiv \pm g^{2\alpha_2} \pmod{p^n},
$$

where $\alpha_2 = \alpha_0 + ((p - 1) p^{n - 1} - 1) \alpha_1$.

The sums over $t$ here

$$
\sum_1 = \sum_{(\alpha_1)} \sum_{(\beta_1)} \sum_{(\beta_2)} e_p^n \left( g^{\alpha_0 + \alpha_1 k} R \left( (u + iv)^{2k\beta_1 + \beta_1'} \right) + g^{\alpha_0 + \alpha_2 k} R \left( (u + iv)^{2k\beta_2 + \beta_2' + \delta k} \right) \right),
$$

(4.16)

here $(\alpha_1)$ denotes that $\alpha_1$ runs the value $0, 1, \ldots, \frac{1}{2} (p - 1) p^{n - 1} - 1$; $(\beta_1)$ runs the value $0, 1, \ldots, (p + 1) p^{n - 1} - 1$, $i = 1, 2$; furthermore, $\delta = 0$ if $h \equiv g^{2\alpha_0} \pmod{p^n}$ and $\delta = 1$ if $h \equiv -g^{2\alpha_0} \pmod{p^n}$.

Similarly,

$$
\sum_2 = \sum_{(\alpha_1)} \sum_{(\beta_1)} \sum_{(\beta_2)} e_p^n \left( g^{\alpha_0 + \alpha_1 k} R \left( (u + iv)^{2k\beta_1 + \beta_1' + 1} \right) + g^{\alpha_0 + \alpha_2 k} R \left( (u + iv)^{2k\beta_2 + \beta_2' + \delta k} \right) \right),
$$

(4.17)

Again we have

$$
\beta_i = (p + 1) t_i + z_i, \quad t_i \pmod{p^{n - 1}}, \quad z_i = 0, 1, \ldots, p, \quad i = 1, 2.
$$

Then

$$
k \beta_i = 2(p + 1) k t_i + k z_i, \quad i = 1, 2.
$$

Now by (4.12), (4.13) and Lemma 2.1, it follows that the sums over $t_i$ are equal zero if the congruences

$$
\beta_0' + 2k z_1 \equiv 0 \pmod{p + 1},
$$

$$
\beta_0'' + 2k z_2 + k \delta \equiv 0 \pmod{p + 1} \quad \text{for a sum} \quad \sum_1,
$$

$$
\beta_0' + 2k z_1 + 1 \equiv 0 \pmod{p + 1},
$$

$$
\beta_0'' + 2k z_2 + k \delta \equiv 0 \pmod{p + 1} \quad \text{for a sum} \quad \sum_2,
$$

(4.18)

are disturb.

Consequently one from the sums $\sum_1$ or $\sum_2$ is equal always zero.

The congruences (4.18) can be true only for $(k, p + 1)^2$ pairs of the value $(z_1, z_2)$.

Let $\mathfrak{B}$ be set of these values $(z_1, z_2)$.

By (4.12) – (4.14) we obtain

$$
\tilde{K}(a, b; h, p^n; k) = \sum_{(\alpha_1)} e_p^n (N_0 g^{\alpha_1} + M_0 g^{\alpha_2}) \times
$$

$$
\times \sum_{(z_1, z_2) \in \mathfrak{B}} \sum_{t_1, t_2 \pmod{p^{n - 1}}} e_p^{n-2} (F_1(k t_1) g^{\alpha_1} + F_2(k t_2) g^{\alpha_2}),
$$

where $F_i(t) = c_1^{(i)} t + c_2^{(i)} t^2 + p^j \lambda_j c_3^{(i)} t^3 + \ldots + p^j \lambda_j c_j^{(i)} t^j$, $(c_j^{(i)}, p) = (c_3^{(i)}, p) = \ldots = 1$,

$\lambda_j > 0$ for $j \geq 3$, $(N_0, p) = (M_0, p) = 1$.

The sums over $t_1, t_2$ calculates equally. Let $k = p^m k_1$, $(k_1, p) = 1$. We break the sum over $t_i$ into blocks of the length $p^{n - 2 - 2m}$ (if $2m < n - 2$). Then applying Lemma 2.3, we obtain
\[ \widetilde{K}(a, b; h, t^n; k) = p^{n+2m} \sum_{(\alpha_2)} e_{p^n}(N_1 g^{\alpha_1} + N_2 g^{\alpha_2}), \] (4.19)

where \((N_1, p) = (N_2, p) = 1\).

From the definition \(\alpha_2\) follows \(g^{\alpha_2} \equiv g^\alpha (g')^{\alpha_1} \pmod{p^n}\).

The sum on the right in (4.19) is the incomplete Kloosterman sum. By a choice of a primitive root \(g\) we have

\[ g^{p-1} = 1 + pu, \quad (u, p) = 1. \]

Then \(g^{p-1} = 1 - pu_1, (u_1, p) = 1, u \equiv u_1 \pmod{p}\). We set

\[ \alpha_1 = (p - 1)t + z, \]
\[ t = 0, 1, \ldots, \frac{1}{2}(p^{n-1} - 1), \quad z = 0, 1, \ldots, p - 2. \]

Thus

\[ g^{\alpha_1} = g^z (1 + a_1 pt + a_2 p^2 t^2 + a_3 p^3 t^3 + \ldots) \pmod{p^n}, \]
\[ a_1 \equiv -u_1, \quad a_2 \equiv -2u^2 \pmod{p}, \quad \lambda_j \geq 3. \]

Similarly

\[ g^{\lambda_2} \equiv g^\alpha g^{\alpha_1} \equiv g^\alpha g^z (1 + b_1 pt + b_2 p^2 t^2 + b_3 p^3 t^3 + \ldots) \pmod{p^n}, \]
\[ b_1 \equiv -u_1, \quad b_2 \equiv -2u^2 \pmod{p}, \quad \mu_j \geq 3. \]

Hence,

\[ N_1 g^{\alpha_1} + N_2 g^{\lambda_2} \equiv c_0 + c_1 pt + c_2 p^2 t^2 + c_3 p^3 t^3 + \ldots \pmod{p^n}, \]

where \(c_i = g^z a_i N_1 + g^{\alpha_1} g^z b_i N_2, i = 1, 2, \ldots, p - 1\).

Since \((N_1, p) = (N_2, p) = 1\), it is easy to observe that two congruence

\[ c_1 \equiv 0 \pmod{p}, \quad c_2 \equiv 0 \pmod{p} \]

cannot realize simultaneously.

But from \(c_1 \equiv 0 \pmod{p}\) follows \(g^{2z} \equiv g^\alpha N_2 N'_1 \pmod{p}\). It is possible only one value \(z\). Let’s designate this value through \(z_0\).

Thus from (4.19) we infer

\[ \widetilde{K}(a, b; h, t^n; k) = \]
\[ = p^{n+2m} \left( \sum_{z=0}^{p-2} \frac{1}{2} (p^n - 1) e^{2\pi i p^z} e_{p^n-1} (c_1 t + c_2 p^2 t^2 + c_3 p^3 t^3 + \ldots) + \right. \]
\[ \left. + \sum_{t=0}^{p^{n-1} - 1} e^{2\pi i p^z} e_{p^n-2} (c'_1 t + c'_2 p^2 t^2 + c'_3 p^3 t^3 + \ldots) \right). \] (4.20)

where \((c_1, p) = (c'_2, p) = 1\).

The sums over \(t\) are the incomplete rational sums, their estimations we obtain by way of estimations complete exponent sums.

We have for an arbitrary polynomial \(\Phi(t) \in \mathbb{Z}[t] : \)

\[ \left| \sum_{t=0}^{q-1} e^{2\pi i \Phi(t)} - \frac{q}{q} \sum_{t=0}^{q-1} e^{2\pi i \Phi(t)} \right| \leq \sum_{r=1}^{q-1} \frac{1}{\min(r, q - r + 1)} \sum_{t=0}^{q-1} e^{2\pi i \Phi(t)/q}. \] (4.21)
Now, if $\Phi(t) = c_1 t + c_2 t^2 + c_3 t^{p-1} t^3 + \ldots$, $(c_1, p) = 1$, $q = p^{n-1}$, then the complete sums in (4.21) are equal to zero for all $r$ except the case $r \equiv c_1 \pmod{p}$. In this special case we have $\Phi(t) = c_1 t + c_2 t^2 + c_3 t^{p-1} t^3 + \ldots$, $(c_1, p) = 1$, $q = p^{n-2}$, and then a complete sum estimates by a value $2p^{\frac{n-2}{2}}$.

Hence,

$$\left| \tilde{K}(a, b; h, p^n; k) \right| \leq p^{n+m} \left[ \sum_{z=0}^{p-2} \frac{1}{|c(z)|} + \sum_{r=1}^{p^n} \frac{1}{k^2 p^{n-2} + pp^{n-2}} \right].$$

At last, we take account that for the distinct values $z$ we have the distinct values $c_1(z) \pmod{p}$, and thus we obtain

$$\left| \tilde{K}(a, b; h, p^n; k) \right| \leq p^{2n+m} \left( \log p + \frac{\log p^n}{p} \right).$$

If $2m > n - 2$ then the assertion of theorem is trivial.

The theorem is proved.

We go towards the estimation of the norm Kloosterman sum $\tilde{K}(a, b; h, p^n; k)$ for the case $p \equiv 1 \pmod{4}$, $k \geq 2$, $(a, p) = (b, p) = 1$. For $p \equiv 1 \pmod{4}$ we have $p = p\bar{p}$, where $p$ and $\bar{p}$ are the complex conjugate Gaussian prime number. Then the reduced residue system mod $p^n$ can write as

$$x = g^{\ell_1} \bar{p}^{\ell_2} p^n, \quad 0 \leq \ell_1, \quad \ell_2 \leq (p - 1)p^{n-1} - 1,$$

where $g$ is a primitive root mod $p^n$ such that

$$g^{p^n-1} = 1 + pH, \quad H \in \mathbb{Z}, \quad (H, p) = 1.$$

Thus

$$N(x) = x\bar{x} = g^{2\ell_1} p^n + g^{\ell_2} p^n + g^{\ell_1+\ell_2} \bar{p}^{2n} + g^{\ell_1+\ell_2} p^{2n} \equiv g^{\ell_1+\ell_2} S(p^{2n}) \pmod{p^n}. \quad (4.22)$$

Therefore, if $p = c + id$ then $(c, p) = (d, p) = 1$, and by the induction we easily obtain

$$p^{2n} \equiv c_n + id_n, \quad n = 1, 2, \ldots,$$

where

$$c_n \equiv \begin{cases} (-1)^{n-1} \cdot 2^m \cdot c \cdot d^{2m-1} \pmod{p}, & \text{if } n = 2m - 1, \\ (-1)^m \cdot 2^{m+2} d^{2m}, & \text{if } n = 2m. \end{cases}$$

$$d_n \equiv \begin{cases} (-1)^{2m-1} \cdot 2^m \cdot d^{2m-1} \pmod{p}, & \text{if } n = 2m - 1, \\ (-1)^{m-1} \cdot 2^{m+2} c \cdot d^{2m-1} \pmod{p}, & \text{if } n = 2m. \end{cases}$$

Hence, for $p \equiv 1 \pmod{4}$ we have

$$\tilde{K}(a, b; h, p^n; k) = \sum_{(U)} e^{p^n} \left( A(g^{\ell_1 k} + g^{\ell_2 k}) + B(g^{\ell_1 k} + g^{\ell_2 k}) \right) = \sum_{(U')} e^{p^n} \left( A(x_1^k + x_2^k) + B(y_1^k + y_2^k) \right), \quad (4.23)$$

where

$$U := \left\{ \ell_1', \ell_2', \ell_1'' \equiv \ell_2' \pmod{(p - 1)p^{n-1}} \right\} g^{\ell_1' k} + \ell_2'' k'$$

\[ H \pmod{p^n}. \]
Let \( p \equiv 1 \pmod{4} \) is a prime number and let \( a, b \in \mathbb{Z}[i] \), \( (a, p) = 1 \). Then

\[
\left| \tilde{K}(a, b; h, p; k) \right| \ll \begin{cases} 
d^2 p^2 & \text{if } (d - 1)^4 < p, \\
d^4 p^2 & \text{if } (d - 1)^4 \geq p,
\end{cases}
\]

where \( d = (k, p - 1) \).

**Proof.** With out loss of generality, we can suppose \( a, b \in \mathbb{Z} \).

By (4.23) and similarly as in the case \( p \equiv 3 \pmod{4} \) we obtain

\[
\tilde{K}(a, b; h, p; k) = \sum_{x_1, x_2, y_1, y_2 \in F_p^*} \chi(h) \left( \chi(x) e_p(Ax^2) \right)^2 \left( \chi(y) e_p(By^2) \right)^2.
\]

Now, for \( (d - 1)^4 < p \) we obtain by analogy with the case \( p \equiv 3 \pmod{4} \)

\[
\sum_{h \in F_p^*} \chi(h) \tilde{K}(a, b; h, p; d) = \left( \sum_{x \in F_p^*} \chi(x) e_p(Ax^2) \right)^2 \left( \sum_{y \in F_p^*} \chi(y) e_p(By^2) \right)^2.
\]

Hence,

\[
\sum_{h \in F_p^*} \left| \tilde{K}(a, b; h, p; d) \right|^2 \leq (d - 1)^4 p^4 \quad \text{if } (d - 1)^4 < p.
\]

Then

\[
\tilde{K}(a, b; h, p; k) \ll d^2 p^2 \quad \text{if } (d - 1)^4 < p.
\]

Let \( (d - 1)^4 \geq p \). Denote through \( g \) a primitive element of field \( F_p \) and let \( x = g^{\ind x} \) for \( x \in F_p^* \).

Let \( G \) is a group of multiplicative characters of \( F_p \). For \( \chi \in G \) we have \( \chi(x) = e_{p-1}(\nu \ind x) \) with some \( \nu \in F_p \). Then using the arguments from Theorem 4.1, we can obtain on a routine way the following relation:

\[
\tilde{K}(a, b; h, p; d) = \frac{1}{p-1} \sum_{\chi \in G} \chi(H) \sum_{s_1, s_2, s_3, s_4 = 0}^{d-1} \chi((A^2B^2)e_d(s_1 + s_2) \ind A + (s_3 + s_4) \ind B) \times
\]

\[
\chi(s_1 + \ldots + s_4 \ind A) \chi(s_1, \ldots, s_4) e_p(s_1 + \ldots + s_4) = \frac{1}{p-1} \sum_{\nu \in F_p} \sum_{s_1, \ldots, s_4 = 0}^{d-1} e_{p-1}(\nu \ind H) e_{p-1}(F_1(\nu, s)) \times
\]

\[
\chi(s_1 + \ldots + s_4 \ind A) e_p(s_1 + \ldots + x_4),
\]

where

\[
F_1(\nu, s) := \left( 2\nu + (s_1 + s_2) \frac{p-1}{d} \right) \ind A + \left( 2\nu + (s_3 + s_4) \frac{p-1}{d} \right) \ind B,
\]

\[
F_2(\nu, s, x) := \left( s_1 \frac{p-1}{d} + \nu \right) \ind x_1 + \ldots + \left( s_4 \frac{p-1}{d} + \nu \right) \ind x_4.
\]
The last sum over \(x_1, \ldots, x_4\) is the product of the Gauss sums of field \(\mathbb{F}_p\). And hence,
\[
\left| \tilde{K}(a, b; h, p; k) \right| \leq d^4p^2.
\]

The theorem is proved.

If \(n \geq 2\) we can use the description of solution of the congruence \(x_1, x_2, x_3, x_4 \equiv H \pmod{p^n}\):
\[
x_i = y_i + p^m z_i, \quad y_i \pmod{p^m}, \quad z_i \pmod{p^{n-m}},
\]
\[
(y, p) = 1, \quad i = 1, 2, 3; \quad m = \left\lceil \frac{n + 1}{2} \right\rceil,
\]
\[
x_4 = H y'_1 y'_2 y'_3 (1 - p^m y'_1 z_1 - p^m y'_2 z_2 - p^m y'_3 z_3), \quad y_1 y'_1 \equiv 1 \pmod{p^m}.
\]

**Theorem 4.5.** Let \(p \equiv 1 \pmod{4}\) be a prime number, \(n \in \mathbb{N}, \ n \geq 2; \ h \in \mathbb{Z}, (h, p) = 1; \ a, b \in \mathbb{Z}[i], (a, p) = (b, p) = 1.\) Then
\[
\left| \tilde{K}(a, b; h, p^n; k) \right| \ll \begin{cases} 
d^4p^\frac{n}{2} & \text{if } (d - 1)^4 < p, \\
d^4p^{n+m} & \text{if } (d - 1)^4 \geq p,
\end{cases}
\]
where \(m = \left\lceil \frac{n + 1}{2} \right\rceil\).

**Proof.** By (4.23), (4.24) we have
\[
\tilde{K}(a, b; h, p^n; k) = \sum_{x_1, x_2, x_3 \pmod{p^{n-m}}} e_{p^n}(f(y_1, y_2, y_3)) \times \\
\sum_{z_1, z_2, z_3 \pmod{p^{n-m}}} e_{p^{n-m}}(F(z_1, z_2, z_3)), \tag{4.25}
\]
where
\[
f(y_1, y_2, y_3) = A y_1^k + a y_2^k + B y_3^k + B H y_1^k y_2^k y_3^k,
\]
\[
F(z_1, z_2, z_3) = k \left[ (A y_1^{k-1} - B y_1^{k+1} y_2^k y_3^k) z_1 + \\
+ (A y_2^{k-1} - B (y_1^{k+1} y_2 y_3^k) z_2 + (A y_3^{k-1} - B (y_1 y_2 y_3^{k+1}) z_3 \right].
\]

Let \((k, p^{n-m}) = p^\ell\). Then we obtain from (4.25)
\[
\tilde{K}(a, b; h, p^n; k) = p^{\ell(n-m)} \sum_{(U)} e_{p^n}(f(y_1, y_2, y_3)),
\]
where \(U := \{ y_1, y_2, y_3 \pmod{p^m} | (y_i, p) = 1, \ i = 1, 2, 3; \ y_1^k \equiv y_2^k \equiv y_3^k \pmod{p^{n-m-\ell}}, y_1 y_2 y_3 \equiv B A' \pmod{p^{n-m-\ell}} \} \).

Now, for \(n = 2m\) we estimate the sum \(\sum_{(U)}\) by the number triples \((y_1, y_2, y_3) \in U\), and for \(n = 2m - 1\) we take into account also the Theorem 2.4. Hence, we have finally
\[
\left| \tilde{K}(a, b; h, p^n; k) \right| \ll \begin{cases} 
d^4p^{2n} & \text{if } (d - 1)^4 < p, \\
d^4p^{n+m} & \text{if } (d - 1)^4 \geq p.
\end{cases}
\]

The theorem is proved.

Collection our previous estimations from the Theorems 4.2 – 4.5 we get the following theorem.
**Theorem 4.6.** Let \( \alpha, \beta \in \mathbb{Z}[i] \) and let \( h, q, k, n \in \mathbb{N} \), \( k \geq 2 \), \( (k, q) = (h, q) = 1 \). Then for \( (\alpha, q) = (\beta, q) = 1 \) we have

\[
\tilde{K}(\alpha, \beta; h, q; k) \ll D(k, q)q^\frac{3}{2},
\]

where

\[
D(k, q) = \prod_{\substack{p | q \\ p \equiv 1(q)}} d^6(k, p) \prod_{\substack{p^3 || q \\ p \equiv 3(q)}} d^3(k, p) \log p^n,
\]

\( d(k, p) = (k, p - 1) \).

We must note that the norm Kloosterman sum \( \tilde{K}(\alpha, \beta; h, q; k) \) has not an analogue in the ring \( \mathbb{Z} \).


Received 17.02.2006