LITTLEWOOD – PALEY THEOREM ON $L^p(t)(\mathbb{R}^n)$ SPACES

1. Introduction. Let $m$ be a bounded function on $\mathbb{R}^n$. The operator $T$ defined by the Fourier transform equation $(\hat{T}f)(x) = m(x)\hat{f}(x)$, $x \in \mathbb{R}^n$, is called a multiplier operator with multiplier $m$. Let $\rho$ be an $(n$-dimensional) rectangle and $\chi_\rho$ the characteristic function of $\rho$. The operator $S_\rho$ having multiplier $m = \rho$ and defined by the equation

$$(S_\rho f)(x) = \chi_\rho(x)\hat{f}(x), \quad x \in \mathbb{R}^n,$$

is called a partial sum operator.

Let a collection of disjoint rectangles $\Delta = \{\rho\}$ be a decomposition of $\mathbb{R}^n$ (i.e., $\bigcup_{\rho \in \Delta} = \mathbb{R}^n$). Given a function $f$ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, define

$$G(f)(x) = \left(\sum_{\rho \in \Delta} \|S_\rho f(x)\|^2\right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Let $\{n_k\}_{k=-\infty}^{+\infty}$, $n_k > 0$, $k \in \mathbb{Z}$, be a lacunary sequence (i.e., there is an $a > 1$ such that $n_{k+1}/n_k \geq a$ for all $k$). Let $\Delta$ be the collection of all intervals of the form $[n_k, n_{k+1}]$ and $[-n_k, n_k]$, $k \in \mathbb{Z}$. Then $\Delta$ is called a lacunary decomposition of $\mathbb{R}$. When $n_k = 2^k$, $k \in \mathbb{Z}$, the resulting $\Delta$ is called the dyadic decomposition of $\mathbb{R}$.

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Let $\Delta_i$, $i = 1, 2, \ldots, n$, be $n$ lacunary decomposition of $\mathbb{R}$. Let $\Delta$ be the collection of the intervals of the form $\rho = \rho_1 \times \rho_2 \times \cdots \times \rho_n$ where $\rho_i \in \Delta_i$. Then $\Delta$ is called a lacunary decomposition of $\mathbb{R}^n$.

The important feature of the classical Littlewood–Paley theory is that a characterization of the spaces $L^{p(n)}(\mathbb{R})$, $1 < p < \infty$. It is well known (see [1, 2]) that if $\Delta$ is a lacunary decomposition of $\mathbb{R}^n$ then \[ \|G(f)\|_p \leq \|f\|_p \leq A \|G(f)\|_p \] for $1 < p < \infty$; i.e., there are constants $A$ and $B$ such that \[ A \|f\|_p \leq \|G(f)\|_p \leq B \|f\|_p. \]

The purpose of this paper is to obtain analogously characterizations of variable exponent Lebesgue spaces $L^{p(t)}(\mathbb{R})$.

Given a measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, $L^{p(t)}(\mathbb{R}^n)$ denotes the set of measurable functions $f$ on $\mathbb{R}^n$ such that for some $\lambda > 0$ \[ \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx < \infty. \]

This set becomes a Banach function space when equipped with the norm \[ \|f\|_{p(t)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}. \]

Given a locally integrable function $f$, we define the Hardy–Littlewood maximal function $Mf$ by \[ Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \] where the supremum is taken over all cubes containing $x$ with sides parallel to the coordinate axes. For conciseness, define $\mathcal{P}(\mathbb{R}^n)$ to be the set of measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that \[ 1 < a \leq p(t) \leq b < \infty : t \in \mathbb{R}^n. \]

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $M$ is bounded on $L^{p(t)}(\mathbb{R}^n)$. Conditions for the boundedness of the Hardy–Littlewood maximal operator on spaces $L^{p(t)}(\mathbb{R}^n)$ have been studied in [3–8]. Diening [8] studied the necessary and sufficient conditions in terms of the conjugate exponent $p'(\cdot)$, $(1/p(t) + 1/p'(t) = 1, t \in \mathbb{R}^n)$. He has proved that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, he also proved that if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ then $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$ for some $q > 1$.

In harmonic analysis a fundamental operator is the Hardy–Littlewood maximal operator. In many applications a crucial step has been to show that operator $M$ is bounded on a variable $L^p$ space. Cruz-Uribe, Fiorenza, Martell and Perez [4] have showed that many classical operators in harmonic analysis such as singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue space $L^{p(t)}(\mathbb{R}^n)$ whenever the Hardy–Littlewood maximal operator is bounded on $L^{p(t)}(\mathbb{R}^n)$.

If we consider, instead, the strong maximal operator $M_{\mathcal{R}}$ defined by \[ M_{\mathcal{R}}(f)(x) = \sup_{x \in \mathcal{R}} \frac{1}{|\mathcal{R}|} \int_{\mathcal{R}} |f(y)| \, dy, \]

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where $R$ is any rectangle in $\mathbb{R}^n$, $n > 1$, with sides parallel to the coordinate axes then the situation is different. For the strong Hardy – Littlewood maximal operator $M_R$, we prove following theorem.

**Theorem 1.** Let $1 \leq p(t) \leq b < \infty$, $t \in \mathbb{R}^n$. The strong Hardy – Littlewood maximal operator $M_R$ is bounded on $L^{p(t)}(\mathbb{R}^n)$ space if and only if $p(t) = \text{const} = p$ and $p > 1$.

For function $f \in L(\mathbb{R}^n)$, the expression

$$Hf(x) = \left(\int \prod_{t=1}^{n} \frac{1}{x_k - y_k} f(y) dy\right)$$

is said to be $n$-dimensional $(n > 1)$ Hilbert operator. Analogously we may prove following theorem.

**Theorem 2.** Let $1 \leq p(t) \leq b < \infty$, $t \in \mathbb{R}^n$. Then $n$-dimensional Hilbert operator $(n > 1)$ is bounded on $L^{p(t)}(\mathbb{R}^n)$ space if and only if $p(t) = \text{const} = p$ and $p > 1$.

We prove following Littlewood – Paley type characterization of $L^{p(t)}(\mathbb{R}^n)$ space.

**Theorem 3.** 1. Let $\Delta$ be a lacunary decomposition of $\mathbb{R}$ and $p(\cdot) \in \mathcal{B}(\mathbb{R})$. Then there are constants $c, C > 0$ such that for all $f \in L^{p(t)}(\mathbb{R})$

$$c \|f\|_{p(t)} \leq \|G(f)\|_{p(t)} \leq C \|f\|_{p(t)}. \quad (1)$$

2. Let $\Delta$ be the dyadic decomposition of $\mathbb{R}^n$, $n > 1$. If $p(\cdot) \neq \text{const}$ then operator $G$ is not bounded on $L^{p(t)}(\mathbb{R}^n)$.

**2. Proof of theorems. Proof of Theorem 1.** According to Jessen, Marcinkiewicz and Zygmund [9] $M_R$ is bounded on all the $L^p$, $p > 1$, spaces and first part of Theorem 1 is trivial.

Let $M_R$ is bounded on $L^{p(t)}(\mathbb{R}^n)$. Virtue of interpolation theorem (see [10]), we have $M_R$ is bounded on $L^{p(t)} = [L^{p(t)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n)]_{\theta}, 0 < \theta < 1$, and without restriction of generality we may assume that $1 < \inf_{\mathbb{R}^n} p(t)$. Let $1/p(t) + 1/p'(t) = 1$, $t \in \mathbb{R}^n$. Note that

$$\sup_{R} \frac{1}{|R|} \int_{R} \|X_R\|_{p(t)} \|X_R\|_{p'(t)} < \infty \quad (2)$$

condition is necessary for boundedness of $M_R$ on $L^{p(t)}(\mathbb{R}^n)$ (see proof below).

We will give the proof of second part of Theorem 1 for the case $n = 2$ for simplicity, since the same argument holds when $n > 2$.

Let $\inf_{\mathbb{R}^2} p(t) < \sup_{\mathbb{R}^2} p(t)$. By Luzin’s theorem we can construct pairwise disjoint family of set $F_i$ with the following condition: 1) $|\mathbb{R}^2 \setminus \bigcup F_i| = 0$, 2) functions $p : F_i \rightarrow \mathbb{R}$ are continuous, 3) for every fixed $i$ all points of $F_i$ are points of density with respect to basis $\mathcal{R}$.

Note that, we can find pair of points $((x_0, y_1), (x_0, y_2))$- or $((x_1, y_0), (x_2, y_0))$-type from $\bigcup F_i$ such that $p(x_0, y_1) \neq p(x_0, y_2)$ or $p(x_1, y_0) \neq p(x_2, y_0)$. Without loss of generality, we may suppose that this pair is $((x_0, y_1), (x_0, y_2)); (x_0, y_1) \in F_1, (x_0, y_2) \in F_2$ and $y_1 < y_2$. 

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Let $0 < \varepsilon < 1$ be fixed. We may find $\delta > 0$ such that for any rectangles $Q_1 \ni (x_0, y_0), \ Q_2 \ni (x_0, y_1)$ with diameters less than $\delta$ the following inequalities are valid:

$$|Q_1 \cap F_1| > (1 - \varepsilon)|Q_1|, \quad |Q_2 \cap F_2| > (1 - \varepsilon)|Q_2|,$$

(3)

$$p_{\Omega_0} = \inf_{Q_2 \cap F_2} p(x, y) < c_1 < c_2 < \sup_{Q_2 \cap F_2} p(x, y) = p_{\Omega_2},$$

(4)

for some constant $c_1, \ c_2$.

Let $Q_1, \tau, Q_2, \tau$ are rectangles with properties (3), (4) of the form $(a, b) \times (c, d), \ a < b < c < d$. We have continuously embedding $L^p(\Omega) \hookrightarrow L^{p_q}(\Omega)$ and $L^{p_l}(\Omega) \hookrightarrow L^{p_{\Omega_0}}(\Omega)$, where $\frac{1}{p_0} + \frac{1}{p_1} = 1$ (see for example [11]). For rectangle $Q_\tau = (x_0 - \tau, x_0 + \tau) \times (a, d)$ we have

$$A_\tau = \frac{1}{|Q_\tau|} \left\| x_{Q_\tau} \cdot p(x) \right\|_{p_{\Omega_0}} \geq \frac{1}{2\tau (d - a)} \left\| x_{Q_\tau} \cdot p(x) \right\|_{p_{\Omega_0}} \geq$$

$$\geq \frac{C}{2\tau (d - a)} (2\tau (d - a))^{-1/p_0} (2\tau (b - a))^{1-1/p_0}.$$

Note that if $\tau \to 0$ $(a, b, c, d$ is fixed) $A_\tau \to \infty$ and consequently (2) is not valid. This completes the proof.

**Proof of Theorem 3.** The inequalities (1) are consequence of the extrapolation theorem given by Cruz-Uribe, Fiorenza, Martell and Perez [4] and the weighted norm inequalities for $G(f)$ function given by Kurtz [12]. We describe this results.

Let $p_\alpha = \text{ess inf} \{ p(x): x \in \mathbb{R} \}$. By a weight we mean a nonnegative, locally integrable function $\omega$. When $1 < p < \infty$, we say $\omega \in A_{p_\alpha}$ if for every interval $Q$

$$\frac{1}{|Q|} \int_Q \omega(x) dx \left( \frac{1}{|Q|} \int_Q \omega(x)^{1-p_{\alpha}} dx \right)^{p-1} \leq C < \infty.$$

The infimum over the constants on the right-hand side of the last inequality we denote by $A_{p_\alpha, \omega}$. By $\mathcal{F}$ will denote a family of ordered pairs of nonnegative, measurable functions $(f, g)$. We say that an inequality

$$\int_{\mathbb{R}} f(x) g_0(x) \omega(x) dx \leq C \int_{\mathbb{R}} g(x) f_0 \omega(x) dx, \quad 0 < p_0 < \infty,$$

(5)

holds for any $(f, g) \in \mathcal{F}$ and $\omega \in A_q$ (for some $q, \ 1 < q < \infty$) if it holds for any pair in $\mathcal{F}$ such that the left-hand side is finite, and the constant $C$ depends only on $p_0$ and the $A_{p_\alpha, \omega}$ constant of $\omega$.

**Theorem 4.** Given a family $\mathcal{F}$, assume that (5) holds for some $1 < p_0 < \infty$, for every weight $\omega \in A_{p_0}$ and for all $(f, g) \in \mathcal{F}$. Let $p(\cdot) \in \mathcal{P}(\mathbb{R})$ be such that there exists $1 < p_1 < p_\alpha$, with $(p(\cdot)/p_1) \in \mathcal{B}(\mathbb{R})$. Then

$$\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}$$

for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R})$.

**Theorem 5** [12]. Let $\Delta$ be a a lacunary decomposition of $\mathbb{R}$, $1 < p < \infty$, and
\( \omega \in A_p \). Then there exist constant \( c, C \) depending only on \( p, A_{p, \omega}, \) and \( \Delta \), such that
\[
c \int_{\mathbb{R}} |f(x)|^{p} \omega(x) dx \leq \int_{\mathbb{R}} (G(f)(x))^{p} \omega(x) dx \leq C \int_{\mathbb{R}} |f(x)|^{p} \omega(x) dx.
\]

From assumption of Theorem 3 we get that there exists \( 1 < p_1 < p \) with \((p(\cdot)) / p_1 \in B(\mathbb{R}) \) (see [8]). Let \( L_{\mathrm{comp}}^\infty(\mathbb{R}) \) be the set of all bounded functions with compact support. From Theorems 4, 5 with the pairs \((Wf, |f|)\) we get right side inequality of (1) if \( f \in L_{\mathrm{comp}}^\infty(\mathbb{R}) \). Note that \( L_{\mathrm{comp}}^\infty(\mathbb{R}) \) is dense in \( L^{p(t)}(\mathbb{R}) \) (see [11]) and consequently this inequality is also valid for all \( f \in L^{p(t)}(\mathbb{R}) \). Analogously we obtain left side inequality of (1).

Let \( n > 1 \). Fix a rectangle \( R = I_1 \times I_2 \times \ldots \times I_n \) and let \( f \) be positive on \( R \) and 0 elsewhere function. Let \( k_j \) be the greatest integer such that \( 2^{kj} \leq (4n|I_j|)^{-1} \) and \( \rho \) be the dyadic rectangle \([2^{kj}, 2^{kj+1}] \times \ldots \times [2^{kj}, 2^{kj+1}]\). Note that (see [12, p. 246]) for all \( x \in R \)
\[
|S_{\rho} f(x)| \geq \frac{C}{|R|} \int_{R} f(x) dx.
\]

Let the operator \( G \) is bounded on \( L^{p(t)}(\mathbb{R}^n) \). Then for some constant \( C \) we have
\[
\frac{1}{|Q|} \int_{Q} f(x) dx \|1_{R}\|_{p(t)} \leq C \|f\|_{p(t)}. \tag{6}
\]

Note that \((L^{p(t)}(\mathbb{R}^n))^*\) is isomorphic to the space \( L^{p(t)}(\mathbb{R}^n) \), where \( 1/p(t) + 1/p'(t) = 1, t \in \mathbb{R}^n \) (see [11]). Therefore, for all rectangle \( R \), from (6) we get condition (2). We use Theorem 1 to obtain the desired result.

3. Applications. We now consider applications of Theorem 3. In [7] is proved following theorem.

**Theorem 6.** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}) \) and exponent \( p(\cdot) \) is constant outside some large ball. Then operator \( M \) is bounded on \( L^{p(t)}(\mathbb{R}) \) if and only if (2) fulfilled for intervals.

The estimate (2) is necessary for boundedness of operator \( M \) in \( L^{p(t)}(\mathbb{R}) \). Combining the Littlewood – Paley type characterization of \( L^{p(t)}(\mathbb{R}) \) space (Theorem 3) with the previous theorem we can obtain the following corollary.

**Corollary.** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}) \) and exponent \( p(\cdot) \) is constant outside some large ball. Let \( \Delta \) be the dyadic decomposition of \( \mathbb{R} \). The following are equivalent:

1) \( p(\cdot) \in \mathcal{B}(\mathbb{R}); \)

2) there are constants \( c, C > 0 \) such that for all \( f \in L^{p(t)}(\mathbb{R}) \)
\[
c \|f\|_{p(t)} \leq \|G(f)\|_{p(t)} \leq C \|f\|_{p(t)}.
\]

Let \( \{f_k\} \) be a sequence of functions defined on \( \mathbb{R} \). By \( \sum_k f_k \in L^{p(t)}(\mathbb{R}) \) we mean the partial sums \( \sum_1^N f_k \) converge in \( L^{p(t)}(\mathbb{R}) \). We now will generalize Theorem 6 of Stein [13].
**Theorem 7.** Let \( p(\cdot) \in \mathcal{B}(\mathbb{R}) \), and \( S_k \) be any collection of lacunary partial sum operators. Then \( f \in L^{p(t)}(\mathbb{R}) \) if and only if \( \sum_k \varepsilon_k S_k f \) converges in \( L^{p(t)}(\mathbb{R}) \) for any sequence \( \{\varepsilon_k\} \in l^\infty \). Moreover, \( \|f\|_{p(t)} \) is equivalent to
\[
\sup_{\|\varepsilon_k\|_p=1} \left\| \sum_k \varepsilon_k S_k f \right\|_{p(t)}.
\]

**Proof.** Let \( p(\cdot) \in \mathcal{B}(\mathbb{R}) \) and \( S_k \) be any collection of lacunary partial sum. For \( f \in L^{p(t)}(\mathbb{R}) \) we have \( \left( \sum_k |S_k f|^2 \right)^{1/2} \in L^{p(t)}(\mathbb{R}) \). Note that if \( \{\varepsilon_k\} \in l^\infty \) then \( \left\| \sum_k |S_k f|^2 \right\|^{1/2} \leq \sup_{\|\varepsilon_k\|_p=1} \left\| \sum_k \varepsilon_k S_k f \right\|_{p(t)} \).

If \( N > M \) using Theorem 3,
\[
\left\| \sum_{k=M+1}^N \varepsilon_k S_k f \right\|_{p(t)} \leq C \left\| \varepsilon_k \right\|_p \left\| \sum_{k=M+1}^N |S_k f|^2 \right\|^{1/2}_{p(t)},
\]
which implies \( \left\{ \sum_k \varepsilon_k S_k f \right\}_1^\infty \) is Cauchy in \( L^{p(t)}(\mathbb{R}) \). From this follows
\[
\left\| \sum \varepsilon_k S_k f \right\|_{p(t)} \leq c \left\| \varepsilon_k \right\|_p \left\| \sum_k |S_k f|^2 \right\|^{1/2}_{p(t)}.
\]
Assume that \( S_k \) be any collection of lacunary partial sum operators and
\[
\sum_k \varepsilon_k S_k f \in L^{p(t)}(\mathbb{R}) \quad \text{for all } \{\varepsilon_k\} \in l^\infty.
\]
We will prove that \( \left( \sum_k |S_k f|^2 \right)^{1/2} \in L^{p(t)}(\mathbb{R}) \) and there exists a constant \( c > 0 \) independent of \( f \) such that
\[
\left\| \left( \sum_k |S_k f|^2 \right)^{1/2} \right\|_{p(t)} \leq c \sup_{\|\varepsilon_k\|_p=1} \left\| \sum_k \varepsilon_k S_k f \right\|_{p(t)}.
\]

First we will prove that \( M = \sup_{\|\varepsilon_k\|_p=1} \left\| \sum_k \varepsilon_k S_k f \right\|_{p(t)} \) is finite. Indeed, consider the collection of maps \( \{G_N : l^\infty \to L^{p(t)}(\mathbb{R})\} \) defined by \( G_N(\{\varepsilon_k\}) = \sum_{k=1}^N \varepsilon_k S_k f \). Let \( \varepsilon_k \to 0 \) as \( k \to \infty \). Each \( G_N \) is continuous and by assumption \( G_N(\{\varepsilon_k\}) \) converges to \( G(\{\varepsilon_k\}) \) in \( L^{p(t)}(\mathbb{R}) \) for each \( \{\varepsilon_k\} \in l^\infty \). Therefore \( \left\| G_N(\{\varepsilon_k\}) \right\|_{p(t)} \) is bounded for each \( \{\varepsilon_k\} \in l^\infty \). By the principle of uniform boundedness, there exists a constant \( c > 0 \) such that \( \|G_N\| \leq c \) for all \( N \). It follows that \( \|G\| \leq c \).

To proof of (7) will use Khinchine’s inequality for Rademacher series. Let \( \eta(t) = \text{sgn}(\sin 2^m \pi t) \), \( m = 0, 1, 2, \ldots \), be the Rademacher functions, and set \( f = \sum_{m=0}^\infty a_m \eta_m \). Then there are constants \( B_p \) and \( C_p \) such that for \( 0 < p < \infty \)
\[
B_p \left( \int_0^1 |f(t)|^p dt \right)^{1/p} \leq \left( \sum_{m=0}^\infty |a_m|^p \right)^{1/2} \leq C_p \left( \int_0^1 |f(t)|^p dt \right)^{1/p},
\]
(see [2]). Let \( \varepsilon_k = \eta(t) \) for \( 0 \leq t < 1 \). Then \( \left\| \varepsilon_k \right\|_p = 1 \) and
\[
M \geq \left\| \sum_k \eta_k(x) S_k f \right\|_{p(t)}.
\]

Using (8) for \( p = 1 \) and Fubini’s theorem we have

\[
\left\| \left( \sum_k |S_k f|^2 \right)^{1/2} \right\|_{p(t)} \leq C_1 \left\| \sum_k \eta_k(x) S_k f \right\|_{p(t)} \leq \]

\[
c C_1 \sup_{\|g\|_{L^p(\Omega)} \leq 1} \int_0^1 \left( \sum_k \eta_k(x) S_k f(z) \right) |g(z)| dz = \]

\[
c C_1 \sup_{\|g\|_{L^p(\Omega)} \leq 1} \int_0^1 \left( \sum_k \eta_k(x) S_k f(z) g(z) \right) dz \leq \]

\[
\leq c C_1 \int_0^1 \sup_{\|g\|_{L^p(\Omega)} \leq 1} \left( \sum_k \eta_k(x) S_k f(z) g(z) \right) dz \leq \]

\[
\leq c C_1 \sum_k \eta_k(x) S_k f \right\|_{p(t)} \leq c C_1 M,
\]

which proves (7). This completes the proof of Theorem 7.


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