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EXISTENCE PRINCIPLES FOR HIGHER ORDER NONLOCAL BOUNDARY-VALUE PROBLEMS AND THEIR APPLICATIONS TO SINGULAR STURM – LIOUVILLE PROBLEMS*

ПРИНЦИПИ ІСНУВАННЯ ДЛЯ НЕЛОКАЛЬНИХ ГРАНИЧНИХ ЗАДАЧ ВИЩОГО ПОРЯДКУ ТА ЇХ ЗАСТОСУВАННЯ ДО СИНГУЛЯРНИХ ЗАДАЧ ШТУРМА – ЛІУВІЛЛЯ

The paper presents existence principles for the nonlocal boundary-value problem $(\phi(u^{(p-1)}))' = g(t, u, \ldots, u^{(p-1)})$, $\alpha_k(u) = 0$, $1 \le k \le p-1$, where $p \ge 2$, $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism, g is a Carathéodory function which is either regular or has singularities in its space variables and $\alpha_k\colon C^{p-1}[0,T] \to \mathbb{R}$ is a continuous functional. An application of the existence principles to singular Sturm – Liouville problems $(-1)^n(\phi(u^{(2n-1)}))' = f(t, u, \ldots, u^{(2n-1)})$, $u^{(2k)}(0) = 0$, $a_k u^{(2k)}(T) + b_k u^{(2k+1)}(T) = 0$, $0 \le k \le n-1$, is given.

Наведено принципи існування для нелокальної граничної задачі $(\phi(u^{(p-1)}))' = g(t, u, \ldots, u^{(p-1)})$, $\alpha_k(u) = 0$, $1 \le k \le p-1$, де $p \ge 2$, $\phi\colon \mathbb{R} \to \mathbb{R}$ — гомеоморфізм, що зростає і є непарним, g — функція Каратеодорі, що або є регулярною, або має особливості за своїми просторовими змінними, а $\alpha_k\colon C^{p-1}[0,T] \to \mathbb{R}$ — неперервний функціонал. Показано застосування принципів існування до сингулярних задач Штурма – Ліувілля $(-1)^n(\phi(u^{(2n-1)}))' = f(t, u, \ldots, u^{(2n-1)})$, $u^{(2k)}(0) = 0$, $a_k u^{(2k)}(T) + b_k u^{(2k+1)}(T) = 0$, $0 \le k \le n-1$.

1. Introduction. Let $T > 0$ and let $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. As usual, $C^j[0,T]$ denotes the set of functions having the jth derivative continuous on $[0,T]$. $AC[0,T]$ and $L_1[0,T]$ is the set of absolutely continuous functions on $[0,T]$ and Lebesgue integrable functions on $[0,T]$, respectively. $C^0[0,T]$ and $L_1[0,T]$ is equipped with the norm

$$\|x\| = \max\big\{|x(t)|\colon t \in [0,T]\big\} \quad \text{and} \quad \|x\|_L = \int_0^T |x(t)|\,dt,$$

respectively.

Assume that $G \subset \mathbb{R}^p$, $p \ge 2$. $\operatorname{Car}\big([0,T] \times G\big)$ stands for the set of functions $f\colon [0,T] \times G \to \mathbb{R}$ satisfying the local Caratéodory conditions on $[0,T] \times G$, that is: (i) for each $(x_0, \ldots, x_{p-1}) \in G$, the function $f(\cdot, x_0, \ldots, x_{p-1})\colon [0,T] \to \mathbb{R}$ is measurable; (ii) for a.e. $t \in [0,T]$, the function $f(t, \cdot, \ldots, \cdot)\colon G \to \mathbb{R}$ is continuous; (iii) for each compact set $K \subset G$, $\sup\{|f(t, x_0, \ldots, x_{p-1})|\colon (x_0, \ldots, x_{p-1}) \in K\} \in L_1[0,T]$.

Let $p \in \mathbb{N}$, $p \ge 2$. Denote by $\mathcal{A}$ the set of functionals $\alpha\colon C^{p-1}[0,T] \to \mathbb{R}$ which are

(a) continuous and

(b) bounded, that is, $\alpha(\Omega)$ is bounded for any bounded $\Omega \subset C^{p-1}[0,T]$.

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Let $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism and let either $g \in \mathrm{Car}([0,T] \times \times \mathbb{R}^p)$ or $g \in \mathrm{Car}([0,T] \times \mathcal{D}_*)$, $\mathcal{D}_* \subset \mathbb{R}^p$ and has singularities only at the value 0 of its space variables. Consider the nonlocal boundary-value problem

$$\big(\phi(u^{(p-1)})\big)' = g(t, u, \ldots, u^{(p-1)}), \tag{1.1}$$

$$\alpha_k(u) = 0, \qquad \alpha_k \in \mathcal{A}, \quad 0 \le k \le p-1, \tag{1.2}$$

where α_k satisfy *a compatibility condition* that for each $\mu \in [0,1]$ there exists a solution of the problem

$$(\phi(u^{(p-1)}))' = 0, \qquad \alpha_k(u) - \mu\alpha_k(-u) = 0, \quad 0 \le k \le p-1.$$

This problem is equivalent to the fact that the system

$$\alpha_k\left(\sum_{i=0}^{p-1} A_i t^i\right) - \mu\alpha_k\left(-\sum_{i=0}^{p-1} A_i t^i\right) = 0, \quad 0 \le k \le p-1, \tag{1.3}$$

has a solution $(A_0, \ldots, A_{p-1}) \in \mathbb{R}^p$ for each $\mu \in [0,1]$.

We say that $u \in C^{p-1}[0,T]$ is a *solution of problem* (1.1), (1.2) if $\phi(u^{(p-1)}) \in \in AC[0,T]$, u satisfies (1.2) and fulfils $\big(\phi(u^{(p-1)}(t))\big)' = g\big(t, u(t), \ldots, u^{(p-1)}(t)\big)$ for a.e. $t \in [0,T]$.

The aim of this paper is

1) to present existence principles for problem (1.1), (1.2) in a regular and a singular case and

2) to give an application of these existence principles to singular Sturm – Liouville boundary-value problems.

Notice that our existence principles stand a generalization of those obtained for second-order differential equations with ϕ-Laplacian in [1, 2].

Our Sturm – Liouville problem consisting of the differential equation

$$(-1)^n\big(\phi(u^{(2n-1)})\big)' = f(t, u, \ldots, u^{(2n-1)}) \tag{1.4}$$

and the boundary conditions

$$u^{(2k)}(0) = 0, \qquad a_k u^{(2k)}(T) + b_k u^{(2k+1)}(T) = 0, \quad 0 \le k \le n-1. \tag{1.5}$$

Here $n \ge 2$, $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $f \in \mathrm{Car}([0,T] \times \mathcal{D})$ is positive where

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \ldots \times \mathbb{R}_+ \times \mathbb{R}_0}_{4\ell-2} & \text{if} \quad n = 2\ell - 1, \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \ldots \times \mathbb{R}_- \times \mathbb{R}_0}_{4\ell} & \text{if} \quad n = 2\ell, \end{cases}$$

f may be singular at the value 0 of all its space variables and

$$a_k > 0, \qquad b_k > 0, \qquad a_k T + b_k = 1 \quad \text{for} \quad 0 \le k \le n-1. \tag{1.6}$$

We say that a function $u \in C^{2n-1}[0,T]$ is a *solution of problem* (1.4), (1.5) if $\phi(u^{(2n-1)}) \in AC[0,T]$, u satisfies the boundary conditions (1.5) and fulfils the equality $(-1)^n\big(\phi(u^{(2n-1)}(t))\big)' = f\big(t, u(t), \ldots, u^{(2n-1)}(t)\big)$ for a.e. $t \in [0,T]$.

Singular problems of the Sturm – Liouville type for higher order differential equations were considered in [3 – 5]. In [3] the authors discuss the differential equation $u^{(n)} + h_1(t, u, \ldots, u^{(n-2)}) = 0$ together with the boundary conditions

$$\begin{gathered} u^{(j)}(0) = 0, \quad 0 \le j \le n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \qquad \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{gathered} \tag{1.7}$$

where $\alpha\gamma + \alpha\delta + \beta\gamma > 0$, $\beta,\ \delta \ge 0$, $\beta + \alpha > 0$, $\delta + \gamma > 0$ and $h_1 \in C^0\big((0,1) \times \mathbb{R}_+^{n-1}\big)$ is positive. The existence of a positive solution $u \in C^{n-1}[0,1] \cap C^n(0,1)$ is proved by a fixed point theorem for mappings that are decreasing with respect to a cone in a Banach space. Paper [4] deals with the problem $u^{(n)} + h_2(t, u, \ldots, u^{(n-1)}) = 0$, (1.7), where $h_2 \in \mathrm{Car}\,\big([0,T] \times \mathcal{D}_*\big)$, $\mathcal{D}_* = \mathbb{R}_+^{n-1} \times \mathbb{R}_0$, is positive. The existence of a positive solution $u \in AC^{n-1}[0,T]$ is proved by a combination of regularization and sequential techniques with a Fredholm type existence theorem. In [5], by constructing some special cones and using a Krasnoselskii fixed point on a cone, the existence of a positive solution $u \in C^{4n-2}[0,1] \cap C^{4n}(0,1)$ is proved for problem $u^{(4n)} = h_3(t, u, u^{(4n-2)})$, $u(0) = u(1) = 0$, $au^{(2k)}(0) - bu^{(2k+1)}(0) = 0$, $cu^{(2k)}(1) + du^{(2k+1)}(1) = 0$, $1 \le k \le 2n-1$. Here $h_3 \in C\big([0,1] \times \mathbb{R}_+ \times \mathbb{R}_-\big)$ is nonnegative, a, b, c, d are nonnegative constants and $ac + ad + bc > 0$.

To the best our knowledge, there is no paper considering singular problems of the Sturm – Liouville type in our generalization (1.4), (1.5). In addition, any solution u of problem (1.4), (1.5) has the maximal smoothness, u and its even derivatives ($\le 2n-2$) ‘start’ at the singular points of f and its odd derivatives ($\le 2n-1$) ‘go throughout’ singularities of f somewhere inside of $[0,T]$.

Throughout the paper we work with the following conditions on the functions ϕ and f in equation (1.4):

(H_1) $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$,

(H_2) $f \in \mathrm{Car}([0,T] \times \mathcal{D})$ and there exists $a > 0$ such that

$$a \le f(t, x_0, \ldots, x_{2n-1})$$

for a.e. $t \in [0,T]$ and all $(x_0, \ldots, x_{2n-1}) \in \mathcal{D}$,

(H_3) $f(t, x_0, \ldots, x_{2n-1}) \le h\Big(t, \sum_{j=0}^{2n-1} |x_j|\Big) + \sum_{j=0}^{2n-1} \omega_j(|x_j|)$ for a.e. $t \in [0,T]$ and all $(x_0, \ldots, x_{2n-1}) \in \mathcal{D}$, where $h \in \mathrm{Car}([0,T] \times [0,\infty))$ is positive and nondecreasing in the second variable, $\omega_j\colon \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing,

$$\limsup_{v\to\infty} \frac{1}{\phi(v)} \int_0^T h(t, 2n + Kv)\, dt < 1 \tag{1.8}$$

with

$$K = \begin{cases} 2n & \text{if} \quad T = 1, \\ \dfrac{T^{2n} - 1}{T - 1} & \text{if} \quad T \neq 1, \end{cases} \tag{1.9}$$

and

$$\int_0^1 \omega_{2n-1}(\phi^{-1}(s))\,ds < \infty, \qquad \int_0^1 \omega_{2j}(s)\,ds < \infty \quad \text{for} \quad 0 \le j \le n-1,$$

$$\int_0^1 \omega_{2j+1}(s^2)\,ds < \infty \quad \text{for} \quad 0 \le j \le n-2.$$

***Remark* 1.1.** If ϕ satisfies (H_1) then $\phi(0) = 0$. Under assumption (H_3) the functions $\omega_{2n-1}(\phi^{-1}(s))$, $\omega_{2j}(s)$, $0 \le j \le n-1$, and $\omega_{2i+1}(s^2)$, $0 \le i \le n-2$, are locally Lebesgue integrable on $[0, \infty)$ since ω_k, $0 \le k \le 2n-1$, is nonincreasing and positive on $\mathbb{R}_+$.

The rest of the paper is organized as follows. In Section 2, we present existence principles for a regular and a singular problem (1.1), (1.2). The regular existence principle is proved by the Leray – Schauder degree (see, e.g., [6]). An application of both principles is given in Section 3 to the Sturm – Liouville problem (1.4), (1.5).

2. Existence principles. The following result states conditions for solvability of problem (1.1), (1.2) where g in equation (1.1) is regular.

Theorem 2.1. *Let (H_1) hold. Let $g \in \operatorname{Car}([0,T] \times \mathbb{R}^p)$ and $\varphi \in L_1[0,T]$. Suppose that there exists a positive constant L independent of λ such that*

$$\|u^{(j)}\| < L, \quad 0 \le j \le p-1,$$

for all solutions u of the differential equations

$$(\phi(u^{(p-1)}))' = (1 - \lambda)\varphi(t), \quad \lambda \in [0,1], \tag{2.1}$$

$$(\phi(u^{(p-1)}))' = \lambda g(t, u, \ldots, u^{(p-1)}) + (1 - \lambda)\varphi(t), \quad \lambda \in [0,1], \tag{2.2}$$

satisfying the boundary conditions (1.2). *Also assume that there exists a positive constant Λ such that*

$$|A_j| < \Lambda, \quad 0 \le j \le p-1, \tag{2.3}$$

for all solutions $(A_0, \ldots, A_{p-1}) \in \mathbb{R}^p$ of system (1.3) *with $\mu \in [0,1]$.*

Then problem (1.1), (1.2) *has a solution $u \in C^{p-1}[0,T]$, $\phi(u^{(p-1)}) \in AC[0,T]$.*

Proof. Let

$$\Omega = \Big\{ x \in C^{p-1}[0,T] \colon \|x^{(j)}\| < \max\{L, \Lambda K_1\} \ \text{ for } \ 0 \le j \le p-1 \Big\},$$

where

$$K_1 = \begin{cases} p & \text{if } T = 1, \\ \dfrac{T^p - 1}{T - 1} & \text{if } T \neq 1. \end{cases}$$

Then Ω is an open and symmetric with respect to $0 \in C^{p-1}[0,T]$ subset of the Banach space $C^{p-1}[0,T]$. Define an operator $\mathcal{P}\colon [0,1] \times \overline{\Omega} \to C^{p-1}[0,T]$ by the formula

$$\mathcal{P}(\rho, x)(t) = \int\limits_0^t \frac{(t-s)^{p-2}}{(p-2)!} \phi^{-1} \left(\phi(x^{(p-1)}(0) + \alpha_{p-1}(x)) + \int\limits_0^s V(\rho, x)(v)\, dv \right) ds +$$

$$+ \sum_{j=0}^{p-2} \frac{x^{(j)}(0) + \alpha_j(x)}{j!} t^j \tag{2.4}$$

where $V(\rho, x)(t) = \rho g(t, x(t), \ldots, x^{(p-1)}(t)) + (1-\rho)\varphi(t)$. It follows from the continuity of ϕ and α_j, $0 \le j \le p-1$, $g \in \mathrm{Car}([0,T] \times \mathbb{R}^p)$ and from the Lebesgue dominated convergence theorem that $\mathcal{P}$ is a continuous operator. We now prove that $\mathcal{P}\big([0,T] \times \overline{\Omega}\big)$ is relatively compact in $C^{p-1}[0,T]$. Notice that the boundedness of $\overline{\Omega}$ in $C^{p-1}[0,T]$ guarantees the existence of a positive constant r and a $\psi \in L_1[0,T]$ such that $|\alpha_k(x)| \le r$ and $\big|g(t, x(t), \ldots, x^{(p-1)}(t))\big| \le \psi(t)$ for a.e. $t \in [0,T]$ and all $x \in \overline{\Omega}$, $0 \le k \le p-1$. Then

$$\big|(\mathcal{P}(\rho, x))^{(j)}(t)\big| \le \big(r + \max\{L, \Lambda K_1\}\big) \sum_{i=0}^{p-j-2} \frac{T^i}{i!} +$$

$$+ \frac{T^{p-j-1}}{(p-j-2)!} \phi^{-1}\big(\phi(r + \max\{L, \Lambda K_1\}\big) + \|\psi\|_L + \|\varphi\|_L\big),$$

$$\big|(\mathcal{P}(\rho, x))^{(p-1)}(t)\big| \le \phi^{-1}\big(\phi\big(r + \max\{L, \Lambda K_1\}\big) + \|\psi\|_L + \|\varphi\|_L\big),$$

$$\Big|\phi((\mathcal{P}(\rho, x))^{(p-1)}(t_2)) - \phi((\mathcal{P}(\rho, x))^{(p-1)}(t_1))\Big| \le \left| \int\limits_{t_1}^{t_2} (\psi(s) + |\varphi(s)|)\, ds \right|$$

for t, t_1, $t_2 \in [0,T]$, $(\rho, x) \in [0,T] \times \overline{\Omega}$ and $0 \le j \le n-2$. Hence $\mathcal{P}\big([0,T] \times \times \overline{\Omega}\big)$ is bounded in $C^{p-1}[0,T]$ and the set $\{\phi((\mathcal{P}(\rho, x))^{(p-1)})\colon (\rho, x) \in [0,1] \times \overline{\Omega}\}$ is equicontinuous on $[0,T]$. Since $\phi\colon \mathbb{R} \to \mathbb{R}$ is increasing and continuous, the set $\Big\{(\mathcal{P}(\rho, x))^{(p-1)}\colon (\rho, x) \in [0,1] \times \overline{\Omega}\Big\}$ is equicontinuous on $[0,T]$ too. Now, by the Arzelà – Ascoli theorem, $\mathcal{P}([0,1] \times \overline{\Omega})$ is relatively compact in $C^{p-1}[0,T]$. We have proved that $\mathcal{P}$ is a compact operator.

Suppose that x_* is a fixed point of the operator $\mathcal{P}(1, \cdot)$. Then

$$x_*(t) = \sum_{j=0}^{p-2} \frac{x_*^{(j)}(0) + \alpha_j(x_*)}{j!} t^j + \int\limits_0^t \frac{(t-s)^{p-2}}{(p-2)!} \phi^{-1} \times$$

$$\times \left(\phi(x_*^{(p-1)}(0) + \alpha_{p-1}(x_*)) + \int\limits_0^s g(v, x_*(v), \ldots, x_*^{(p-1)}(v)) dv \right) ds$$

for $t \in [0,T]$. Hence $\alpha_k(x_*) = 0$ for $0 \le k \le p-1$ and x_* is a solution of equation (1.1). Consequently, x_* is a solution of problem (1.1), (1.2). In order to prove the assertion of our theorem it suffices to show that

$$\deg(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega, 0) \neq 0 \tag{2.5}$$

where "deg" stands for the Leray – Schauder degree and $\mathcal{I}$ is the identical operator on $C^{p-1}[0,T]$. To show this let the compact operator $\mathcal{K}\colon [0,2]\times\overline{\Omega} \to C^{p-1}[0,T]$ be defied by

$$\mathcal{K}(\mu,x)(t) = \begin{cases} \sum\limits_{j=0}^{p-1}\Big[x^{(j)}(0) + \alpha_{j+1}(x) - (1-\mu)\alpha_j(-x)\Big]\dfrac{t^j}{j!} & \text{if } \mu \in [0,1], \\ \int\limits_0^t \dfrac{(t-s)^{p-2}}{(p-2)!}\phi^{-1}\Bigg(\phi(x^{(p-1)}(0) + \alpha_{p-1}(x)) + \\ \qquad +(\mu-1)\int\limits_0^s \varphi(v)\,dv\Bigg)ds + \sum\limits_{j=0}^{p-2}\dfrac{x^{(j)}(0)+\alpha_j(x)}{j!}t^j & \text{if } \mu \in (1,2]. \end{cases}$$

Then $\mathcal{K}(0,\cdot)$ is odd (that is $\mathcal{K}(0,-x) = -\mathcal{K}(0,x)$ for $x \in \overline{\Omega}$) and

$$\mathcal{K}(2,x) = \mathcal{P}(0,x) \quad \text{for} \quad x \in \overline{\Omega}. \tag{2.6}$$

Assume that $\mathcal{K}(\mu_0,u_0) = u_0$ for some $(\mu_0,u_0) \in [0,1]\times\overline{\Omega}$. Then

$$u_0(t) = \sum_{j=0}^{p-1}\Big[u_0^{(j)}(0) + \alpha_j(u_0) - (1-\mu_0)\alpha_j(-u_0)\Big]\frac{t^j}{j!}, \quad t \in [0,T],$$

and therefore $u_0(t) = \sum_{j=0}^{p-1}\tilde{A}_j\dfrac{t^j}{j!}$ where $\tilde{A}_j = u_0^{(j)}(0) + \alpha_j(u_0) - (1-\mu_0)\alpha_j(-u_0)$. Consequently, $u_0^{(j)}(0) = \tilde{A}_j$ and so $\alpha_j(u_0) - (1-\mu_0)\alpha_j(-u_0) = 0$ for $0 \le j \le p-1$, which means

$$\alpha_k\left(\sum_{j=0}^{p-1}\tilde{A}_j\frac{t^j}{j!}\right) - (1-\mu_0)\alpha_k\left(-\sum_{j=0}^{p-1}\tilde{A}_j\frac{t^j}{j!}\right) = 0, \quad 0 \le k \le p-1.$$

Then, by our assumption, $\left|\dfrac{\tilde{A}_j}{j!}\right| < \Lambda$ for $0 \le j \le p-1$ and we have

$$\big\|u_0^{(j)}\big\| < \Lambda\sum_{j=0}^{p-1}T^j = \Lambda K_1, \quad 0 \le j \le p-1.$$

Hence $u_0 \notin \partial\Omega$ and therefore, by the Borsuk antipodal theorem and the homotopy property,

$$\deg(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) \neq 0 \tag{2.7}$$

and

$$\deg(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) = \deg(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega, 0). \tag{2.8}$$

We come to show that

$$\deg(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega, 0) = \deg(\mathcal{I} - \mathcal{K}(2,\cdot), \Omega, 0). \tag{2.9}$$

If $\mathcal{K}(\mu_1, u_1) = u_1$ for some $(\mu_1, u_1) \in (1,2] \times \overline{\Omega}$ then

$$u_1(t) = \sum_{j=0}^{p-2} \frac{u_1^{(j)}(0) + \alpha_j(u_1)}{j!} t^j +$$

$$+ \int_0^t \frac{(t-s)^{p-2}}{(p-2)!} \phi^{-1} \left(\phi(u_1^{(p-1)}(0) + \alpha_{p-1}(u_1)) + (\mu_1 - 1) \int_0^s \varphi(v)\, dv \right) ds$$

for $t \in [0,T]$. Hence u_1 satisfies the boundary conditions (1.2) and u_1 is a solution of the differential equation (2.1) with $\lambda = 2 - \mu_1 \in [0,1)$. By our assumptions, $\|u_1^{(j)}\| < L$ for $0 \le j \le p-1$. Therefore $u_1 \notin \partial\Omega$ and equality (2.9) follows from the homotopy property. Finally, suppose that $\mathcal{P}(\tilde{\rho}, \tilde{u}) = \tilde{u}$ for some $(\tilde{\rho}, \tilde{u}) \in [0,1] \times \overline{\Omega}$. Then $\tilde{u}$ is a solution of problem (2.2), (1.2) with $\lambda = \tilde{\rho}$ and therefore $\|\tilde{u}^{(j)}\| < L$ for $0 \le j \le p-1$. Hence $\tilde{u} \notin \partial\Omega$ and, by the homotopy property, $\deg(\mathcal{I} - \mathcal{P}(0,\cdot), \Omega, 0) = \deg(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega, 0)$. From this and from (2.6) – (2.9) it follows that (2.5) holds, which completes the proof.

***Remark* 2.1.** If functional $\alpha_k \in \mathcal{A}$ is linear for $0 \le k \le p-1$ then system (1.3) has the form

$$\sum_{j=0}^{p-1} A_j \alpha_k(t^j) = 0, \quad 0 \le k \le p-1.$$

All of its solutions $(A_0, \ldots, A_{p-1}) \in \mathbb{R}^p$ are bounded exactly if $\det(\alpha_k(t^j))_{k,j=0}^{p-1} \neq 0$ (and then $A_j = 0$ for $0 \le j \le p-1$), which is equivalent to the fact that problem $\big(\phi(u^{(p-1)})\big)' = 0$, (1.2) has only the trivial solution.

If the function $g \in \mathrm{Car}([0,T] \times \mathcal{D}_*)$, $\mathcal{D}_* \subset \mathbb{R}^p$ in equation (1.1) has singularities only at the value 0 of its space variables, then the following result for the solvability of problem (1.1), (1.2) holds.

Theorem 2.2. *Let condition* (H_1) *hold. Let* $g \in \mathrm{Car}([0,T] \times \mathcal{D}_*)$, $\mathcal{D}_* \subset \mathbb{R}^p$, *have singularities only at the value* 0 *of its space variables. Let the function* $g_m \in \mathrm{Car}\big([0,T] \times \mathbb{R}^p\big)$ *in the differential equation*

$$\big(\phi(u^{(p-1)})\big)' = g_m(t, u, \ldots, u^{(p-1)}) \tag{2.10}$$

satisfy

$$\begin{cases} 0 \le \nu g_m(t, x_0, \ldots, x_{p-1}) \le q\big(t, |x_0|, \ldots, |x_{p-1}|\big) \\ \textit{for a.e.} \quad t \in [0,T] \quad \textit{and all} \quad (x_0, \ldots, x_{p-1}) \in \mathbb{R}_0^p, \quad m \in \mathbb{N}, \\ \textit{where} \quad q \in \mathrm{Car}([0,T] \times \mathbb{R}_+^p) \quad \textit{and} \quad \nu \in \{-1, 1\}. \end{cases} \tag{2.11}$$

Suppose that for each $m \in \mathbb{N}$, *the regular problem* (2.10), (1.2) *has a solution* u_m *and there exists a subsequence* $\{u_{k_m}\}$ *of* $\{u_m\}$ *converging in* $C^{p-1}[0,T]$ *to some* u.

Then $\phi(u^{(p-1)}) \in AC[0,T]$ *and* u *is a solution of the singular problem* (1.1), (1.2) *if* $u^{(j)}$ *has a finite number of zeros for* $0 \le j \le p-1$ *and*

$$\lim_{m\to\infty} g_{k_m}\big(t, u_{k_m}(t), \ldots, u_{k_m}^{(p-1)}(t)\big) = g\big(t, u(t), \ldots, u^{(p-1)}(t)\big) \tag{2.12}$$

for a.e. $t \in [0,T]$.

Proof. Assume that (2.12) holds for a.e. $t \in [0,T]$ and let $0 \le \xi_1 < \ldots < \xi_\ell \le T$ are all zeros of $u^{(j)}$ for $0 \le j \le p-1$. Since $\|u_{k_m}^{(j)}\| \le L$ for each $m \in \mathbb{N}$ and $0 \le j \le p-1$, where L is a positive constant, it follows that

$$\int_0^T \nu g_{k_m}\big(t, u_{k_m}(t), \ldots, u_{k_m}^{(p-1)}(t)\big)\,dt = \nu\Big[\phi\big(u_{k_m}^{(p-1)}(T)\big) - \phi\big(u_{k_m}^{(p-1)}(0)\big)\Big] \le 2\phi(L)$$

for $m \in \mathbb{N}$. Now (2.11), (2.12) and the Fatou lemma [7, 8] give

$$\int_0^T \nu g(t, u(t), \ldots, u^{(p-1)}(t))\,dt \le 2\phi(L).$$

Hence $\nu g\big(t, u(t), \ldots, u^{(p-1)}(t)\big) \in L_1[0,T]$ and so $g\big(t, u(t), \ldots, u^{(p-1)}(t)\big) \in L_1[0,T]$. Put $\xi_0 = 0$ and $\xi_{\ell+1} = T$. We show that the equality

$$\phi(u^{(p-1)}(t)) = \phi\left(u^{(p-1)}\left(\frac{\xi_{i+1}+\xi_i}{2}\right)\right) + \int_{(\xi_{i+1}+\xi_i)/2}^{t} g(s, u(s), \ldots, u^{(p-1)}(s))\,ds \tag{2.13}$$

is satisfied on $[\xi_i, \xi_{i+1}]$ for each $i \in \{0, \ldots, \ell\}$ such that $\xi_i < \xi_{i+1}$. Indeed, let $i \in \in \{0, \ldots, \ell\}$, $\xi_i < \xi_{i+1}$. Choose an arbitrary $\rho \in \left(0, \dfrac{\xi_{i+1}+\xi_i}{2}\right)$ and let us look at the interval $[\xi_i + \rho, \xi_{i+1} - \rho]$. We know that $|u^{(j)}| > 0$ on (ξ_i, ξ_{i+1}) for $0 \le j \le p-1$ and therefore $|u^{(j)}(t)| \ge \varepsilon$ for $t \in [\xi_i + \rho, \xi_{i+1} - \rho]$ and $0 \le j \le p-1$ where ε is a positive constant. Hence there exists $m_0 \in \mathbb{N}$ such that $\big|u_{k_m}^{(j)}(t)\big| \ge \dfrac{\varepsilon}{2}$ for $t \in [\xi_i + \rho, \xi_{i+1} - \rho]$, $0 \le j \le p-1$ and $m \ge m_0$. This gives (see (2.11))

$$\big|g_{k_m}(t, u_{k_m}(t), \ldots, u_{k_m}^{(p-1)}(t))\big| \le$$

$$\le \sup\left\{q(t, x_0, \ldots, x_{p-1})\colon t \in [0,T],\ \ x_j \in \left[\frac{\varepsilon}{2}, L\right]\ \text{ for }\ 0 \le j \le p-1\right\} \in L_1[0,T]$$

for a.e. $t \in [\xi_i + \rho, \xi_{i+1} - \rho]$ and all $m \ge m_0$. Letting $m \to \infty$ in

$$\phi\big(u_{k_m}^{(p-1)}(t)\big) = \phi\left(u_{k_m}^{(p-1)}\left(\frac{\xi_{i+1}+\xi_i}{2}\right)\right) +$$

$$+ \int_{(\xi_{i+1}+\xi_i)/2}^{t} g_{k_m}\big(s, u_{k_m}(s), \ldots, u_{k_m}^{(p-1)}(s)\big)\,ds$$

yields (2.13) for $t \in [\xi_i + \rho,\ \xi_{i+1} + \rho]$ by the Lebesgue dominated convergence theorem. Since $\rho \in \left(0, \dfrac{\xi_{i+1} + \xi_i}{2}\right)$ is arbitrary, equality (2.13) holds on the interval (ξ_i, ξ_{i+1}) and using the fact that $g\big(t, u(t), \ldots, u^{(p-1)}(t)\big) \in L_1[0,T]$, (2.13) is satisfied also at $t = \xi_i$ and ξ_{i+1}. From equality (2.13) on $[\xi_i, \xi_{i+1}]$ (for $0 \le i \le \ell$), we deduce that $\phi(u^{(p-1)}) \in AC[0,T]$ and u is a solution of equation (1.1). Finally, it follows from $\alpha_j(u_{k_m}) = 0$ for $0 \le j \le p-1$ and $m \in \mathbb{N}$, and from the continuity of α_j that $\alpha_j(u) = 0$ for $0 \le j \le p-1$. Consequently, u is a solution of problem (1.1), (1.2).

The theorem is proved.

3. Sturm – Liouville problem. ***3.1. Auxiliary results.*** Throughout the next part of this paper we assume that numbers $a_k,\ b_k$ in the boundary conditions (1.5) fulfil condition (1.6). For each $j \in \{0, \ldots, n-2\}$, denote by G_j the Green function of the Sturm – Liouville problem

$$-u'' = 0, \qquad u(0) = 0, \qquad a_j u(T) + b_j u'(T) = 0.$$

Then

$$G_j(t,s) = \begin{cases} s(1 - a_j t) & \text{for } \ 0 \le s \le t \le T, \\ t(1 - a_j s) & \text{for } \ 0 \le t < s \le T. \end{cases}$$

Hence $G_j(t,s) > 0$ for $(t,s) \in (0,T] \times (0,T]$ and $G_j(t,s) = G_j(s,t)$ for $(t,s) \in$ $\in [0,T] \times [0,T]$. Put $G^{[1]}(t,s) = G_{n-2}(t,s)$ for $(t,s) \in [0,T] \times [0,T]$ and define $G^{[j]}$ recurrently by the formula

$$G^{[j]}(t,s) = \int_0^T G_{n-j-1}(t,v) G^{[j-1]}(v,s)\, dv, \quad (t,s) \in [0,T] \times [0,T], \tag{3.1}$$

for $2 \le j \le n-1$. It follows from the definition of the function $G^{[j]}$ that the equalities

$$u^{(2n-2j)}(t) = (-1)^{j-1} \int_0^T G^{[j-1]}(t,s) u^{(2n-2)}(s)\, ds, \quad 2 \le j \le n, \tag{3.2}$$

are true on $[0,T]$ for each $u \in C^{2n-2}[0,T]$ satisfying the boundary conditions (1.5).

Lemma 3.1. *For $1 \le j \le n-1$, the inequality*

$$G^{[j]}(t,s) \ge \frac{T^{2j-3}(1 - \alpha T)^j}{3^{j-1}} ts \quad \textit{for} \quad (t,s) \in [0,T] \times [0,T] \tag{3.3}$$

holds where

$$\alpha = \max\{a_k : 0 \le k \le n-2\} \quad \left(< \frac{1}{T}\right). \tag{3.4}$$

Proof. Since

$$G_j(t,s) = \begin{cases} s(1-a_jt) \geq s(1-a_jT) & \text{for } 0 \leq s \leq t \leq T, \\ t(1-a_js) \geq t(1-a_jT) & \text{for } 0 \leq t < s \leq T \end{cases}$$

for $0 \leq j \leq n-2$, we have $G_j(t,s) \geq \dfrac{1-a_jT}{T}st \geq \dfrac{1-\alpha T}{T}st$ for $(t,s) \in [0,T]\times[0,T]$ and $0 \leq j \leq n-2$. Consequently, $G^{[1]}(t,s) = G_{n-2}(t,s) \geq \dfrac{1-\alpha T}{T}st$ for $(t,s) \in \in [0,T]\times[0,T]$ and therefore inequality (3.3) is true for $j=1$. We now proceed by induction. Assume that (3.3) is true for $j = i$ $(< n-1)$. Then

$$G^{[i+1]}(t,s) = \int_0^T G_{n-i-2}(t,v)G^{[i]}(v,s)\,dv \geq$$

$$\geq \int_0^T \frac{1-\alpha T}{T}tv\frac{T^{2i-3}(1-\alpha T)^i}{3^{i-1}}vs\,dv =$$

$$= \frac{T^{2i-4}(1-\alpha T)^{i+1}}{3^{i-1}}ts\int_0^T v^2 ds = \frac{T^{2i-1}(1-\alpha T)^{i+1}}{3^i}ts$$

for $(t,s) \in [0,T]\times[0,T]$. Therefore (3.3) is true with $j = i+1$.

The lemma is proved.

Let ϕ satisfy (H_1). Choose an arbitrary $a > 0$ and put

$$\mathcal{B}_a = \Big\{u \in C^{2n-1}[0,T]\colon \phi(u^{(2n-1)}) \in AC[0,T],\ \ (-1)^n\big(\phi(u^{(2n-1)}(t))\big)' \geq a$$
$$\text{for a.e.} \quad t \in [0,T] \quad \text{and} \quad u \quad \text{satisfies (1.5)}\Big\}. \tag{3.5}$$

The properties of functions belonging to the set $\mathcal{B}_a$ are given in the following lemma.

Lemma 3.2. *Let $u \in \mathcal{B}_a$. Then there exists $\{\xi_{2j+1}\}_{j=0}^{n-1} \subset (0,T)$ such that*

$$u^{(2j+1)}(\xi_{2j+1}) = 0, \quad 0 \leq j \leq n-1, \tag{3.6}$$

and

$$\big|u^{(2n-1)}(t)\big| \geq \phi^{-1}\big(a|t-\xi_{2n-1}|\big), \tag{3.7}$$

$$\big|u^{(2n-2j+1)}(t)\big| \geq \frac{T^{2j-4}S}{2\cdot 3^{j-2}}(1-\alpha T)^{j-2}(t-\xi_{2n-2j+1})^2, \quad 2 \leq j \leq n, \tag{3.8}$$

$$(-1)^{n+j}u^{(2n-2j)}(t) \geq \frac{T^{2j-2}S}{3^{j-1}}(1-\alpha T)^{j-1}t, \quad 1 \leq j \leq n, \tag{3.9}$$

for $t \in [0,T]$, where

$$S = \frac{1}{T}\min\left\{ b_{n-1}\int\limits_0^{T/2}\phi^{-1}(at)\,dt,\ \frac{b_{n-1}}{a_{n-1}}\phi^{-1}\left(\frac{aT}{2}\right)\right\} \tag{3.10}$$

and α *is given in* (3.4).

Proof. Since ϕ is increasing and $\big(\phi((-1)^n u^{(2n-1)}(t))\big)' = (-1)^n\big(\phi(u^{(2n-1)}(t))\big)' \geq$ $\geq a$ for a.e. $t \in [0,T]$, it follows that $(-1)^n u^{(2n-1)}$ is increasing on $[0,T]$ and $(-1)^{n-1}u^{(2n-2)}$ is concave on this interval. If $u^{(2n-1)}(t) \neq 0$ for $t \in (0,T)$, then

$$\left|a_{n-1}u^{(2n-2)}(T) + b_{n-1}u^{(2n-1)}(T)\right| =$$

$$= \left|a_{n-1}\int\limits_0^T u^{(2n-1)}(t)dt + b_{n-1}u^{(2n-1)}(T)\right| > 0,$$

contrary to $a_{n-1}u^{(2n-2)}(T) + b_{n-1}u^{(2n-1)}(T) = 0$ by (1.5) with $k = n-1$. Hence $u^{(2n-1)}(\xi_{2n-1}) = 0$ for a unique $\xi_{2n-1} \in (0,T)$. Now integrating the equality $\big(\phi((-1)^n u^{(2n-1)}(t))\big)' \geq a$ over $[t, \xi_{2n-1}]$ and $[\xi_{2n-1}, t]$ gives

$$(-1)^{n-1}u^{(2n-1)}(t) \geq \phi^{-1}\big(a(\xi_{2n-1} - t)\big), \quad t \in [0, \xi_{2n-1}], \tag{3.11}$$

$$(-1)^n u^{(2n-1)}(t) \geq \phi^{-1}\big(a(t - \xi_{2n-1})\big), \quad t \in [\xi_{2n-1}, T], \tag{3.12}$$

which shows that (3.7) holds. In order to prove inequality (3.9) for $j = 1$ we consider two cases, namely $\xi_{2n-1} < \dfrac{T}{2}$ and $\xi_{2n-1} \geq \dfrac{T}{2}$.

Case 1. Let $\xi_{2n-1} < \dfrac{T}{2}$. Then (see (3.12))

$$(-1)^n u^{(2n-1)}(T) \geq \phi^{-1}(a(T - \xi_{2n-1})) > \phi^{-1}\left(\frac{aT}{2}\right),$$

and therefore (see (1.5) with $k = n-1$)

$$(-1)^{n-1}u^{(2n-2)}(T) = (-1)^n\frac{b_{n-1}}{a_{n-1}}u^{(2n-1)}(T) > \frac{b_{n-1}}{a_{n-1}}\phi^{-1}\left(\frac{aT}{2}\right). \tag{3.13}$$

Case 2. Let $\xi_{2n-1} \geq \dfrac{T}{2}$. Then (3.11) yields

$$(-1)^{n-1}u^{(2n-2)}\left(\frac{T}{2}\right) = (-1)^{n-1}\int\limits_0^{T/2} u^{(2n-1)}(t)\,dt \geq \int\limits_0^{T/2}\phi^{-1}\big(a(\xi_{2n-1} - t)\big)\,dt \geq$$

$$\geq \int\limits_0^{T/2}\phi^{-1}\left(a\left(\frac{T}{2} - t\right)\right)dt = \int\limits_0^{T/2}\phi^{-1}(at)\,dt =: L.$$

Let $\varepsilon := (-1)^n u^{(2n-1)}(T)$. We know that $(-1)^n u^{(2n-1)}$ is increasing on $[0,T]$ and $u^{(2n-1)}(\xi_{2n-1}) = 0$. Hence $\varepsilon > 0$ and

$$(-1)^{n-1}u^{(2n-2)}(t) = (-1)^{n-1}u^{(2n-2)}(\xi_{2n-1}) + (-1)^{n-1}\int\limits_{\xi_{2n-1}}^{t} u^{(2n-1)}(s)\,ds >$$

$$> (-1)^{n-1}u^{(2n-2)}(\xi_{2n-1}) - \varepsilon(t-\xi_{2n-1}) \geq$$

$$\geq (-1)^{n-1}u^{(2n-2)}\left(\frac{T}{2}\right) - \varepsilon(t-\xi_{2n-1})$$

for $t \in (\xi_{2n-1}, T]$. Consequently, $(-1)^{n-1}u^{(2n-2)}(T) > L - \varepsilon(T-\xi_{2n-1}) > L - \varepsilon T$. Then $\dfrac{b_{n-1}}{a_{n-1}}\varepsilon = (-1)^n \dfrac{b_{n-1}}{a_{n-1}}u^{(2n-1)}(T) = (-1)^{n-1}u^{(2n-2)}(T) > L - \varepsilon T$, and so (see (1.6)) $\varepsilon > L\left(\dfrac{b_{n-1}}{a_{n-1}} + T\right)^{-1} = a_{n-1}L$. It follows that

$$(-1)^{n-1}u^{(2n-2)}(T) = (-1)^n \frac{b_{n-1}}{a_{n-1}}u^{(2n-1)}(T) = \frac{b_{n-1}}{a_{n-1}}\varepsilon > b_{n-1}L. \tag{3.14}$$

Now (3.13) and (3.14) imply that $(-1)^{n-1}u^{(2n-2)}(T) > ST$ where S is given in (3.10). This and $u^{(2n-2)}(0) = 0$ and the fact that $(-1)^{n-1}u^{(2n-2)}$ is concave on $[0,T]$ guarantee that $(-1)^{n-1}u^{(2n-2)}(t) \geq St$ for $t \in [0,T]$, which proves (3.9) for $j=1$.

Combining (3.2), (3.3) and (3.9) (with $j=1$), we get

$$(-1)^{n+j}u^{(2n-2j)}(t) = (-1)^{n-1}\int\limits_0^T G^{[j-1]}(t,s)u^{(2n-2)}(s)\,ds \geq$$

$$\geq \frac{T^{2j-5}S}{3^{j-2}}(1-\alpha T)^{j-1}t\int\limits_0^T s^2\,ds = \frac{T^{2j-2}S}{3^{j-1}}(1-\alpha T)^{j-1}t$$

for $t \in [0,T]$ and $2 \leq j \leq n$. We have proved that (3.9) is true.

Since, by (3.9), $|u^{(2n-2j)}| > 0$ on $(0,T]$ for $1 \leq j \leq n$ and u satisfies (1.5), essentially the same reasoning as in the beginning of this prove shows that $u^{(2j+1)}(\xi_{2j+1}) = 0$ for a unique $\xi_{2j+1} \in (0,T)$, $0 \leq j \leq n-2$. Using (3.9) we obtain

$$\left|u^{(2n-2j+1)}(t)\right| = \left|\int\limits_{\xi_{2n-2j+1}}^{t} u^{(2n-2j+2)}(s)\,ds\right| \geq$$

$$\geq \frac{T^{2j-4}S}{3^{j-2}}(1-\alpha T)^{j-2}\left|\int\limits_{\xi_{2n-2j+1}}^{t} s\,ds\right| =$$

$$= \frac{T^{2j-4}S}{2\cdot 3^{j-2}}(1-\alpha T)^{j-2}|t^2 - \xi_{2n-2j+1}^2| \geq \frac{T^{2j-4}S}{2\cdot 3^{j-2}}(1-\alpha T)^{j-2}(t-\xi_{2n-2j+1})^2$$

for $t \in [0,T]$ and $2 \leq j \leq n$. Hence (3.8) is true, which finishes the proof.

3.2. Auxiliary regular problems. Let (H_2) and (H_3) hold. For each $m \in \mathbb{N}$, define $\chi_m, \varphi_m, \tau_m \in C^0(\mathbb{R})$ and $\mathbb{R}_m \subset \mathbb{R}$ by the formulas

$$\chi_m(v) = \begin{cases} v & \text{for } v \geq \dfrac{1}{m}, \\ \dfrac{1}{m} & \text{for } v < \dfrac{1}{m}, \end{cases} \qquad \varphi_m(v) = \begin{cases} -\dfrac{1}{m} & \text{for } v > -\dfrac{1}{m}, \\ v & \text{for } v \leq -\dfrac{1}{m}, \end{cases}$$

$$\tau_m = \begin{cases} \chi_m & \text{if } n = 2k-1, \\ \varphi_m & \text{if } n = 2k, \end{cases} \qquad \mathbb{R}_m = \mathbb{R} \setminus \left(-\frac{1}{m}, \frac{1}{m}\right).$$

Choose $m \in \mathbb{N}$ and use the function f to define $f_m \in \mathrm{Car}\left([0,T] \times \mathbb{R}^{2n}\right)$ by the formula

$$f_m(t, x_0, x_1, x_2, x_3, \ldots, x_{2n-2}, x_{2n-1}) =$$

$$= \begin{cases} f(t, \chi_m(x_0), x_1, \varphi_m(x_2), x_3, \ldots, \tau_m(x_{2n-2}), x_{2n-1}) \\ \qquad \text{for } \quad (t, x_0, x_1, x_2, x_3, \ldots, x_{2n-2}, x_{2n-1}) \in \\ \qquad \in [0,T] \times \mathbb{R} \times \mathbb{R}_m \times \mathbb{R} \times \mathbb{R}_m \times \ldots \times \mathbb{R} \times \mathbb{R}_m, \\ \dfrac{m}{2}\left[f_m\left(t, x_0, \dfrac{1}{m}, x_2, x_3, \ldots, x_{2n-2}, x_{2n-1}\right)\left(x_1 + \dfrac{1}{m}\right) - \right. \\ \qquad \left. - f_m\left(t, x_0, -\dfrac{1}{m}, x_2, x_3, \ldots, x_{2n-2}, x_{2n-1}\right)\left(x_1 - \dfrac{1}{m}\right)\right] \\ \qquad \text{for } \quad (t, x_0, x_1, x_2, x_3, \ldots, x_{2n-2}, x_{2n-1}) \in \\ \qquad \in [0,T] \times \mathbb{R} \times \left[-\dfrac{1}{m}, \dfrac{1}{m}\right] \times \mathbb{R} \times \mathbb{R}_m \times \ldots \times \mathbb{R} \times \mathbb{R}_m, \\ \dfrac{m}{2}\left[f_m\left(t, x_0, x_1, x_2, \dfrac{1}{m}, \ldots, x_{2n-2}, x_{2n-1}\right)\left(x_3 + \dfrac{1}{m}\right) - \right. \\ \qquad \left. - f_m\left(t, x_0, x_1, x_2, -\dfrac{1}{m}, \ldots, x_{2n-2}, x_{2n-1}\right)\left(x_3 - \dfrac{1}{m}\right)\right] \\ \qquad \text{for } \quad (t, x_0, x_1, x_2, x_3, \ldots, x_{2n-2}, x_{2n-1}) \in \\ \qquad \in [0,T] \times \mathbb{R}^3 \times \left[-\dfrac{1}{m}, \dfrac{1}{m}\right] \times \ldots \times \mathbb{R} \times \mathbb{R}_m, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{m}{2}\left[f_m\left(t, x_0, x_1, x_2, \ldots, x_{2n-2}, \dfrac{1}{m}\right)\left(x_{2n-1} + \dfrac{1}{m}\right) - \right. \\ \qquad \left. - f_m\left(t, x_0, x_1, x_2, \ldots, x_{2n-2}, -\dfrac{1}{m}\right)\left(x_{2n-1} - \dfrac{1}{m}\right)\right] \\ \qquad \text{for } \quad (t, x_0, x_1, x_2, \ldots, x_{2n-2}, x_{2n-1}) \in [0,T] \times \mathbb{R}^{2n-1} \times \left[-\dfrac{1}{m}, \dfrac{1}{m}\right]. \end{cases}$$

Then conditions (H_2) and (H_3) give

$$a \le (1-\lambda)a + \lambda f_m(t, x_0, \ldots, x_{2n-1}) \tag{3.15}$$

for a.e. $t \in [0,T]$ and all $(x_0, \ldots, x_{2n-1}) \in \mathbb{R}^{2n}$, $\lambda \in [0,1]$, and

$$(1-\lambda)a + \lambda f_m(t, x_0, \ldots, x_{2n-1}) \le h\left(t, 2n + \sum_{j=0}^{2n-1} |x_j|\right) + \sum_{j=0}^{2n-1} \omega_j\big(|x_j|\big) \tag{3.16}$$

for a.e. $t \in [0,T]$ and all $(x_0, \ldots, x_{2n-1}) \in \mathbb{R}_0^{2n}$, $\lambda \in [0,1]$.

Consider the family of approximate regular differential equations

$$(-1)^n\big(\phi(u^{(2n-1)})\big) = \lambda f_m(t, u, \ldots, u^{(2n-1)}) + (1-\lambda)a, \quad \lambda \in [0,1]. \tag{3.17}$$

Lemma 3.3. *Let $(H_1)-(H_3)$ hold. Then there exists a positive constant W independent of $m \in \mathbb{N}$ and $\lambda \in [0,1]$ such that*

$$\|u^{(j)}\| < W, \quad 0 \le j \le 2n-1, \tag{3.18}$$

for all solutions u of problem (3.17), (1.5).

Proof. Let u be a solution of problem (3.17), (1.5). Then $(-1)^n\big(\phi(u^{(2n-1)}(t))\big)' \ge$ $\ge a$ for a.e. $t \in [0,T]$ by (3.15) and consequently, $u \in \mathcal{B}_a$ where the set $\mathcal{B}_a$ is given in (3.5). Hence, by Lemma 3.2, u satisfies (3.6) and (3.7) where $\xi_{2j+1} \in (0,T)$ is the unique zero of $u^{(2j+1)}$, $0 \le j \le n-1$, and

$$\big|u^{(2n-2j+1)}(t)\big| \ge Q_j(t-\xi_{2n-2j+1})^2, \quad 2 \le j \le n,$$
$$(-1)^{n+i}u^{(2n-2i)}(t) \ge P_i t, \quad 1 \le i \le n,$$

for $t \in [0,T]$, where

$$Q_j = \frac{T^{2j-4}S}{2\cdot 3^{j-2}}(1-\alpha T)^{j-2}, \qquad P_i = \frac{T^{2i-2}S}{3^{i-1}}(1-\alpha T)^{i-1} \tag{3.19}$$

with α and S given in (3.4) and (3.10), respectively. Accordingly,

$$\sum_{j=0}^{2n-1}\int_0^T \omega_j\big(|u^{(j)}(t)|\big)\,dt \le \sum_{j=1}^{n}\int_0^T \omega_{2n-2j}(P_j t)\,dt +$$
$$+\sum_{j=2}^{n}\int_0^T \omega_{2n-2j+1}\big(Q_j(t-\xi_{2n-2j+1})^2\big)\,dt + \int_0^T \omega_{2n-1}(\phi^{-1}(a|t-\xi_{2n-1}|))\,dt <$$
$$< \sum_{j=1}^{n}\frac{1}{P_j}\int_0^{P_jT}\omega_{2n-2j}(s)\,ds + 2\sum_{j=2}^{n}\frac{1}{\sqrt{Q_j}}\int_0^{\sqrt{Q_j}T}\omega_{2n-2j+1}(s^2)\,ds +$$
$$+\frac{2}{aT}\int_0^{aT}\omega_{2n-1}(\phi^{-1}(s))\,ds =: \Lambda. \tag{3.20}$$

By (H_3), $\Lambda < \infty$. Since $u^{(2j)}(0) = 0$ and $u^{(2j+1)}(\xi_{2j+1}) = 0$ for $0 \le j \le n-1$, we have

$$\|u^{(j)}\| \le T^{2n-j-1}\|u^{(2n-1)}\|, \quad 0 \le j \le 2n-2. \tag{3.21}$$

Combining (3.16), (3.20), (3.21) and $u^{(2n-1)}(\xi_{2n-1}) = 0$, we obtain

$$\phi\big(|u^{(2n-1)}(t)|\big) = \left| \int\limits_{\xi_{2n-1}}^{t} [(1-\lambda)a + \lambda f_m(s, u(s), \ldots, u^{(2n-1)}(s))]\, ds \right| <$$

$$< \int\limits_0^T h\left(t, 2n + \sum_{j=0}^{2n-1} |u^{(j)}(t)|\right) dt + \sum_{j=0}^{2n-1} \int\limits_0^T \omega_j\big(|u^{(j)}(t)|\big)\, dt <$$

$$< \int\limits_0^T h\left(t, 2n + \|u^{(2n-1)}\| \sum_{j=0}^{2n-1} T^j\right) dt + \Lambda =$$

$$= \int\limits_0^T h(t, 2n + K\|u^{(2n-1)}\|)\, dt + \Lambda$$

for $t \in [0, T]$, where K is given in (1.9). Hence

$$\phi\big(\|u^{(2n-1)}\|\big) < \int\limits_0^T h\big(t, 2n + K\|u^{(2n-1)}\|\big)\, dt + \Lambda. \tag{3.22}$$

It follows from condition (1.8) that there exists a positive constant W_* such that $\int_0^T h(t, 2n + Kv)\, dt < \phi(v)$ whenever $v \ge W_*$. This and (3.22) yields $\|u^{(2n-1)}\| <$ $< W_*$. Consequently, (3.21) shows that (3.18) is fulfilled with $W = W_* \max\big\{1, T^{2n-1}\big\}$.

The lemma is proved.

***Remark* 3.1.** Let $c > 0$. If follows from the proof of Lemma 3.3 that any solution u of problem $(-1)^n\big(\phi(u^{(2n-1)})\big)' = c$, (1.5) satisfies the inequality $\|u^{(j)}\| <$ $< \phi^{-1}(cT)\max\{1, T^{2n-1}\}$ for $0 \le j \le 2n-1$.

We are now in a position to show that for each $m \in \mathbb{N}$ there exists a solution u_m of the regular differential equation

$$(-1)^n\big(\phi(u^{(2n-1)})\big)' = f_m(t, u, \ldots, u^{(2n-1)}) \tag{3.23}$$

satisfying the boundary conditions (1.5).

Lemma 3.4. *Let (H_1)–(H_3) hold. Then for each $m \in \mathbb{N}$ there exists a solution $u_m \in C^{2n-1}[0, T]$, $\phi(u^{(2n-1)}) \in AC[0, T]$, of problem* (3.23), (1.5) *and*

$$\|u_m^{(j)}\| < W \quad \textit{for} \quad m \in \mathbb{N} \quad \textit{and} \quad 0 \le j \le 2n-1, \tag{3.24}$$

where W is a positive constant. In addition, the sequence $\{u_m^{(2n-1)}\}$ is equicontinuous on $[0, T]$.

Proof. Choose an arbitrary $m \in \mathbb{N}$. Let W be a positive constant in Lemma 3.3. In order to prove the existence of a solution of problem (3.23), (1.5) we use Theorem 2.1 with $p = 2n$, $g = (-1)^n f_m$ and $\varphi = (-1)^n a$ in equations (2.1), (2.2) and with

$$\alpha_{2k}(u) = u^{(2k)}(0), \qquad \alpha_{2k+1}(u) = a_k u^{(2k)}(T) + b_k u^{(2k+1)}(T), \quad 0 \le k \le n-1, \tag{3.25}$$

in the boundary conditions (1.2).

Due to Lemma 3.3 and Remark 3.1, all solutions u of problems (3.17), (1.5) and $(-1)^n\big(\phi(u^{(2n-1)})\big)' = \lambda a$, (1.5) $(0 \le \lambda \le 1)$ satisfy inequality (3.18). Moreover, α_k (defined in (3.25)) belongs to the set $\mathcal{A}$ (with $p = 2n$) for $0 \le k \le 2n-1$. The system (see (1.3))

$$\alpha_k\left(\sum_{i=0}^{2n-1} A_i t^i\right) - \mu\alpha_k\left(-\sum_{i=0}^{2n-1} A_i t^i\right) = 0, \quad 0 \le k \le 2n-1, \tag{3.26}$$

has the form (see (3.25))

$$(1+\mu)\left(\sum_{i=0}^{2n-1} A_i t^i\right)^{(2k)}\Bigg|_{t=0} = 0, \quad 0 \le k \le n-1, \tag{3.27}$$

$$(1+\mu)\left[a_k\left(\sum_{i=0}^{2n-1} A_i t^i\right)^{(2k)}\Bigg|_{t=T} + \right.$$

$$\left. + b_k\left(\sum_{i=0}^{2n-1} A_i t^i\right)^{(2k+1)}\Bigg|_{t=T}\right] = 0, \quad 0 \le k \le n-1. \tag{3.28}$$

It follows from (3.27) that $A_{2k} = 0$ for $0 \le k \le n-1$ and then we deduce from (3.28) and from $a_k T + b_k = 1$ that $A_{2j+1} = 0$ for $0 \le j \le n-1$. Consequently, $(A_0, \ldots, A_{2n-1}) = (0, \ldots, 0) \in \mathbb{R}^{2n}$ is the unique solution of (3.26) for each $\mu \in [0,1]$. Hence all the assumptions of Theorem 2.1 are satisfied and therefore for each $m \in \mathbb{N}$, there exists a solution $u_m \in C^{2n-1}[0,T]$, $\phi(u^{(2n-1)}) \in AC[0,T]$, of problem (3.23), (1.5) fulfilling inequality (3.24).

It remains to show that the sequence $\{u_m^{(2n-1)}\}$ is equicontinuous on $[0,T]$. Notice that $u_m \in \mathcal{B}_a$ for all $m \in \mathbb{N}$ where the set $\mathcal{B}_a$ is given in (3.5). Then, by Lemma 3.2, there exists $\{\xi_{2j+1,m}\}_{j=0}^{n-1} \subset (0,T)$, $m \in \mathbb{N}$, such that

$$u_m^{(2j+1)}(\xi_{2j+1,m}) = 0, \qquad 0 \le j \le n-1, \quad m \in \mathbb{N}, \tag{3.29}$$

and

$$\big|u_m^{(2n-1)}(t)\big| \ge \phi^{-1}\big(a|t-\xi_{2n-1,m}|\big),$$

$$\big|u_m^{(2n-2j+1)}(t)\big| \ge Q_j(t-\xi_{2n-2j+1,m})^2, \quad 2 \le j \le n, \tag{3.30}$$

$$(-1)^{n+j}u_m^{(2n-2j)}(t) \ge P_j t, \quad 1 \le j \le n,$$

for $t \in [0,T]$ and $m \in \mathbb{N}$, where Q_j, P_j are given in (3.19). Let $0 \le t_1 < t_2 \le T$. Then (see (3.16) with $\lambda = 1$, (3.24) and (3.30))

$$\left|\phi\big(u_m^{(2n-1)}(t_2)\big) - \phi\big(u_m^{(2n-1)}(t_1)\big)\right| =$$

$$= \int_{t_1}^{t_2} f_m\big(t, u_m(t), \ldots, u_m^{(2n-1)}(t)\big)\, dt \le$$

$$\le \int_{t_1}^{t_2} h\left(t, 2n + \sum_{j=0}^{2n-1} \|u_m^{(j)}\|\right) dt + \sum_{j=0}^{2n-1} \int_{t_1}^{t_2} \omega_j\big(|u_m^{(j)}(t)|\big)\, dt \le$$

$$\le \int_{t_1}^{t_2} h(t, 2n(1+W))\, dt + \int_{t_1}^{t_2} \omega_{2n-1}\big(\phi^{-1}(a|t - \xi_{2n-1,m}|\big)\, dt +$$

$$+ \sum_{j=2}^{n} \int_{t_1}^{t_2} \omega_{2n-2j+1}\big(Q_j(t - \xi_{2n-2j+1,m})^2\big)\, dt +$$

$$+ \sum_{j=1}^{n} \int_{t_1}^{t_2} \omega_{2n-2j}(P_j t)\, dt \tag{3.31}$$

for $m \in \mathbb{N}$. By (H_3), $h(t, 2n(1+W)) \in L_1[0,T]$ and $\omega_{2n-1}(\phi^{-1}(s))$, $\omega_{2j}(s)$, $0 \le j \le \le n-1$, $\omega_{2i+1}(s^2)$, $0 \le i \le n-2$, are locally integrable on $[0,\infty)$. From these facts and from (3.31) and from the relations

$$\int_{t_1}^{t_2} \omega_{2n-1}\big(\phi^{-1}(a|t - \xi_{2n-1,m}|)\big)\, dt =$$

$$= \begin{cases} \dfrac{1}{a} \displaystyle\int_{a(\xi_{2n-1,m}-t_2)}^{a(\xi_{2n-1,m}-t_1)} \omega_{2n-1}\big(\phi^{-1}(t)\big)\, dt, & \text{if} \quad t_2 \le \xi_{2n-1,m}, \\ \dfrac{1}{a} \left[\displaystyle\int_0^{a(\xi_{2n-1,m}-t_1)} \omega_{2n-1}\big(\phi^{-1}(t)\big)\, dt + \right. & \\ \left. \quad + \displaystyle\int_0^{a(t_2-\xi_{2n-1,m})} \omega_{2n-1}\big(\phi^{-1}(t)\big)\, dt \right] & \text{if} \quad t_1 < \xi_{2n-1,m} < t_2, \\ \dfrac{1}{a} \displaystyle\int_{a(t_1-\xi_{2n-1,m})}^{a(t_2-\xi_{2n-1,m})} \omega_{2n-1}\big(\phi^{-1}(t)\big)\, dt & \text{if} \quad \xi_{2n-1,m} \le t_1, \end{cases}$$

$$\int_{t_1}^{t_2} \omega_{2n-2j+1}\big(Q_j(t - \xi_{2n-2j+1,m})^2\big)\, dt =$$

$$= \begin{cases} \dfrac{1}{\sqrt{Q_j}} \displaystyle\int\limits_{\sqrt{Q_j}(\xi_{2n-2j+1,m}-t_2)}^{\sqrt{Q_j}(\xi_{2n-2j+1,m}-t_1)} \omega_{2n-2j+1}(t^2)\,dt & \text{if} \quad t_2 \le \xi_{2n-2j+1,m}, \\ \dfrac{1}{\sqrt{Q_j}} \left[\displaystyle\int\limits_0^{\sqrt{Q_j}(\xi_{2n-2j+1,m}-t_1)} \omega_{2n-2j+1}(t^2)\,dt + \right. & \\ \left. \qquad + \displaystyle\int\limits_0^{\sqrt{Q_j}(t_2-\xi_{2n-2j+1,m})} \omega_{2n-2j+1}(t^2)\,dt \right] & \text{if} \quad t_1 < \xi_{2n-2j+1,m} < t_2, \\ \dfrac{1}{\sqrt{Q_j}} \displaystyle\int\limits_{\sqrt{Q_j}(t_1-\xi_{2n-2j+1,m})}^{\sqrt{Q_j}(t_2-\xi_{2n-2j+1,m})} \omega_{2n-2j+1}(t^2)\,dt & \text{if} \quad \xi_{2n-2j+1,m} \le t_1, \end{cases}$$

it follows that $\{\phi(u_m^{(2n-1)})\}$ is equicontinuous on $[0,T]$. We now deduce the equicontinuity of $\{u_m^{(2n-1)}\}$ on $[0,T]$ from the equality

$$\left|u_m^{(2n-1)}(t_2) - u_m^{(2n-1)}(t_1)\right| = \left|\phi^{-1}\big(\phi(u_m^{(2n-1)}(t_2))\big) - \phi^{-1}\big(\phi(u_m^{(2n-1)}(t_1))\big)\right|$$

for $0 \le t_1 < t_2 \le$T, $m \in \mathbb{N}$, and the facts that $\{\phi(u_m^{(2n-1)})\}$ is bounded in $C^0[0,T]$ and ϕ^{-1} is continuous and increasing on $\mathbb{R}$.

The lemma is proved.

3.3. Existence result and an example. The main result is presented in the following theorem.

Theorem 3.1. *Let* (H_1) – (H_3) *hold. Then problem* (1.4), (1.5) *has a solution* $u \in$ $\in C^{2n-1}[0,T]$, $\phi(u^{(2n-1)}) \in AC[0,T]$ *and* $(-1)^k u^{(2k)} > 0$ *on* $(0,T]$, $u^{(2k+1)}(\xi_{2k+1}) =$ $= 0$ *for* $0 \le k \le n-1$ *where* $\xi_{2k+1} \in (0,T)$.

Proof. By Lemma 3.4, for each $m \in \mathbb{N}$ there exists a solution u_m of problem (3.23), (1.5). Consider the sequence $\{u_m\}$. Then inequality (3.24) is satisfied with a positive constant W and since $u_m \in \mathcal{B}_a$, Lemma 3.2 guarantees the existence of $\{\xi_{2j+1,m}\}_{j=0}^{n-1} \subset$ $\subset (0,T)$ such that (3.29) and (30) hold for $t \in [0,T]$ and $m \in \mathbb{N}$, where Q_j and P_j are given in (3.19). Moreover, the sequence $\{u_m^{2n-1}\}$ is equicontinuous on $[0,T]$ by Lemma 3.4. Hence there exist a subsequence $\{u_{k_m}\}$ converging in $C^{2n-1}[0,T]$ and a subsequence $\{\xi_{2j+1,k_m}\}$, $1 \le j \le n-1$, converging in $\mathbb{R}$. Let $\lim_{m\to\infty} u_{k_m} = u$ and $\lim_{m\to\infty} \xi_{2j+1,k_m} = \xi_{2j+1}$, $1 \le j \le n-1$. Letting $m \to \infty$ in (3.24), (3.29) and (3.30) (with k_m instead of m) yields (for $t \in [0,T]$)

$$\left|u^{(2n-1)}(t)\right| \ge \phi^{-1}\big(a|t-\xi_{2n-1}|\big),$$

$$u^{(2j+1)}(\xi_{2j+1}) = 0 \quad \text{for} \quad 0 \le j \le n-1,$$

$$\left|u^{(2n-2j+1)}(t)\right| \ge Q_j(t-\xi_{2n-2j+1})^2 \quad \text{for} \quad 2 \le j \le n-1,$$

$$\|u^{(j)}\| \le W \quad \text{for} \quad 0 \le j \le 2n-1$$

and

$$(-1)^{n+j}u^{(2n-2j)}(t) \geq P_j t \quad \text{for} \quad 1 \leq j \leq n. \tag{3.32}$$

Hence $u^{(j)}$ has exactly one zero in $[0,T]$ for $0 \leq j \leq 2n-1$ and

$$\lim_{m\to\infty} f_{k_m}\big(t, u_{k_m}(t), \ldots, u_{k_m}^{(2n-1)}(t)\big) =$$

$$= f\big(t, u(t), \ldots, u^{(2n-1)}(t)\big) \quad \text{for a.e.} \quad t \in [0,T].$$

In addition, by (3.32), $(-1)^k u^{(2k)} > 0$ on $(0,T]$ and $(-1)^k u^{(2k+1)}(0) \geq P_{n-k} > 0$ for $0 \leq k \leq n-1$. Hence $(-1)^k u^{(2k+1)}(T) < 0$ for $0 \leq k \leq n-1$ by (1.5), which combining with $(-1)^k u^{(2k+1)}(0) > 0$ implies $\xi_{2k+1} \in (0,T)$ for $0 \leq k \leq n-1$. Finally, having in mind the definition of the function f_m and inequality (3.16) we have

$$0 \leq f_m(t, x_0, \ldots, x_{2n-1}) \leq q\big(t, |x_0|, \ldots, |x_{2n-1}|\big)$$

$$\text{for a.e.} \quad t \in [0,T] \quad \text{and all} \quad (x_0, \ldots, x_{2n-1}) \in \mathbb{R}_0^{2n}$$

where $q(t, x_0, \ldots, x_{2n-1}) = h\left(t, 2n + \sum_{j=0}^{2n-1} x_j\right) + \sum_{j=0}^{2n-1} \omega_j(x_j)$ for $t \in [0,T]$ and $(x_0, \ldots, x_{2n-1}) \in \mathbb{R}_+^{2n}$. Clearly, $q \in \mathrm{Car}([0,T] \times \mathbb{R}_+^{2n})$. Hence problem (1.4), (1.5) satisfies the assumptions of Theorem 2.2 with $p = 2n$, $g = (-1)^n f$, $g_m = f_m$ (that is $\nu = (-1)^n$ in (2.11)) and with the boundary conditions (3.25) which are the special case of the boundary conditions (1.2). Consequently, Theorem 2.2 guarantees that $\phi(u^{(2n-1)}) \in AC[0,T]$ and u is a solution of problem (1.4), (1.5).

The theorem is proved.

***Example* 3.1.** Let $p > 1$, $\alpha_{2n-1} \in (0, p-1)$, $\alpha_{2j} \in (0,1)$ for $0 \leq j \leq n-1$, $\alpha_{2j+1} \in \left(0, \dfrac{1}{2}\right)$ for $0 \leq j \leq n-2$, $\beta_k \in (0, p-1)$, $c_k > 0$, $d_k \in L_1[0,T]$ for $0 \leq k \leq 2n-1$, d_k is nonnegative and $r \in L_1[0,T]$, $r(t) \geq a > 0$ for a.e. $t \in [0,T]$. Consider the differential equation

$$(-1)^n\big(|u^{(2n-1)}|^{p-2}u^{(2n-1)}\big)' = r(t) + \sum_{k=0}^{2n-1}\left(\frac{c_k}{|u^{(k)}|^{\alpha_k}} + d_k(t)|u^{(k)}|^{\beta_k}\right). \tag{3.33}$$

Equation (3.33) satisfies conditions $(H_1)-(H_3)$ with $\phi(v) = |v|^{p-2}v$, $h(t,v) = r(t) + (2n + v^{\gamma})\sum_{j=0}^{2n-1} d_k(t)$ where $\gamma = \max\{\beta_k \colon 0 \leq k \leq 2n-1\} < p-1$ and $\omega_k(v) = \dfrac{c_k}{v^{\alpha_k}}$, $0 \leq k \leq 2n-1$. Hence Theorem 3.1 guarantees that problem (3.33), (1.5) has a solution $u \in C^{2n-1}[0,T]$, $\phi(u^{(2n-1)}) \in AC[0,T]$ and $(-1)^k u^{(2k)} > 0$ on $(0,T]$, $u^{(2k+1)}(\xi_{2k+1}) = 0$ for $0 \leq k \leq n-1$ where $\xi_{2k+1} \in (0,T)$.

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