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Armin Shams (Univ. Manchester, UK; Ferdowsi Univ. Mashhad, Iran)

RE-EXTENDING CHEBYSHEV'S THEOREM ABOUT BERTRAND'S CONJECTURE

ПОВТОРНЕ РОЗШИРЕННЯ ТЕОРЕМИ ЧЕБИШОВА ЩОДО ГІПОТЕЗИ БЕРТРАНА

In this paper, Chebyshev's theorem (1850) about Bertrand's conjecture is re-extended using a theorem about Sierpinski's conjecture (1958). The theorem had been extended before several times, but this extension is a major extension far beyond the previous ones. At the beginning of the proof, maximal gaps table is used to verify initial states. The extended theorem contains a constant, r, which can be reduced if more initial states can be checked. Therefore, the theorem can be even more extended when maximal gaps table is extended. The main extension idea is not based on r, though.

У даній статті теорему Чебишова (1850) щодо гіпотези Бертрана повторно розширено за допомогою теореми щодо гіпотези Серпінського (1958). Раніше теорему розширювали декілька разів, але розглядуване розширення є найголовнішим із попередніх. Доведення починається з використання таблиці максимальних проміжків для перевірки початкових станів. Розширена теорема містить константу r, яка може бути зменшена при можливості перевірки більшої кількості початкових станів. Отже, теорему може бути розширено навіть більше у випадку розширення таблиці максимальних проміжків. Проте основна ідея розширення не базується на r.

1. Introduction. We are going go introduce and prove the following two theorems (mainly the first one):

Theorem 1. *For each real number m not less than 2, there exists at least one prime number p such that*:

$$m < p < m + 2 \times \frac{m}{\log_{re} m}, \quad r = 1.207.$$

Theorem 2. *Each natural number n can be written as sum of distinct prime-based squares and/or base-2 powers of distinct prime numbers or* 1:

$$\forall\, n \in \mathrm{N} \quad \exists\; a_1, a_2, \ldots, a_x,\; b_1, b_2, \ldots, b_y \in \mathrm{P} \bigcup \{1\}:$$

$$\forall\; 1 \le k,\; l \le x \quad a_k \ne a_l, \qquad \forall\; 1 \le k,\; l \le y \quad b_k \ne b_l,$$

$$n = \sum_{i=1}^{x} a_i^2 + \sum_{j=1}^{y} 2^{b_j},$$

P *is the set of prime numbers and* N *is the set of natural numbers.*

In our proofs, we somehow deal with one conjecture and two theorems:

Conjecture 1 (Sierpinski, 1958). *If we form an* $N \times N$ $(N \in \mathrm{N} - \{1\})$ *square using consecutive natural numbers from* 1 *to* N^2, *then there exists at least one prime number in each row of this square* (*the square is called Sierpinski's square*) [1, *p.* 732].

Theorem 3. *For each* $n > e^k$ $(n, k \in \mathrm{N})$, *there exists at least one prime number in each first* k *row of Sierpinski's square* [1, p. 732].

Theorem 4 (Chebyshev, 1850). *For each natural number* n, *there exists at least one prime number* p *such that:*

$$n < p < 2n \quad (\text{Erdos (1932): } n < p < 2n - 2,\ n > 3) \ [1, \text{p. } 791].$$

Note that this theorem was extended at least three times after 1932; but extensions were made such that they were true for numbers upper that a specific non-small number:

$$\forall\, n \in \mathrm{N}\ \exists\, p\colon\ n < p < \left(1 + \frac{1}{5}\right)n \ \text{ for } n \geq P_{10} \ \ (\text{Nagura, 1952}) \ [2],$$

$$\forall\, n \in \mathrm{N}\ \exists\, p\colon\ n < p < \left(1 + \frac{1}{13}\right)n \ \text{ for } n \geq P_{119} \ \ (\text{Rohrbach \& Weis, 1964}) \ [2],$$

$$\forall\, n \in \mathrm{N}\ \exists\, p\colon\ n < p < \left(1 + \frac{1}{16597}\right)n \ \text{ for } n \geq P_{2010761} \ \ (\text{Schoenfeld, 1976}) \ [2].$$

Above theorems are better than our suggested re-extended theorem for some limited initial states, but after those states, they are weaker than ours for all other unlimited remaining states.

2. Extending the Theorem 4. The Theorem 4 can be extended as follows:

For first 1693182318746371 numbers computer can check the theorem, but it takes a very long time if an ordinary computer is used. If we consider the table of maximal gaps between primes not greater than 1693182318746371 [3], obtained by super computers, then we can soon realize that the theorem is acceptable for initial numbers up to 1693182318746371.

If we assume $r = 1.207$ and $m \geq 1693182318746371$ (m is a positive integer), then we can write $m \geq (re)^{29}$ and thus $\log_{re} m \geq 29$. On the other hand, first derivative of $1.207^x - 8 \times x$ is positive for $x \geq 29$. Thus $1.207^x - 8 \times x$ is an increasing function for $x \geq 29$ and we know that this function is positive for $x = 29$. Therefore $1.207^x - 8 \times x > 0$, for $x \geq 29$, that concludes $8 \times x < 1.207^x$, $x \geq 29$. We can substitute 1.207 by r, and x by $\log_{re} m$. We will have

$$\log_{re} m \times 8 \ < \ r^{\log_{re} m}. \tag{1}$$

Reversing both sides and multiplying them by m we can claim that

$$\frac{m}{r^{\log_{re} m}} \ < \ \frac{m}{\log_{re}(m) \times 8}. \tag{2}$$

If we replace m with $(re)^{\log_{re} m}$ in left side, we will have

$$\frac{(re)^{\log_{re} m}}{r^{\log_{re} m}} \ < \ \frac{m}{\log_{re}(m) \times 8}. \tag{3}$$

If we take natural logarithm from both sides, we will have

$$\log_{re} m < \ln\left(\frac{m}{\log_{re} m}\right) - 2.07. \tag{4}$$

So we can write

$$m < \frac{m}{\log_{re} m}\left(\ln\left(\frac{m}{\log_{re} m}\right) - 2.07\right). \tag{5}$$

According to $m \geq 1693182318746371$, we can write $\ln\left(\frac{m}{\log_{re} m}\right) - \ln\left[\frac{m}{\log_{re} m}\right] <$ < 0.01 $\Big($to prove this we can raise e to the power of both sides and rewrite the sides according to $x = \frac{m}{\log_{re} m} > e^{30}$ and $[x] \geq x - 1$ and $\frac{x}{x-1} = 1 + \frac{1}{x-1}$ and $e^{0.01} >$ $> 1.005\Big)$.

And then we can write

$$m < \frac{m}{\log_{re} m}\left(\ln\left(\frac{m}{\log_{re} m}\right) - 2.07 + 0.01 - \left(\ln\left(\frac{m}{\log_{re} m}\right) - \ln\left[\frac{m}{\log_{re} m}\right]\right)\right). \tag{6}$$

Obviously

$$m < \frac{m}{\log_{re} m}\left(\ln\left(\frac{m}{\log_{re} m}\right) - 2.06\right). \tag{7}$$

If $x \geq e^n$, $n \geq 1$, then $\ln x < \frac{x}{n}$, therefore according to $m \geq 1693182318746371$, which means $\frac{m}{\log_{re} m} > e^{30}$ we can write $\ln\frac{m}{\log_{re} m} < 0.06 \times \frac{m}{\log_{re} m}$; therefore $\ln\left[\frac{m}{\log_{re} m}\right] - 2 < 0.06 \times \frac{m}{\log_{re} m}$; so

$$m < \frac{m}{\log_{re} m}\left(\ln\left[\frac{m}{\log_{re} m}\right] - 2.06\right) + 0.06 \times \frac{m}{\log_{re} m} - \left(\ln\left[\frac{m}{\log_{re} m}\right] - 2\right). \tag{8}$$

We will have

$$m < \left(\frac{m}{\log_{re} m} - 1\right)\left(\ln\left[\frac{m}{\log_{re} m}\right] - 2\right). \tag{9}$$

We can write

$$m < \left[\frac{m}{\log_{re} m}\right]\left(\ln\left[\frac{m}{\log_{re} m}\right] - 2\right). \tag{10}$$

Assuming $n = \frac{m}{\log_{re} m}$ we conclude $m < n(\ln n - 2)$. According to $[\ln n] \geq \ln n - 1$ we will have $m < n([\ln n] - 1)$. Using the Theorem 3 we can write $\exists\, k:\ k = [\ln n]$.

So we conclude

$$m < n(k-1). \tag{11}$$

Using the found n, m is somewhere in the first $k-1$ rows of the Sierpinski's square where there is at least one prime number in each row. The worst case takes place when m is prime and the next prime is in the last column of the next row. So we can write

$$\forall\, m \in \mathrm{N} \quad \exists\, p:\ m < p < m + 2n, \quad m \geq 2. \tag{12}$$

So

$$\forall\, m \in \mathrm{R} \quad \exists\, p:\ [m] < p < [m] + 2 \times \frac{[m]}{\log_{re}[m]}, \quad m \geq 2, \tag{13}$$

p is natural, $m - [m]$ is less than 1 and by taking derivative we can prove that $\dfrac{m}{\log_{re} m}$ and thus $f(m) = m + 2 \times \dfrac{m}{\log_{re} m}$ are increasing functions for $m \geq re$, so, because $m \geq [m]$, $f(m)$ is not less than $f([m]) = [m] + 2 \times \dfrac{[m]}{\log_{re}[m]}$, $m \geq re$. Therefore we have

$$\forall\, m \in \mathrm{R} \quad \exists\, p:\ [m] + (m - [m]) < p < m + 2 \times \frac{m}{\log_{re} m}, \quad m \geq re. \tag{14}$$

So

$$\forall\, m \in \mathrm{R} \quad \exists\, p:\ m < p < m + 2 \times \frac{m}{\log_{re} m}, \quad m \geq re. \tag{15}$$

Therefore, based on $2 \times \dfrac{m}{\log_{re} m} > 2$ if $m > 1$, the theorem can be extended as:

The extended theorem. *For each real number* $m > 1$, *there exists at least one prime number* p *such that*:

$$m < p < m + 2 \times \frac{m}{\log_{re} m}, \quad r = 1.207.$$

Important note. We can reduce r if we satisfy part (1) with a large m. According to the computational power of the computers which have been used to produce the maximal gaps table (in Appendix), it is provable that $r = 1.207$ (or "r" greater than 1.207 that we do not need).

3. The new theorem (Theorem 2, provable using the extended theorem). We will use following lemma to prove Theorem 2:

Lemma. *For each* $n \geq 121$ *there exists* p *such that* $\dfrac{n}{2} < p^2 < n$.

Proof. It is sufficient to check a few initial states, and then use the Nagura theorem (page 2) to obtain $\forall\, m \geq 11 \ \exists\, p:\ m < p < \sqrt{2}\, m$. For each natural n not less than 121, if we assume $m = \sqrt{\dfrac{n}{2}}$, then $\sqrt{\dfrac{n}{2}} \geq 11$ and $\sqrt{\dfrac{n}{2}} \in \mathrm{R}$, therefore there exists p such that $\dfrac{\sqrt{n}}{2} < p < \sqrt{2}\sqrt{\dfrac{n}{2}}$. So: $\dfrac{n}{2} < p^2 < n$, $n \geq 121$.

Theorem 5. *Each natural number* n *can be written as sum of distinct prime-based squares and/or base-*2 *powers of distinct prime numbers or* 1:

$$\forall\, n \in \mathrm{N} \quad \exists\, a_1, a_2, \ldots, a_x,\; b_1, b_2, \ldots, b_y \in \mathrm{P} \cup \{1\}:$$

$$\forall\, 1 \le k,\; l \le x \quad a_k \ne a_l, \qquad \forall\, 1 \le k,\; l \le y \quad b_k \ne b_l,$$

$$n = \sum_{i=1}^{x} a_i^2 + \sum_{j=1}^{y} 2^{b_j}.$$

Besides, it can be shown that b_j *can be kept less than or equal to* 5 *and thus* y *can be kept less than or equal to* 4.

Proof. For n less than 121, the theorem can be easily verified. To use strong induction, it's enough to prove that if the theorem is true for 1 to n, then it is also true for $n+1$. According to the lemma, we can find p such that $\frac{n+1}{2} < p^2 < n+1$. It is sufficient to assume that, p is one of "a_i"s and obtain other "a_i"s from $n+1-p^2$ for which, the truth of theorem is previously proved according to induction assumption. Obviously based on induction assumption, obtained "a_i"s from $n+1-p^2$ are different (as well as "b_i"s) and p is different with all "a_i"s (it follows from the relation $n+1 = p^2 + q$ that $0 \le q \le \frac{n+1}{2} \le p^2$, and hence, the representation $q = \sum_{i=1}^{x} a_i'^2 + \sum_{j=1}^{y} 2^{b_j'}$ contains $a_i' < p$ for every $i = 1, \ldots, x$). Therefore out theorem is true for $n+1$; so, based on strong induction, the theorem is true for all natural n.

4. Conclusions. The extended theorem is a fundamental theorem in prime number theory. Based on this extension, several other extensions can be made to related theorems.

In the extended theorem, if the truth of the theorem is proved for all $m < C$ then the coefficient r can be reduced to approach 1+. C depends on r and is a very large number if r approaches 1. C can be obtained from part (1) of the proof and is the least m that satisfies the "<" sign according to r. The number "8" in part (1) can be a little reduced currently by minor changes to the proof.

For large C (but not very large), the truth of the theorem for all $m < C$ can be checked by super computer. Reduction of r is limited by the computational power of the computer and the efficiency of the algorithm. By using more efficient programs or more powerful computers we can check initial states for a greater C (for all $m < C$) and reduce r even more, approaching $r = 1$.

One important result of the article is that if Sierpinski's conjecture is proved, the theorem can be considerably re-extended by a proof method similar to the one discussed in the article.

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Appendix. In the following table we list the **maximal gaps** through 1131. These are the first occurrences of gaps of at least of this length. For example, there is a gap of 879 composites after the prime 277900416100927. This is the first occurrence of a gap of this length, but still is not a maximal gap since 905 composites follow the smaller prime 218209405436543 [5].

Gap	Following the prime	Reference
0	2	
1	3	
3	7	
5	23	
7	89	
13	113	
17	523	
19	887	
.	.	
.	.	
.	.	
581	1346294310749	
587	1408695493609	
601	1968188556461	
651	2614941710599	
673	7177162611713	
715	13829048559701	[4]
765	19581334192423	[4]
777	42842283925351	[4]
803	90874329411493	[5]
805	171231342420521	[5]
905	21820940543643	[5]
915	1189459969825483	[6]
923	1686994940955803	[6]
1131	1693182318746371	[6]

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