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D. Simson (Nicolaus Copernicus Univ., Toruń, Poland)

TAME COMODULE TYPE, ROITER BOCSES, AND A GEOMETRY CONTEXT FOR COALGEBRAS[*]

РУЧНИЙ КОМОДУЛЬНИЙ ТИП, БОКСИ РОЙТЕРА І ГЕОМЕТРИЧНИЙ КОНТЕКСТ ДЛЯ КОАЛГЕБР

Dedicated to the memory of Andrey Vladimirovich Roiter

We study the class of coalgebras C of fc-tame comodule type introduced by the author. To any basic computable K-coalgebra C and a bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$, we associate a bimodule matrix problem $\mathbf{Mat}_C^v(\mathbb{H})$, an additive Roiter bocs $\mathbf{B}_v^C$, an affine algebraic K-variety $\mathbf{Comod}_v^C$, and an algebraic group action $\mathbf{G}_v^C \times \mathbf{Comod}_v^C \longrightarrow \mathbf{Comod}_v^C$. We study the fc-tame comodule type and the fc-wild comodule type of C by means of $\mathbf{Mat}_C^v(\mathbb{H})$, the category $\operatorname{rep}_K(\mathbf{B}_v^C)$ of K-linear representations of $\mathbf{B}_v^C$, and geometry of $\mathbf{G}_v^C$-orbits of $\mathbf{Comod}_v^C$. For computable coalgebras C over an algebraically closed field K, we give an alternative proof of the fc-tame-wild dichotomy theorem. A characterisation of fc-tameness of C is given in terms of geometry of $\mathbf{G}_v^C$-orbits of $\mathbf{Comod}_v$. In particular, we show that C is fc-tame of discrete comodule type if and only if the number of $\mathbf{G}_v^C$-orbits in $\mathbf{Comod}_v^C$ is finite, for every $v = (v'|v'') \in K_0(C) \times K_0(C)$.

Вивчено клас коалгебр C fc-ручного комодульного типу, що введений автором. Кожну базову зліченну K-коалгебру C та дводольний вектор $v = (v'|v'') \in K_0(C) \times K_0(C)$ пов'язано з бімодульною матричною задачею $\mathbf{Mat}_C^v(\mathbb{H})$, адитивними боксами Ройтера $\mathbf{B}_v^C$, афінним алгебраїчним K-різновидом $\mathbf{Comod}_v^C$ та алгебраїчним груповим оператором $\mathbf{G}_v^C \times \mathbf{Comod}_v^C \longrightarrow \mathbf{Comod}_v^C$. Дослідження fc-ручного та fc-дикого комодульних типів C проведено з використанням $\mathbf{Mat}_C^v(\mathbb{H})$, категорії $\operatorname{rep}_K(\mathbf{B}_v^C)$ K-лінійних зображень $\mathbf{B}_v^C$ та геометрії $\mathbf{G}_v^C$-орбіт $\mathbf{Comod}_v^C$. Для зліченних коалгебр C над алгебраїчно замкненим полем K наведено альтернативне доведення теореми про fc-ручну дику дихотомію. Характеризацію fc-ручної властивості для C подано через геометрію $\mathbf{G}_v^C$-орбіт $\mathbf{Comod}_v$. Показано, зокрема, що C належить до fc-ручного дискретного комодульного типу тоді і тільки тоді, коли кількість $\mathbf{G}_v^C$-орбіт в $\mathbf{Comod}_v^C$ скінченна для кожного $v = (v'|v'') \in K_0(C) \times K_0(C)$.

1. Introduction. Throughout this paper, we use the terminology and notation introduced in [21, 22, 28]. We fix a field K. Given a K-coalgebra C, we denote by C-Comod and C-comod the categories of left C-comodules and left C-comodules of finite K-dimension. We recall that C is said to be **basic** if the left C-comodule ${}_CC$ has a decomposition

$$ {}_CC = \bigoplus_{j \in I_C} E(j) \tag{1.1}$$

into a direct sum of pairwise non-isomorphic indecomposable injective left comodules $E(j)$. Throughout this paper, given $j \in I_C$, we denote by $S(j)$ the unique simple subcomodule of $E(j)$. Hence, $\operatorname{soc} C = \bigoplus_{j \in I_C} S(j)$. Following [26], the coalgebra C is called Hom-**computable** (or computable, in short) if $\dim_K \mathrm{Hom}_C(E(i), E(j))$ is finite, for all $i, j \in I_C$. A left C-comodule M is said to be **computable** if $\dim_K \mathrm{Hom}_C(M, E(j))$ is finite, for all $j \in I_C$.

Given a computable comodule M, we denote by $\mathbf{lgth} M = (\ell_j(M))_{j \in I_C} \in \mathbb{Z}^{I_C}$ the **composition length vector** of M, where $\ell_j(M) < \infty$ is the number of simple composition factors of M isomorphic to the simple comodule $S(j)$. It is clear that $\mathbf{lgth} M \in \mathbb{Z}^{(I_C)}$, if M is of finite K-dimension. We recall from [21] that the map

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$M \mapsto \mathbf{lgth} M$ defines a group isomorphism $\mathbf{lgth}\colon K_0(C) \xrightarrow{\simeq} \mathbb{Z}^{(I_C)}$, where $K_0(C) = K_0(C\text{-comod})$ is the Grothendieck group of the category $C\text{-comod}$ and $\mathbb{Z}^{(I_C)}$ is the direct sum of I_C copies of $\mathbb{Z}$.

We recall from [21] and [25] that an arbitrary K-coalgebra C is defined to be of **K-wild comodule type** (or **K-wild**, in short), if the category $C\text{-comod}$ of finite dimensional C-comodules is of K-wild representation type [18, 21, 23] in the sense that there exists an exact K-linear representation embedding $T\colon \text{mod}\Gamma_3(K) \longrightarrow C\text{-comod}$, where $\Gamma_3(K) = \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix}$. A K-coalgebra C is defined to be of **K-tame comodule type** [25] (or **K-tame**, in short), if the category $C\text{-comod}$ of finite dimensional left C-comodules is of K-tame representation type ([18], Section 14.4, [22]), that is, for every vector $v \in K_0(C) \cong \mathbb{Z}^{(I_C)}$, there exist C-$K[t]$-bicomodules $L^{(1)}, \ldots, L^{(r_v)}$, that are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable left C-comodules M with $\mathbf{lgth} M = v$ are of the form $M \cong L^{(s)} \otimes K^1_\lambda$, where $s \leq r_v$ and

$$K^1_\lambda = K[t]/(t - \lambda), \quad \lambda \in K. \tag{1.2}$$

Equivalently, there exist a non-zero polynomial $h(t) \in K[t]$ and C-$K[t]_h$-bicomodules $L^{(1)}, \ldots, L^{(r_v)}$, that are finitely generated free $K[t]_h$-modules, such that all but finitely many indecomposable left C-comodules M with $\mathbf{lgth} M = v$ are of the form $M \cong L^{(s)} \otimes K^1_\lambda$, where $s \leq r_v$ and $K[t]_h = K[t, h(t)^{-1}]$ is a rational K-algebra, see [7] or [18] (Section 14.4). In this case, we say that $L^{(1)}, \ldots, L^{(r_v)}$ form an **almost parametrising family** for the family $\text{ind}_v(C\text{-comod})$ of all indecomposable C-comodules M with $\mathbf{lgth} M = v$.

Here, by a C-$K[t]_h$-bicomodule ${}_C L_{K[t]_h}$ we mean a K-vector space L equipped with a left C-comodule structure and a right $K[t]_h$-module structure satisfying the obvious associativity conditions. In [28], a K-tame-wild dichotomy theorem is proved for left (or right) semiperfect coalgebras and for acyclic hereditary coalgebras over an algebraically closed field K by reducing the problem to the fc-tame-wild dichotomy theorem [28] (Theorem 2.11) and, consequently, to the tame-wild dichotomy theorem for finite dimensional K-algebras proved in [7] and [3].

The aim of the paper is to study the classes of coalgebras C of fc-tame comodule type and of fc-wild comodule type introduced in [28]. We recall that C is of of fc-tame comodule type if, for every coordinate vector $v = (v'|v'') \in K_0(C) \times K_0(C)$, the indecomposable finitely copresented C-comodules N such that $\mathbf{cdn}(N) = (v'|v'')$ form at most finitely many one-parameter families, see Section 2 for a precise definition.

We study mainly computable fc-tame and fc-wild basic coalgebras C by means of a bimodule matrix problem $\mathbf{Mat}^v_C(\mathbb{H})$, the additive category $\text{rep}_K(\mathbf{B}^C_v)$ of K-linear representations an additive Roiter bocs $\mathbf{B}^C_v$, an affine algebraic K-variety $\mathbf{Map}^C_v$, an algebraic (parabolic) group action $\mathbf{G}^C_v \times \mathbf{Map}^C_v \longrightarrow \mathbf{Map}^C_v$, and a Zariski open $\mathbf{G}^C_v$-invariant subset $\mathbf{Comod}^C_v \subseteq \mathbf{Map}^C_v$, associated to C and to any bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$. It is shown in Section 4 that there is a bijection between the $\mathbf{G}^C_v$-orbits of $\mathbf{Comod}^C_v$ and the isomorphism classes of comodules in $C\text{-Comod}_{fc}$. On this way, we get in Theorem 4.1 a characterisation of fc-tameness and fc-wildness of computable colagebras by means of $\mathbf{Mat}^v_C(\mathbb{H})$, the K-linear representations of the Roiter bocs $\mathbf{B}^C_v$, and in terms of geometry of the $\mathbf{G}^C_v$-orbits of $\mathbf{Comod}^C_v$.

We show in Section 4 that a computable colagebra C is fc-tame of discrete comodule type if and only if the number of $\mathbf{G}_v^C$-orbits in $\mathbf{Comod}_v^C$ is finite, for every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$. Moreover, we prove that a computable colagebra C is fc-tame if and only if, for every bipartite vector $v = = (v'|v'') \in K_0(C) \times K_0(C)$, there exists a constructible subset $\mathcal{C}(v)$ of the constructible set $\mathbf{indComod}_v^C \subseteq \mathbf{Comod}_v^C$ (defined by the indecomposable C-comodules) such that $\mathbf{G}_v^C * \mathcal{C}(v) = \mathbf{indComod}_v^C$ and $\dim \mathcal{C}(v) \leq 1$, see Theorem 4.1.

We also give an alternative proof of the following fc-tame-wild dichotomy theorem proved in [28]: *If C is a basic computable coalgebra over an algebraically closed field K then C is either fc-tame or fc-wild, and these two types are mutually exclusive.*

We prove it in Section 3 by a reduction to the tame-wild dichotomy theorem of Drozd [7] for representations of additive Roiter bocses, by applying the bimodule problems technique introduced in [5] and developed in [3, 4, 9, 17, 19, 20].

Throughout this paper we freely use the coalgebra representation theory notation and terminology introduced in [2, 16, 21, 22, 28]. The reader is referred to [1, 8, 10, 18] for representation theory terminology and notation, and to [3, 4, 7, 9, 13] for a background on the representation theory of bocses.

In particular, given a ring R with an identity element, we denote by Mod(R) the category of all unitary right R-modules, and by mod$(R) \supseteq \mathrm{fin}(R)$ the full subcategories of Mod(R) formed by the finitely generated R-modules and the finite dimensional R-modules, respectively. Given a K-coalgebra C and a left C-comodule M, we denote by $\mathrm{soc}\, M$ the **socle** of M, that is, the sum of all simple C-subcomodules of M.

A comodule N in C-Comod is said to be **socle-finite** if N is a subcomodule of a finite direct sum of indecomposable injective comodules, or equivalently, $\dim_K \mathrm{soc}\, N$ is finite. We say that N is **finitely copresented** if N admits a socle-finite injective copresentation, that is, an exact sequence $0 \longrightarrow N \longrightarrow E_0 \xrightarrow{\psi} E_1$ in $C\text{-Comod}$, where each of the comodules E_0 and E_1 is a finite direct sum of indecomposable injective comodules. If $E_0, E_1 \in \mathrm{add}(E)$, for some socle-finite injective C-comodule E, the comodule N is called **finitely E-copresented**. We denote by $C\text{-Comod}_{fc} \supseteq C\text{-Comod}_{fc}^E$ the full subcategories of $C\text{-Comod}$ whose objects are the finitely copresented comodules and finitely E-copresented comodules, respectively. Here by add(E) we mean the full additive subcategory of $C\text{-Comod}$ whose objects are finite direct sum of indecomposable injective comodules isomorphic to direct summands of E.

2. Preliminaries on fc-comodule types for coalgebras. Throughout we assume that K is an algebraically closed field and C is a basic K-coalgebra with a fixed decomposition (1.1). Following [28], given a finitely copresented C-comodule N in $C\text{-Comod}_{fc}$, with a minimal injective copresentation $0 \longrightarrow N \longrightarrow E_0^N \xrightarrow{g} E_1^N$, we define the **coordinate vector** of N to be the bipartite vector

$$\mathbf{cdn}(N) = (\mathbf{cdn}_0^N \mid \mathbf{cdn}_1^N) \in K_0(C) \times K_0(C) = \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}, \qquad (2.1)$$

where $\mathbf{cdn}_0^N = \mathbf{lgth}(\mathrm{soc}\, E_0^N)$ and $\mathbf{cdn}_1^N = \mathbf{lgth}(\mathrm{soc}\, E_1^N)$. We call a bipartite vector $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$ **proper** if $v' \neq 0$ and v'' has non-negative coordinates. Note that an indecomposable comodule N in $C\text{-Comod}_{fc}$ is injective if and only if the vector $\mathbf{cdn}(N)$ is proper and has the form $v = (e_j|v'')$, where $v'' = 0$ and e_j is the jth standard basis vector of $\mathbb{Z}^{(I_C)}$, for some $j \in I_C$.

The **support** of a bipartite vector $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$ is the finite subset $\mathbf{supp}(v) = \{j \in I_C;\, v'_j \neq 0$ or $v''_j \neq 0\}$ of I_C.

We recall from [28] that K-coalgebra C is defined to be of fc**-wild comodule type** (or fc**-wild**, in short), if the category $C\text{-Comod}_{fc}$ of finitely copresented C-comodules is of K-wild representation type [18, 23, 25] in the sense that there exists an exact K-linear representation embedding $T\colon \text{mod}\Gamma_3(K) \longrightarrow C\text{-Comod}_{fc}$, where $\Gamma_3(K) = \begin{bmatrix} K & K^3 \\ 0 & K \end{bmatrix}$.

A C-$K[t]_h$-bicomodule ${}_C L_{K[t]_h}$ is defined to be **finitely copresented** if there is a C-$K[t]_h$-bicomodule exact sequence $0 \to {}_C L_{K[t]_h} \to E' \otimes K[t]_h \xrightarrow{\psi} E'' \otimes K[t]_h$, such that E', E'' are socle-finite injective C-comodules. If E', E'' are finitely E-copresented, we call ${}_C L_{K[t]_h}$ **finitely** E**-copresented**.

A K-coalgebra C is defined to be of fc**-tame comodule type** (or fc**-tame**, in short), if the category $C\text{-Comod}_{fc}$ is of fc-tame representation type [18] (Section 14.4), that is, for every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C) \cong \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$, there exist C-$K[t]_h$-bicomodules $L^{(1)}, \ldots, L^{(r_v)}$, that are finitely copresented, such that all but finitely many indecomposable left C-comodules N in $C\text{-Comod}_{fc}$, with $\mathbf{cdn}(N) = v$, are of the form $N \cong L^{(s)} \otimes K^1_\lambda$, where $s \leq r_v$,

$$K^1_\lambda = K[t]/(t - \lambda),$$

and $\lambda \in K$. In this case, we say that $L^{(1)}, \ldots, L^{(r_v)}$ is a **finitely copresented almost parametrising family** for the family $\text{ind}_v(C\text{-Comod}_{fc})$ of all indecomposable C-comodules N with $\mathbf{cdn}(N) = v$. Obviously, one can restrict the definition to proper bipartite vectors $v = (v'|v'')$.

We recall from [28] that the **growth function** $\widehat{\boldsymbol{\mu}}^1_C\colon K_0(C) \times K_0(C) \longrightarrow \mathbb{N}$ of C associates to any bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$, the minimal number $\widehat{\boldsymbol{\mu}}^1_C(v) = r_v \geq 1$ of non-zero finitely copresented C-$K[t]_h$-bicomodules $L^{(1)}, \ldots, L^{(r_v)}$ forming an almost parametrising family for $\text{ind}_v(C\text{-Comod}_{fc})$. We set $\widehat{\boldsymbol{\mu}}^1_C(v) = r_v = 0$, if there is no such a family of bicomodules, that is, there is only a finite number of comodules N in $\text{ind}_v(C\text{-Comod}_{fc})$, up to isomorphism.

An fc-tame coalgebra C is defined to be of fc**-discrete comodule type** if $\widehat{\boldsymbol{\mu}}^1_C = 0$, that is, the number of the isomorphism classes of the indecomposable C-comodules N in $C\text{-Comod}_{fc}$ with $\mathbf{cdn}(N) = v$ is finite, for every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$.

By the main result in [28], the definition is left-right symmetric, for any computable coalgebra C. Note also that the K-tameness and K-wildness of a coalgebra are defined by means of finite dimensional comodules, but the fc-tame comodule type and fc-wild comodule type are defined by means of the category $C\text{-Comod}_{fc}$ of finitely copresented comodules that usually contains a lot of infinite dimensional comodules.

In the proof of our main results, we need the following construction that associates to any $v = (v'|v'') \in K_0(C) \times K_0(C)$ and any finitely copresented C-$K[t]_h$-bicomodule ${}_C L_{K[t]_h}$ a new one ${}_C \widetilde{L}_{K[t]_h}$, called fc-localising v-corrected C-$K[t]_h$-bicomodule.

Construction 2.1. Let C be a basic K-coalgebra with a decomposition (1.1), and let $v = (v'|v'') \in K_0(C) \times K_0(C) = \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$ be a proper bipartite vector.

Let $U_v = \mathbf{supp}(v) \subseteq I_C$ be the support of $v = (v'|v'')$. We call the socle-finite injective C-comodules

$$\mathbf{E}(v') = \bigoplus_{i \in I_C} E(i)^{v'_i} \quad \text{and} \quad \mathbf{E}(v'') = \bigoplus_{j \in I_C} E(j)^{v''_j} \tag{2.2}$$

the **standard injective C-comodules** with $\mathbf{cdn}\mathbf{E}(v') = (v'|0)$ and $\mathbf{cdn}\mathbf{E}(v'') = (v''|0)$.

We fix a rational K-algebra $S = K[t]_h$ and note that

$$E_v = E_{U_v} = \bigoplus_{a \in U_v} E(a) \tag{2.3}$$

is a socle-finite injective direct summand of ${}_CC$.

Assume that ${}_CL_S$ is a finitely copresented C-S-bicomodule with a fixed injective C-S-bicomodule copresentation

$$0 \longrightarrow {}_CL_S \longrightarrow E_0 \otimes S \xrightarrow{\psi} E_1 \otimes S \tag{2.4}$$

where E_0, E_1 are socle-finite injective comodules such that $\mathbf{E}(v') \subseteq E_0$ and $\mathbf{E}(v'') \subseteq$ $\subseteq E_1$.

We construct in three steps a finitely E_v-copresented C-S-bicomodule ${}_C\widetilde{L}_S$, called a **localising fc-correction** of ${}_CL_S$ as follows.

Step 1°. Fix a decomposition $E_0 = E'_0 \oplus E''_0$, where E'_0 is the injective envelope of the semisimple subcomodule $S(v)$ generated by the simple subcomodules of E_0 that are isomorphic to $S(j)$, with $j \in U_v$. Obviously, every simple subcomodule S of E''_0 has the form $S \cong S(a)$, where $a \notin U_v$.

Step 2°. Define a C-S-subbicomodule ${}_CL'_S$ of ${}_CL_S$ to be the kernel of the composite C-S-bicomodule homomorhism $E'_0 \otimes S \xrightarrow{u'_0 \otimes S} E_0 \otimes S \xrightarrow{\psi} E_1 \otimes S$, where $u'_0 \colon E'_0 \hookrightarrow E_0$ is the canonical embedding.

Step 3°. Let $e_v \colon C \to K$ be the idempotent of the algebra $C^* = \mathrm{Hom}_K(C, K)$ defined by the direct summand E_v of ${}_CC$. An **fc-localising correction** of ${}_CL_S$ is the C-S-bicomodule

$${}_C\widetilde{L}_S = e_vC\square_{e_vCe_v}[\mathrm{res}_{E_v}({}_CL')_S], \tag{2.5}$$

where $\mathrm{res}_{E_v} \colon C\text{-}\mathrm{Comod}_{fc} \longrightarrow e_vCe_v\text{-}\mathrm{Comod}_{fc}$ is the exact restriction functor and $e_vC\square_{e_vCe_v}(-) \colon e_vCe_v\text{-}\mathrm{Comod}_{fc} \longrightarrow C\text{-}\mathrm{Comod}_{fc}$ is the left exact cotensor product functor defined in [11] and [25] ((2.9), see also [29]).

The following fc-localising correction lemma is of importance.

Lemma 2.1. *Let K be an algebraically closed field, C a basic K-coalgebra with the decomposition* (1.1), *$v = (v'|v'') \in K_0(C) \times K_0(C) = \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$ a proper bipartite vector, $S = K[t]_h$, and ${}_CL_S$ a finitely copresented C-S-bicomodule with a fixed injective C-S-bicomodule copresentation* (2.4) *as in Construction* 2.1.

(a) *The C-S-bicomodule ${}_C\widetilde{L}_S$* (2.5) *has an injective C-S-bicomodule copresentation*

$$0 \longrightarrow {}_C\widetilde{L}_S \longrightarrow \widetilde{E}_0 \otimes S \xrightarrow{\widetilde{\psi}} \widetilde{E}_1 \otimes S \tag{2.6}$$

and the comodules $\widetilde{E}_0 = E'_0$, $\widetilde{E}_1$ lie in $\mathrm{add}(E_{U_v})$.

(b) *If N is an indecomposable comodule in C-*Comod_{fc} *such that* $\mathbf{cdn}(N) = v$ *and $N \cong {}_CL_S \otimes K^1_\lambda$, with $\lambda \in K$, then the restriction $\widehat{u}'_0 \colon {}_CL'_S \hookrightarrow {}_CL_S$ of the splitting monomorphism $u'_0 \otimes S \colon E'_0 \otimes S \hookrightarrow E_0 \otimes S$ to ${}_CL'_S$ is an embedding of C-S-bicomodules and induces isomorphisms ${}_C\widetilde{L}_S \otimes K^1_\lambda \cong {}_CL'_S \otimes K^1_\lambda \cong N$ of C-comodules.*

Proof. (a) By the construction, there are a decomposition $E_0 = E'_0 \oplus E''_0$ and exact sequence

$$0 \longrightarrow {}_C L'_S \longrightarrow E'_0 \otimes S \xrightarrow{\psi'} E_1 \otimes S$$

of C-S-bicomodules, where $\psi' = \psi \circ (u'_0 \otimes S)$ and $u'_0 = (\mathrm{id}_{E'_0}, 0)\colon E'_0 \hookrightarrow E_0 = E'_0 \oplus E''_0$ is the canonical embedding into the direct summand E'_0 of E_0. We recall from [11] and [25] (Section 2) that the restriction functor $\mathrm{res}_{E_v}\colon C\text{-}\mathrm{Comod}_{fc} \longrightarrow e_vCe_v\text{-}\mathrm{Comod}_{fc}$ is exact and the cotensor product functor $e_vC\Box_{e_vCe_v}(-)\colon e_vCe_v\text{-}\mathrm{Comod}_{fc} \longrightarrow C\text{-}\mathrm{Comod}_{fc}$ is left exact. Then we derive an exact sequence

$$0 \longrightarrow {}_C \widetilde{L}_S \longrightarrow \widetilde{E}_0 \otimes S \xrightarrow{\psi'} E_1^{\vee} \otimes S$$

of C-S-bicomodules, where $\widetilde{E}_0 = e_vC\Box_{e_vCe_v}\mathrm{res}_{E_v}(E'_0)$ and

$$E_1^{\vee} = e_vC\Box_{e_vCe_v}\mathrm{res}_{E_v}(E_1).$$

Since E_0 is a direct summand of E_{U_v}, then by [11] and [25] (Proposition 2.7 and Theorem 2.10), there is an isomorphism $\widetilde{E}_0 \cong E_0$, the socle of $\mathrm{res}_{E_v}(E_1)$ is a finite dimensional subcomodule of the coalgebra e_vCe_v and the socle of $E_1^{\vee} = e_vC\Box_{e_vCe_v}\mathrm{res}_{E_v}(E_1)$ is a finite direct sum of comodules $S(a)$, with $a \in U_v$. It follows that the injective envelope $\widetilde{E}_1 = E_C(E_1^{\vee})$ of the C-comodule $E_1^{\vee}$ lies in $\mathrm{add}(E_{U_v})$. Hence we get the exact sequence (2.6) and (a) follows.

(b) The canonical embedding $u'_0 = (\mathrm{id}_{E'_0}, 0)\colon E'_0 \hookrightarrow E_0 = E'_0 \oplus E''_0$ into the direct summand E'_0 of E_0 induces the commutative diagram of C-S-bicomodules

$$\begin{tikzcd}
0 \arrow[r] & {}_C L_S \arrow[r] & (E'_0 \oplus E''_0) \otimes S \arrow[r, "\psi"] & E_1 \otimes S \\
0 \arrow[r] & {}_C L'_S \arrow[r] \arrow[u, "\widehat{u}'_0"] & E'_0 \otimes S \arrow[r, "\psi'"] \arrow[u, "u'_0 \otimes S"] & E_1 \otimes S \arrow[u, "\mathrm{id}_{E_1} \otimes S"]
\end{tikzcd}$$

with exact rows, where $\widehat{u}'_0$ is the restriction of the monomorphism $u'_0 \otimes S\colon E'_0 \otimes S \hookrightarrow E_0 \otimes S$ to ${}_C L'_S$. Obviously, $\widehat{u}'_0$ is an embedding of C-S-bicomodules.

Let N be an indecomposable comodule in C-Comod_{fc} such that $\mathbf{cdn}(N) = v = (v'|v'')$ and $N \cong {}_C L_S \otimes K^1_\lambda$, with $\lambda \in K$. Then N has a minimal injective copresentation $0 \longrightarrow N \longrightarrow \mathbf{E}(v') \xrightarrow{g} \mathbf{E}(v'')$. Recall that $\mathbf{cdn}\mathbf{E}(v') = (v'|0)$ and $\mathbf{cdn}\mathbf{E}(v'') = (v''|0)$. Then we get a commutative diagram of C-comodules in C-Comod_{fc}

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & \mathbf{E}(v') \arrow[r, "g"] \arrow[d, "f_0"'] & \mathbf{E}(v'') \arrow[d, "f_1"'] \\
0 \arrow[r] & {}_C L \otimes_S K^1_\lambda \arrow[r] & (E'_0 \oplus E''_0) \otimes K^1_\lambda \arrow[r, "\psi \otimes \mathrm{id}"] & E_1 \otimes K^1_\lambda \\
0 \arrow[r] & {}_C L' \otimes_S K^1_\lambda \arrow[r] \arrow[u, "\widehat{u}'_0"] & E'_0 \otimes K^1_\lambda \arrow[r, "\psi'"] \arrow[u, "u'_0 \otimes \mathrm{id}"] & E_1 \otimes K^1_\lambda \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

with exact rows. Since the upper row is a minimal injective copresentation of N, then f_0 and f_1 are monomorphisms, and f_0 has a factorisation $\mathbf{E}(v') \xrightarrow{f'_0} E'_0 \otimes K^1_\lambda \xrightarrow{u'_0 \otimes S} E'_0 \otimes K^1_\lambda$ through the subcomodule $E'_0 \otimes K^1_\lambda$ of $E_0 \otimes K^1_\lambda$, because the socle of $E''_0 \otimes K^1_\lambda$ contains no simple comodules $S(a)$, with $a \in U_v$. It follows that f'_0 restricts to a monomorphism

$\widehat{f}'_0\colon N \to {}_CL'\otimes_S K^1_\lambda$ such that the composite map $N \xrightarrow{\widehat{f}_0} {}_CL'\otimes_S K^1_\lambda \xrightarrow{\widehat{u}'_0} {}_CL\otimes_S K^1_\lambda$ is an isomorphism. Consequently, $\widehat{f}'_0\colon N \to {}_CL'\otimes_S K^1_\lambda$ is an isomorphism of C-comodules. Hence, in the notation of Construction 2.1, we get the isomorphisms

$$ {}_CL\otimes_S K^1_\lambda = [e_vC\square_{e_vCe_v}\mathrm{res}_{E_v}({}_CL')]\otimes_S K^1_\lambda \cong $$

$$ \cong e_vC\square_{e_vCe_v}[\mathrm{res}_{E_v}({}_CL'\otimes_S K^1_\lambda)] \cong e_vC\square_{e_vCe_v}[\mathrm{res}_{E_v}(N)] \cong N $$

of C-comodules, because N is finitely E_{U_v}-copresented and [25] (Theorem 2.10 (d)) applies to N.

The lemma is proved.

3. fc-Tameness, fc-wildness and Roiter bocses for coalgebras.

We show in this section how the study of fc-tame and fc-wild coalgebras can be reduced to the study of bimodule matrix problems in the sense of Drozd [5], to representations of additive Roiter bocses [3 – 7], and to the study of propartite modules over a class of bipartite algebras [19, 20].

To formulate our main results on fc-tame and fc-wild computable coalgebras, we recall some notation, see [25] and [26]. Given a socle-finite injective direct summand

$$ E = E_U = \bigoplus_{u\in U} E(u) \tag{3.1} $$

of ${}_CC = \bigoplus_{j\in I_C} E(j)$, with a finite subset U of I_C, we define the category $C\text{-Comod}^{E_U}_{fc}$ to be fc-**tame** if for every bipartite vector $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U$, there is a finitely E-copresented almost parametrising family for $\mathrm{ind}_v(C\text{-Comod}^{E_U}_{fc})$.

We start with the following fc-parametrisation correction lemma.

Lemma 3.1. *Let K be an algebraically closed field, C a basic K-coalgebra with the decomposition* (1.1), *and $E = E_U$ a socle-finite injective direct summand* (3.1) *of ${}_CC$.*

(a) *If C is fc-tame then the category $C\text{-Comod}^{E_U}_{fc}$ is fc-tame.*

(b) *If $v = (v'|v'') \in K_0(C)\times K_0(C) = \mathbb{Z}^{(I_C)}\times\mathbb{Z}^{(I_C)}$ is a proper bipartite vector, $S = K[t]_h$, and $L^{(1)},\ldots,L^{(r_v)}$ is a finitely copresented almost parametrising family of C-S-bicomodules for $\mathrm{ind}_v(C\text{-Comod}^{E_U}_{fc})$ then the fc-localising v-corrected C-S-bicomodules $\widetilde{L}^{(1)},\ldots,\widetilde{L}^{(r_v)}$ in the sense of Construction* 2.1 *form a finitely E_U-copresented almost parametrising family for* $\mathrm{ind}_v(C\text{-Comod}^{E_U}_{fc})$.

Proof. It is sufficient to prove (b), because (a) is a direct consequence of (b). Assume that $v = (v'|v'') \in K_0(C)\times K_0(C) = \mathbb{Z}^{(I_C)}\times\mathbb{Z}^{(I_C)}$ is a proper bipartite vector and $L^{(1)},\ldots,L^{(r_v)}$ is a finitely copresented almost parametrising family for $\mathrm{ind}_v(C\text{-Comod}^{E_U}_{fc})$. Assume that $r_v \geq 0$ is a minimal number of such non-zero bimodules. If $r_v = 0$ then there is nothing to prove, because the number of the isomorphism classes of indecomposable comodules in $\mathrm{ind}_v(C\text{-Comod}^{E_U}_{fc})$ is finite.

Assume that $r_v \geq 1$. Then, for each $1 \leq j \leq r_v$, there is an indecomposable comodule N such that $\mathbf{cdn}(N) = v$ and $N \cong L^{(j)}\otimes_S K^1_{\lambda(j)}$, for some $\lambda(j)\in K$. Then N has a minimal injective copresentation $0 \longrightarrow N \longrightarrow \mathbf{E}(v') \xrightarrow{g} \mathbf{E}(v')$.

Since ${}_CL^{(j)}_S$ is a finitely copresented C-S-bicomodule then it has an injective C-S-bicomodule copresentation

$$ 0 \longrightarrow {}_CL^{(j)}_S \longrightarrow E^{(j)}_0\otimes S \xrightarrow{\psi^{(j)}} E^{(j)}_1\otimes S $$

where $E_0^{(j)}, E_1^{(j)}$ are socle-finite injective C-comodules. Since $N \cong L^{(j)} \otimes_S K^1_{\lambda(j)}$ then there are C-comodule monomorphisms $\mathbf{E}(v') \subseteq E_0$ and $\mathbf{E}(v'') \subseteq E_1$, because the sequence

$$0 \longrightarrow {}_C L^{(j)} \otimes_S K^1_{\lambda(j)} \longrightarrow E_0^{(j)} \otimes K^1_{\lambda(j)} \xrightarrow{\widehat{\psi}^{(j)}} E_1^{(j)} \otimes K^1_{\lambda(j)}$$

induced by the previous one is exact and is a socle-finite injective copresentation of $N \cong L^{(j)} \otimes_S K^1_{\lambda(j)}$. Then the Construction 2.1 applies to ${}_C L_S^{(j)}$, for $j = 1, \ldots, r_v$.

By applying Lemma 2.1 to the finitely copresented C-S-bicomodule $L^{(j)}$ we get a finitely E_U-copresented C-S-bicomodule $\widetilde{L}^{(j)}$ such that the fc-localising v-corrected C-S-bicomodules $\widetilde{L}^{(1)}, \ldots, \widetilde{L}^{(r_v)}$ form a finitely E_U-copresented almost parametrising family for $\operatorname{ind}_v(C\text{-}\mathrm{Comod}_{fc}^{E_U})$.

The lemma is proved.

Following [25, 26, 28] given a socle-finite injective direct summand $E = E_U$ (3.1), we consider the K-algebra

$$R_E = \mathrm{End}_C E = \bigoplus_{u \in U} e_u R_E, \tag{3.2}$$

where $e_u R_E = \mathrm{Hom}_C(E, E(u))$ is viewed as an indecomposable projective right ideal of R_E and e_u is the primitive idempotent of R_E defined by the summand $E(u)$ of E. Since the set U is finite then $\sum_{u \in U} e_u$ is the identity of R_E. It is easy to see that the Jacobson radical $J(R_E)$ of R_E has the form $J(R_E) = \{h \in \mathrm{End}_C E; h(\mathrm{soc}\, E) = 0\}$. It follows that the algebra R_E is semiperfect and pseudocompact with respect to the K-linear topology defined by the left ideals $\mathfrak{a}_\beta = \mathrm{Hom}_C(E/V_\beta, E) \subseteq R_E$, where $\{V_\beta\}_\beta$ is the directed set of all finite dimensional subcomodules of E. Since $E = \bigcup_\beta V_\beta$, then there are isomorphisms

$$R_E = \mathrm{End}_C E \cong \varprojlim_\beta \mathrm{Hom}_C(V_\beta, E) \cong \varprojlim_\beta R_E/\mathfrak{a}_\beta. \tag{3.3}$$

Following [3, 7, 28], we consider the homomorphism category $\mathcal{M}ap_1(E)$ whose objects are the triples (E_0, E_1, ψ) with E_0, E_1 comodules in $\mathrm{add}(E)$ and $\psi\colon E_0 \longrightarrow E_1$ a homomorphism of C-comodules such that $\psi(\mathrm{soc}\, E_0) = 0$; and whose morphisms are the pairs (f_0, f_1), where $f_0\colon E_0 \longrightarrow E_0'$, $f_1\colon E_1 \longrightarrow E_1'$ and $\psi' \circ f_0 = f_1 \circ \psi$. Denote by $\mathcal{M}ap_2(E)$ the full subcategory of $\mathcal{M}ap_1(E)$ whose objects are the triples (E_0, E_1, ψ) such that $\mathrm{soc}\, \mathrm{Im}\, \psi = \mathrm{soc}\, E_1$. or equivalently, $\psi\colon E_0 \longrightarrow E_1$ has no non-zero direct summand of the form $0 \longrightarrow E''$. We define the **coordinate vector** of (E_0, E_1, ψ) to be the bipartite vector

$$\mathbf{cdn}(E_0, E_1, \psi) = (\mathbf{lgth}(\mathrm{soc}\, E_0) | \mathbf{lgth}(\mathrm{soc}\, E_1)) \in \mathbb{Z}^U \times \mathbb{Z}^U = K_0(R_E) \times K_0(R_E). \tag{3.4}$$

Following [7], [3] (Section 6) and [28], we denote by $\mathcal{P}_1(R_E^{op})$ the category whose objects are the triples (P_1, P_0, ϕ) with P_0, P_1 finitely generated projective left R_E-modules and $\phi\colon P_1 \longrightarrow \mathrm{rad}(P_0) = P_0 J(R_E)$ a homomorphism of left R_E-modules; and whose morphisms are the pairs (g_1, g_0), where $g_0\colon P_0 \longrightarrow P_0'$, $g_1\colon P_1 \longrightarrow P_1'$ and $\phi' \circ g_1 = g_0 \circ \phi$. Denote by $\mathcal{P}_2(R_E^{op})$ the full subcategory of $\mathcal{P}_1(R_E^{op})$ whose objects are the triples (P_1, P_0, ϕ) with $\mathrm{Ker}\phi \subseteq \mathrm{rad}(P_1)$. or equivalently, $\phi\colon P_1 \longrightarrow P_0$ has no non-zero direct summand of the form $P \longrightarrow 0$. We define the **coordinate vector** of (P_1, P_0, ϕ) to be the bipartite vector

$$\mathbf{cdn}(P_1, P_0, \phi) = (\mathbf{lgth}(\mathrm{top}P_1) | \mathbf{lgth}(\mathrm{top}P_0)) \in \mathbb{Z}^U \times \mathbb{Z}^U = K_0(R_E^{op}) \times K_0(R_E^{op}).$$

We call $\mathbf{cdn}(\mathrm{Coker}\phi) = \mathbf{cdn}(P_1, P_0, \phi)$ the **coordinate vector** of the R_E-module $\mathrm{Coker}\phi$.

We start with the following important result. Here we freely use the terminology and notation introduced in [3] (Section 6), [7], and [28].

Theorem 3.1. *Let K be an algebraically closed field, C a basic K-coalgebra with the decomposition* (1.1), *E a socle-finite injective direct summand* (3.1) *of ${}_CC$, and assume that the K-algebra $R_E = \mathrm{End}_C E$* (3.2) *is finite-dimensional. Let $\mathbf{B}_E = (A, {}_AV_A)$ be the additive Roiter bocs associated to the K-algebra R_E^{op} in* [3] (*Proposition* 6.1). *Then there is a commutative diagram*

$$\begin{tikzcd} \mathcal{M}ap_1(E) \arrow[r, "H_E", "\simeq"'] \arrow[d, "\mathbf{ker}_E"'] & \mathcal{P}_1(R_E^{op}) \arrow[d, "\mathbf{cok}_E"'] & \mathrm{rep}_K(\mathbf{B}_E) \arrow[l, "G"', "\simeq"] \\ C\text{-}\mathrm{Comod}_{fc}^E \arrow[r, "h_E^{\bullet}", "\simeq"'] & \mathrm{mod}(R_E^{op}), & \end{tikzcd} \tag{3.5}$$

where H_E and $h_E^{\bullet} = \mathrm{Hom}_C(\bullet, E)$ are K-linear contravariant equivalences of categories, G is a covariant K-linear equivalence of categories, $h_E^{\bullet}$ is an exact functor, $\mathbf{ker}_E(E_0, E_1, \psi) = \mathrm{Ker}\psi$, $\mathbf{cok}_E(P_1, P_0, \phi) = \mathrm{Coker}\phi$, *and the following conditions are satisfied.*

(a) *The functors $\mathbf{cok}_E$ and $\mathbf{ker}_E$ are full dense and restrict to the representation equivalences $\mathbf{ker}_E \colon \mathcal{M}ap_2(E) \longrightarrow C\text{-}\mathrm{Comod}_{fc}^E$ and $\mathbf{cok}_E \colon \mathcal{P}_2(R_E^{op}) \longrightarrow \mathrm{mod}(R_E^{op})$. The right-hand part in the diagram is defined as in* [7] (*Section* 5) *and* [3, *p.* 476, 478], *with R_E^{op}, G, $\mathbf{cok}_E$ and Λ, Ξ, $\mathbf{cok}$ interchanged.*

(b) *If N is an indecomposable comodule in $C\text{-}\mathrm{Comod}_{fc}^E$ then there exists a unique, up to isomorphism, indecomposable object (E_0, E_1, ψ) in $\mathcal{M}ap_1(E)$ such that $\mathbf{ker}_E(E_0, E_1, \psi) \cong N$. In this case (E_0, E_1, ψ) lies in $\mathcal{M}ap_2(E)$ and*

$$\mathbf{cdn}(N) = \mathbf{cdn}(E_0, E_1, \psi) = \sigma(\mathbf{cdn} H_E(E_0, E_1, \psi)) = \underline{\mathrm{dim}} G^{-1} H_E(E_0, E_1, \psi)),$$

where we set $\sigma(v'|v'') = (v''|v')$.

(c) *If the category $C\text{-}\mathrm{Comod}_{fc}^E$ is not of K-wild representation type* (*shortly, K-wild*) *then the additive category $\mathrm{rep}_K(\mathbf{B}_E)$ of the K-linear representations of $\mathbf{B}_E$ is not wild and, given a non-negative vector*

$$v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U \subseteq \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} \cong K_0(C) \times K_0(C),$$

there exist minimal bocses $\mathbf{B}_1, \ldots, \mathbf{B}_n$, with $\mathbf{B}_i = (B_i, W_i)$, finitely E-copresented C-B_i-bicomodules T_i and full functors $F_i \colon \mathrm{rep}_K(\mathbf{B}_i) \longrightarrow C\text{-}\mathrm{Comod}_{fc}$ which reflect isomorphisms such that

(c_1) *$F_i(X) = T_i \otimes_{B_i} X$, for all representations X in $\mathrm{rep}_K(\mathbf{B}_i)$,*

(c_2) *every indecomposable comodule N in $C\text{-}\mathrm{Comod}_{fc}^E$, with $\mathbf{cdn}(N) = v$, is isomorphic to $F_i(X)$, for some i and some representation X in $\mathrm{rep}_K(\mathbf{B}_i)$,* (c$_3$) *the functors* F_i *induce group homomorphisms* $K_0(\mathbf{B}_i) \longrightarrow \mathbb{Z}^U \subseteq \mathbb{Z}^{(I_C)} \cong$ $\cong K_0(C)$ *taking the dimension vector* $\underline{\dim}(X)$ *of* X *to* $\mathbf{cdn}F_i(X)$.

Proof. By our assumption, the injective comodule $E = E_U$ is socle-finite and the K-algebra $R_E = \mathrm{End}_C E$ is finite dimensional. Let $D\colon \mathrm{mod} R_E^{op} \longrightarrow \mathrm{mod} R_E$ be the standard duality given by $L \mapsto D(L) = \mathrm{Hom}_K(L, K)$, for any L in $\mathrm{mod} R_E^{op}$. We define the contravariant functor $h_E^{\bullet}$ by setting $h_E^{(-)} = \mathrm{Hom}_C(-, E)$. Since E is injective, the functor $h_E^{\bullet}$ is exact and, by [26] (Proposition 2.13), $h_E^{\bullet}$ is an equivalence of categories such that $(\mathbf{lgth}N)_u = (\underline{\dim} h_E^N)_u = \dim_K(h_E^N)e_u$, for any comodule N in $C\text{-}\mathrm{Comod}_{fc}^{E_U}$ and all $u \in U$, where $\underline{\dim} N'$ is the dimension vector of a left R_U-module N'. This means that $\mathrm{res}_U(\mathbf{lgth}N) = \underline{\dim} h_E^N$, for any comodule N in $C\text{-}\mathrm{Comod}_{fc}^{E_U}$, where $\mathrm{res}_U\colon \mathbb{Z}^{(I_E)} \longrightarrow \mathbb{Z}^U$ is the restriction homomorphism.

We define the functor H_E on objects by setting $H_E(E_0, E_1, \psi) = (h_E^{E_1}, h_E^{E_0}, h_E^{\psi})$, and on morphisms by setting $H_E(f_0, f_1) = (h_E^{f_1}, h_E^{f_0})$. A direct calculation shows that $(h_E^{E_1}, h_E^{E_0}, h_E^{\psi})$ belongs to $\mathcal{P}_1(R_E^{op})$, if $(E_0, E_1, \psi) \in \mathcal{M}ap_1(E)$ and that H_E is well defined.

For a purpose of next steps of the proof (and in order to see a nature of $\mathcal{M}ap_1(E)$ as the bimodule problem in the sense of Drozd [5], see also [4, 17]), we give a different detailed proof of the above fact.

Let $\mathbb{K} = \mathrm{add}(E)$ be the full additive subcategory of C-Comod formed by finite direct sums of the injective C-comodules $E(u)$, with $u \in U$, and let $\mathbb{H} = \mathbb{H}^E$ be the $\mathbb{K}$-$\mathbb{K}$-bimodule $\mathbb{H}(-, \cdot) = \mathbb{H}^E(-, \cdot)\colon \mathbb{K}^{op} \times \mathbb{K} \longrightarrow \mathrm{mod} K$ defined by the formula

$$\mathbb{H}(E', E'') = \{g \in \mathrm{Hom}_C(E', E''); \psi(\mathrm{soc}\, E') = 0\} \subseteq \mathrm{Hom}_C(E', E''),$$

with $E', E'' \in \mathbb{K}$. Note that $\mathbb{H}(E, E) = \{\psi \in \mathrm{End}_C E;\ \psi(\mathrm{soc}\, E) = 0\} = J(R_E)$ is the Jacobson radical of the algebra R_E.

We construct H_E as the composite functor

$$\mathcal{M}ap_1(E) \xrightarrow[\simeq]{H'} \mathbf{Mat}({}_{\mathbb{K}}\mathbb{H}_{\mathbb{K}}^{E}) \xrightarrow[\simeq]{H'} \mathcal{P}_1(R_E^{op}), \tag{3.6}$$

where $\mathbf{Mat}({}_{\mathbb{K}}\mathbb{H}_{\mathbb{K}}^{E})$ is the additive K-category of ${}_{\mathbb{K}}\mathbb{H}_{\mathbb{K}}^{E}$-matrices in the sense of Drozd [5], see also [4], [10], [18] (Chapter 17), [20] (Section 2) for details. Recall that the objects of $\mathbf{Mat}({}_{\mathbb{K}}\mathbb{H}_{\mathbb{K}}^{E})$ are the triples (E', E'', ψ), where $E', E'' \in \mathrm{ob}\mathbb{K}$ and $\psi \in \mathbb{H}(E', E'')$, and morphisms are defined in a natural way.

The functor H' is defined by attaching to any object (E_0, E_1, ψ) of $\mathcal{M}ap_1(E)$, with $\psi \in \mathrm{Hom}_C(E_0, E_1) = \mathbb{H}(E_0, E_1)$ and $E_0, E_1 \in \mathbb{K}$, the triple $H'(E_0, E_1, \psi) =$ $= (E_0, E_1, \psi)$, viewed as an object of $\mathbf{Mat}({}_{\mathbb{K}}\mathbb{H}_{\mathbb{K}}^{E})$. Given a morphism (f_0, f_1): $(E_0, E_1, \psi) \longrightarrow (E_0', E_1', \psi')$, we set $H'(f_0, f_1) = (f_0, f_1)$. It is easy to see that H' is a K-linear equivalence of categories.

Now we construct the functor H''. In the notation of [20] (Section 2), we denote by R_E-pr the category of finitely generated projective left R_E-modules and we define the Nakayama equivalence $\omega\colon \mathbb{K} \xrightarrow{\simeq} (R_E\text{-pr})^{op}$ that associates, to any object x of $\mathbb{K}$, the finitely generated projective left R_E-module $\omega(x) = h_E^x = \mathrm{Hom}_C(x, E)$. Hence, by applying the formula (2.9) in [20] to $\mathbf{K} = \mathbf{L} = \mathbb{K} = \mathrm{add}(E)$ and the bimodule $\mathbf{M} = \mathbb{H}$, we conclude that, for any pair $x = E', y = E''$ of objects in $\mathbb{K}$, the (contravariant!) functor ω induces the natural isomorphisms

$$\mathbb{H}(E', E'') = \mathbb{H}(x, y) \cong \mathrm{Hom}_{R_E}(h_E^y, \mathbb{H}(x, E)) \cong$$

$$\cong \mathrm{Hom}_{R_E}(h_E^y, \mathbb{H}(E,E) \otimes_{R_E} h_E^x) \cong$$

$$\cong \mathrm{Hom}_{R_E}(h_E^y, J(R_E) \otimes_{R_E} h_E^x) \cong$$

$$\cong \mathrm{Hom}_{R_E}(h_E^y, \mathrm{rad} h_E^x) = \mathrm{Hom}_{R_E}(h_E^{E''}, \mathrm{rad} h_E^{E'}) \cong$$

$$\cong \mathrm{Hom}_{R_E}(J(R_E)^+ \otimes_{R_E} h_E^y, h_E^x) \cong$$

$$\cong \mathrm{Hom}_{R_E}(J(R_E)^+ \otimes_{R_E} h_E^{E''}, h_E^{E'}), \tag{3.7}$$

where $J(R_E)^+ = \mathrm{Hom}_{R_E}(J(R_E), R_E)$ is viewed as an R_E-R_E-bimodule.

Hence, if (E_0, E_1, ψ) is an object of $\mathcal{M}ap_1(E)$ (or of $\mathbf{Mat}({}_{\mathbb{K}}\mathbb{H}_{\mathbb{K}})$) then $\psi \in \mathbb{H}(E', E'')$ and its image $\widehat{\psi}\colon h_E^{E''} \longrightarrow \mathrm{rad} h_E^{E'}$ under the composite isomorphism (3.7) is such that $h_E^{\psi} = u \cdot \widehat{\psi}$, where $u : \mathrm{rad} h_E^{E'} \hookrightarrow h_E^{E'}$ is the embedding. It follows that $(h_E^{E''}, h_E^{E'}, h_E^{\psi})$ lies in $\mathcal{P}_1(R_E)$ if and only if (E_0, E_1, ψ) lies in $\mathcal{M}ap_1(E)$. We define H'' (and H_E) on objects (E_0, E_1, ψ) by setting

$$H''(E_0, E_1, \psi) = H_E(E_0, E_1, \psi) = (h_E^{E''}, h_E^{E'}, h_E^{\psi}),$$

and on morphisms (f_0, f_1) by $H''(f_0, f_1) = H^E(f_0, f_1) = \big(h_E^{f_1}, h_E^{f_0}\big)$. Obviously, $H = H'' \circ H'$. Since, up to isomorphism, all objects of $\mathcal{P}_1(R_E)$ are of the form $(h_E^{E''}, h_E^{E'}, h_E^{\psi})$, with $(E_0, E_1, \psi) \in \mathcal{M}ap_1(E)$, then the functors H'' and H_E are equivalences of categories making the square in (3.5) commutative.

(a) The fact that the functors **ker** and **cok** are full and dense follows immediately form the definitions. It is easy to see that (P_1, P_0, ϕ) is an object of $\mathcal{P}_1(R_E)$ if and only if $P_1 \xrightarrow{\phi} P_0 \to \mathrm{Coker}\phi \to 0$ is a minimal projective presentation of $\mathrm{Coker}\psi$ in $\mathrm{mod}(R_E^{op})$. Analogously, (E_0, E_1, ψ) is an object of $\mathcal{M}ap_1(E)$ if and only if $0 \to \mathrm{Ker}\psi \to E_0 \xrightarrow{\psi} E_1$ is a minimal injective E-copresentation of $\mathrm{Ker}\psi$. Hence easily follows that the functors $\mathbf{cok}_E$ and $\mathbf{ker}_E$ restrict to the representation equivalences $\mathbf{ker}_E\colon \mathcal{M}ap_2(E) \longrightarrow C\text{-}\mathrm{Comod}_{fc}^E$ and $\mathbf{cok}_E\colon \mathcal{P}_2(R_E^{op}) \longrightarrow \mathrm{mod}(R_E^{op})$. The remaining statements in (a) follow from the definitions and [3] (Section 6).

(b) Let N be an indecomposable comodule in $C\text{-}\mathrm{Comod}_{fc}^E$. Then N admits a minimal injective E-copresentation $0 \to N \to E_0 \xrightarrow{\psi} E_1$ in C-Comod, with $E_0, E_1 \in \mathrm{add}(E)$ and, therefore, (E_0, E_1, ψ) is an object of $\mathcal{M}ap_1(E)$. It follows that

$$H_E(E_0, E_1, \psi) = (h_E^{E_1}, h_E^{E_0}, h_E^{\psi}) \in \mathcal{P}_2(R_E)$$

and, hence, $h_E^{E_1} \xrightarrow{h_E^{\psi}} h_E^{E_0} \longrightarrow h_E^N \to 0$ is a minimal projective presentation of h_E^N in $\mathrm{mod} R_E^{op}$. Hence the equalities $\mathbf{cdn}(N) = \mathbf{cdn}(E_0, E_1, \psi) = \sigma(\mathbf{cdn} H_E(E_0, E_1, \psi))$ easily follow. The equality $\sigma(\mathbf{cdn} H_E(E_0, E_1, \psi)) = \underline{\mathrm{dim}} G^{-1} H_E(E_0, E_1, \psi))$ is proved in [7] (Section 5) and [3] (Section 6).

(c) First we show that the functor G in (3.5) is the composite functor

$$\mathcal{P}_1(R_E^{op}) \xleftarrow[\simeq]{G'} \widehat{R}_E\text{-}\mathrm{mod}_{pr}^{pr} \xleftarrow[\simeq]{G''} \mathrm{rep}_K(\mathbf{B}_E), \tag{3.8}$$

where $\widehat{R}_E\text{-}\mathrm{mod}_{pr}^{pr}$ is the additive K-category of finite dimensional propartite left modules over the finite dimensional bipartite K-algebra

$$\widehat{R}_E = \begin{bmatrix} R_E & J(R_E)^+ \\ 0 & R_E \end{bmatrix} \tag{3.9}$$

in the sense of [20], with $J(R_E)^+ = \mathrm{Hom}_{R_E}(J(R_E), R_E)$. First we note that if $X = (X', X'', \xi\colon J(R_E)^+ \otimes_{R_E} X' \longrightarrow X'')$, is a propartite left $\widehat{R}_E$-module then, up to isomorphism, the projective left R_E-modules X', X'' have the forms $X' = h_E^{E''}$, $X'' = h_E^{E'}$, where $E', E'' \in \mathrm{add}(E)$. Then, in view of the isomorphisms

$$\mathrm{Hom}_{R_E}(J(R_E)^+ \otimes_{R_E} h_E^{E''}, h_E^{E'}) \cong$$

$$\cong \mathrm{Hom}_{R_E}(h_E^{E''}, J(R_E) \otimes_{R_E} h_E^{E'}) \cong \mathrm{Hom}_{R_E}(h_E^{E''}, \mathrm{rad} h_E^{E'})$$

given in (3.7), we can view X as the triple $X = (X', X'', \widetilde{\xi})$, where $\widetilde{\xi} = u \circ \overline{\xi}$ is the composition $h_E^{E''} \xrightarrow{\overline{\xi}} \mathrm{rad} h_E^{E'} \xrightarrow{u} h_E^{E'}$ of the image $\overline{\xi}$ of $\xi \in \mathrm{Hom}_{R_E}(J(R_E)^+ \otimes_{R_E} h_E^{E''}, h_E^{E'})$, under the composite isomorphism, with the canonical embedding u. In other words, the triple $G'(X) = (X', X'', \phi) = (h_E^{E''}, h_E^{E'}, \phi)$ is an object of $\mathcal{P}_1(R_E^{op})$. This defines the equivalence G', and we set $G'' = G \circ (G')^{-1}$. It is clear that the functor $T_K = (G'')^{-1}$ is the equivalence $T_K\colon \widehat{R}_E\text{-mod}_{pr}^{pr} \xrightarrow{\simeq} \mathrm{rep}_K(\mathbf{B}_E)$ defined in [20] ((4.11)).

Following an observation of Drozd [7] (see also [3] and [20, p. 44, 45]), given a finitely generated K-algebra S, the category $\mathrm{rep}(\mathbf{B}_E, S)$ of right **S-module representations of the bocs $\mathbf{B}_E = (A, {}_A V_A)$** has as objects the A-S-bimodules ${}_A X_S$ in $\mathrm{mod}_{fp}(A \otimes S^{op})$ (the category of finitely presented left $(A \otimes S^{op})$-modules), which are finitely generated projective, when viewed as right S-modules, see [7], [3] and [20, p. 44, 45] for details. We set $\mathrm{rep}_K(\mathbf{B}_E) = \mathrm{rep}(\mathbf{B}_E, K)$.

By [20] (Proposition 4.9), there is an equivalence of categories

$$T_S\colon (\widehat{R}_E \otimes S^{op})\text{-mod}_{pr}^{pr} \xrightarrow{\simeq} \mathrm{rep}(\mathbf{B}_E, S), \tag{3.10}$$

for any finitely generated K-algebra S, where

$$(\widehat{R}_E \otimes S^{op}) = \begin{bmatrix} R_E \otimes S^{op} & J(R_E)^+ \otimes S^{op} \\ 0 & R_E \otimes S^{op} \end{bmatrix}.$$

The objects of $(\widehat{R}_E \otimes S^{op})\text{-mod}_{pr}^{pr}$ are $\widehat{R}_E$-S-bimodules that are $(R_E \otimes S^{op})$-$(R_E \otimes S^{op})$-propartite and finitely generated projective as left S-modules.

Following the above construction of the functor G', we can construct equivalences of categories

$$\mathcal{P}_1((R_E \otimes S^{op})^{op}) \xleftarrow{G'_{E,S}} (\widehat{R}_E \otimes S^{op})\text{-mod}_{pr}^{pr} \xleftarrow{G''_{E,S}} \mathrm{rep}(\mathbf{B}_E, S), \tag{3.11}$$

and we extend the diagram (3.5) to the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}ap_1(E \otimes S^{op}) \arrow[r, "H_{E,S}", "\simeq"'] \arrow[d, "\mathbf{ker}"'] & \mathcal{P}_1((R_E \otimes S^{op})^{op}) \arrow[d, "\mathbf{cok}"'] & \mathrm{rep}(\mathbf{B}_E, S) \arrow[l, "G_{E,S}"', "\simeq"] \\
(C \otimes S^{op})\text{-}\mathrm{Comod}_{fc}^{E \otimes S^{op}} \arrow[r, "h_S^{\bullet}", "\simeq"'] & \mathrm{mod}((R_E \otimes S^{op})^{op}), &
\end{tikzcd} \tag{3.12}$$

where $G_{E,S} = G'_{E,S} \circ G''_{E,S}$ and $T_S^{-1} = G''_{E,S}$. We set $\widehat{C} = C \otimes S^{op}$ and view it as an S^{op}-coalgebra with the comultiplication $\widehat{\Delta} = \Delta \otimes S^{op}$ and the counit $\widehat{\varepsilon} = \varepsilon \otimes S^{op}$. Then $\widehat{E} = E \otimes S^{op}$ is an injective object in the category $\widehat{C}$-Comod of left $\widehat{C}$-comodules, which is projective, when viewed as a right S-module.

We define $\widehat{C}\text{-Comod}_{fc}^{\widehat{E}} = (C \otimes S^{op})\text{-Comod}_{fc}^{E\otimes S^{op}}$ to be the full subcategory $\widehat{C}\text{-Comod}$ whose objects are the finitely $\widehat{E}$-copresented $\widehat{C}$-comodules, that is, finitely $E \otimes S^{op}$-copresented $\widehat{C}$-bicomodules. The categories $\mathcal{M}ap_1(E \otimes S^{op})$, $\mathcal{P}_1(R_E \otimes S^{op})$, and the functors $\mathbf{ker} = \mathbf{ker}_{E\otimes S^{op}}$, $\mathbf{cok} = \mathbf{cok}_{R_E\otimes S^{op}}$ are defined in an obvious way.

We only prove that the functor $h_S^{\bullet}\colon (C \otimes S^{op})\text{-Comod}_{fc}^{E\otimes S^{op}} \longrightarrow \text{mod}((R_E \otimes {}\otimes S^{op})^{op})$ in (3.12) defined by $Z \mapsto h_S^Z = \text{Hom}_{\widehat{C}}(Z, \widehat{E})$, is an equivalence of categories. The fact that $H_{E,S}$ is an equivalence of categories can be proved by applying the properties of $h_S^{\bullet}$ and the isomorphism

$$\chi_{E',E''}\colon \text{Hom}_C(E', E'') \otimes S^{op} \longrightarrow \text{Hom}_{\widehat{C}}(E' \otimes S^{op}, E'' \otimes S^{op}), \tag{3.13}$$

with $E', E'' \in \text{add}(E)$, given by $g \otimes s \mapsto [(g \otimes \text{id}) \cdot s\colon E' \otimes S^{op} \longrightarrow E'' \otimes S^{op}]$, because the bimodule problem arguments used above extend almost verbatim to our situation. The homomorphism $\chi_{E',E''}$ is an isomorphism of S-modules, for each pair E', E'' of comodules in $\text{add}(E)$, because it is functorial with respect to homomorphisms $E' \to E_1'$ and $E'' \to E_1''$ of C-comodules and it is proved in [28] ((2.10)) that $\chi_{E',E''}$ is bijective, for $E' = E'' = E$, if the algebra R_E is finite dimensional.

Hence easily follows that a left $\widehat{C}$-comodule Z lies in $(C \otimes S^{op})\text{-Comod}_{fc}^{E\otimes S^{op}}$ if and only if there is an exact sequence $0 \longrightarrow Z \longrightarrow E_0 \otimes S^{op} \longrightarrow E_1 \otimes S^{op}$, with $E_0, E_1 \in \text{add}(E)$. By applying $\text{Hom}_{\widehat{C}}(-, E \otimes S^{op})$ and the isomorphism $\chi_{E',E''}$, we get the exact sequence

$$h_E^{E_1'} \otimes S^{op} \longrightarrow h_E^{E_0} \otimes S^{op} \longrightarrow h_S^Z \longrightarrow 0$$

of left $(R_E \otimes S^{op})$-modules, that is a projective presentation of $h_S^Z = \text{Hom}_{\widehat{C}}(Z, E \otimes S^{op})$. Hence, we conclude that the functor $h_S^{\bullet}$ in (3.12) is an equivalence of categories. It follows that the functor $H_{E,S}$ in (3.12) is an equivalence of categories making the diagram (3.12) commutative.

Note that, by [20] (Proposition 4.9(b)) and the definition of the functors $G_{E,S}$, $H_{E,S}$ in (3.12) and the functors G_E and H_E in (3.5), for every module L in the category $\text{fin}(S^{op})$ of finite dimensional left S-modules and every $\widehat{R}_E$-S-bimodule ${}_{\widehat{R}_E}X_S$ in the category $\mathcal{R}_1((R_E \otimes S^{op})^{op})$ there exist isomorphisms

$$G_E^{-1}({}_{\widehat{R}_E}X \otimes_S L) \cong G_{E,S}^{-1}({}_{\widehat{R}_E}X_S) \otimes_S L,$$

and

$$H_E^{-1}({}_{\widehat{R}_E}X \otimes_S L) \cong H_{E,S}^{-1}({}_{\widehat{R}_E}X_S) \otimes_S L$$

that are functorial with respect to the S-module homomorphisms $L \to L'$ and $\widehat{R}_E$-S–bimodule homomorphisms ${}_{\widehat{R}_E}X_S \to {}_{\widehat{R}_E}X_S'$.

By applying the diagram (3.12), we reduce the proof of (c) to [7] (Propositions 11 and 13), and to [3] (Theorem B). Here we follow closely the notation and the proof of [3] (Theorem B). We recall that our functor G_E in (3.5) is just the functor $\Xi\colon \text{rep}_K(\mathbf{B}_E) \longrightarrow \mathcal{P}_1(R_E)$ in [3, p. 476], where $\text{rep}_K(\mathbf{B}_E) = \text{rep}(\mathbf{B}_E, K)$.

Assume that the category $C\text{-Comod}_{fc}^E$ is not K-wild. Then the category $C\text{-Comod}_{fc}^E$ is not K-wild and, by [28] (Proposition 2.8 (a)), the finite dimensional K-algebras R_E and R_E^{op} are not wild. Hence, according to [3] (Theorem B) and its proof, the category $\text{rep}_K(\mathbf{B}_E)$ is not wild and there exist minimal bocses $\mathbf{B}_1, \ldots, \mathbf{B}_n$, with $\mathbf{B}_i = (B_i, W_i)$, finitely generated R_E-B_i^{op}-bimodules T_i' and full functors $F_i'\colon \text{rep}_K(\mathbf{B}_i) \longrightarrow R_E\text{-mod}$

which reflect isomorphisms such that the conditions (c1), (c2) and (c3) stated in (c) are satisfied with $C\text{-Comod}^E_{fc}$, $F_i\colon \operatorname{rep}_K(\mathbf{B}_i) \longrightarrow C\text{-Comod}^E_{fc}$ and R_E-mod, $F'_i\colon \operatorname{rep}_K(\mathbf{B}_i) \longrightarrow R_E$-mod interchanged. Moreover, it is shown in the proof of [3] (Theorem B) that, for each $i = 1, \ldots, n$, the R_E-B_i^{op}-bimodules T'_i are of the form $T'_i = \mathbf{cok}_{B_i}(\widehat{T}'_i)$, where $\widehat{T}'_i \in \mathcal{P}_1(R_E \otimes B_i{}^{op})$, and $\widehat{F}'_i(X) = \widehat{T}'_i \otimes_{B_i} X$, for all representations X of the bocs $\mathbf{B}_i$.

Let $\widehat{T}_i = H^{-1}_{E,B_i}(T'_i) \in \mathcal{M}ap_1(E \otimes B_i{}^{op})$ be the preimage of T'_i under the functor $H_{E,S}$ in (3.12), with $S = B_i$. Finally, let $T_i = \mathbf{ker}(\widehat{T}_i) \in \widehat{C}\text{-Comod}^{\widehat{E}}_{fc}$ be the image of $\widehat{T}_i$ under the functor $\mathbf{ker}$ in (3.12), applied to $S = B_i$. Then T_i is a finitely E-copresented C-B_i-bicomodule and we set $F_i(-) = T_i \otimes_{B_i} (-)$.

In view of (a), (b) and the properties of the functors $F'_i : \operatorname{rep}_K(\mathbf{B}_i) \longrightarrow R_E$-mod listed above, the conditions $(\mathrm{c}_1)-(\mathrm{c}_3)$ are satisfied, because the arguments given in the proof of [3] (Theorem B) extends almost verbatim. The details are left to the reader.

***Corollary* 3.1.** *Under the assumption made in Theorem* 3.1, *for a given socle-finite injective direct summand E of ${}_CC$ such that* $\dim_K \operatorname{End}_C E < \infty$, *the following conditions are equivalent.*

(a) *The category $C\text{-Comod}^E_{fc}$ is K-wild.*

(b) *$C\text{-Comod}^E_{fc}$ is properly fc-wild* (*or smooth*) [20] (*Section* 6), *that is, for every finitely generated K-algebra Λ* (*equivalently, for $\Lambda = K\langle t_1, t_2\rangle$, or $\Lambda = \Gamma_3(K)$*) *there exists a finitely E-copresented C-Λ-bicomodule ${}_CN_\Lambda$ that induces a representation embedding ${}_CN \otimes_\Lambda (-)\colon \operatorname{fin}(\Lambda^{op}) \longrightarrow C\text{-Comod}^E_{fc}$.*

(c) *The finite dimensional K-algebras R_E^{op} and R_E are wild.*

(d) *The additive K-category $\operatorname{rep}_K(\mathbf{B}_E)$ is wild, where $\mathbf{B}_E$ is the Roiter bocs of R_E^{op}, see* (3.5).

(e) *The additive K-category $\widehat{R}_E\text{-mod}^{pr}_{pr}$ is wild, where $\widehat{R}_E$ is the bipartite algebra* (3.9).

Proof. Since the functor $h^{\bullet}_E\colon C\text{-Comod}^E_{fc} \longrightarrow R_E$-mod in (3.5) is an exact equivalence of categories then the condition (a) implies (c). The inverse implication (c) ⇒ (a) and the equivalence of (a) and (b) follows from [28] (Corollary 2.12). The implication (d) ⇒ (a) follows from Theorem 3.1 (c). The equivalence (d) ⇔ (e) follows from [20] (Proposition 4.9). Since (c) ⇔ (d) follows from [7] (Section 5) and [3], then the proof is complete.

In the proof of the fc-tame-wild dichotomy we use the following lemma.

Lemma 3.2. *Under the assumption made in Theorem* 3.1, *for a given socle-finite injective direct summand $E = E_U$ of ${}_CC$ such that $R_E = \operatorname{End}_C E$ is of finite dimension,*

(a) *the fc-tameness of the category $C\text{-Comod}^E_{fc}$ implies the tameness of the additive K-categories $\mathcal{M}ap_1(E) \cong \operatorname{rep}_K(\mathbf{B}_E) \cong \widehat{R}_E\text{-mod}^{pr}_{pr}$ and the tameness of the algebras R_E and R^{op}, where $\widehat{R}_E$ is the bipartite algebra* (3.9) *and $\mathbf{B}_E$ is the Roiter bocs of R_E^{op}, see* (3.5),

(b) *given a proper bipartite vector $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U \subseteq K_0(C) \times K_0(C)$ we have*

$$\widehat{\boldsymbol{\mu}}^1_C(v) = \widehat{\boldsymbol{\mu}}^1_{\widehat{R}_E}(\sigma(v)) = \widehat{\boldsymbol{\mu}}^1_{R_E^{op}}(\sigma(v)),$$

where $\widehat{\boldsymbol{\mu}}^1_{\widehat{R}_E}(\sigma(v))$ and $\widehat{\boldsymbol{\mu}}^1_{R_E^{op}}(\sigma(v))$ is the minimal cardinality of an almost parametrising family for $\operatorname{ind}_{\sigma(v)}(\widehat{R}_E\text{-mod}^{pr}_{pr})$ and $\operatorname{ind}_{\sigma(v)}(\operatorname{mod}(R_E^{op}))$, respectively.

Proof. Assume that the category $C\text{-Comod}_{fc}^{E}$ is fc-tame, that is, for any proper non-negative bipartite vector $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U \subseteq K_0(C) \times K_0(C)$, there exist a non-zero polynomial $h \in K[t]$, C-$K[t]_h$-bicomodules $L^{(1)}, \ldots, L^{(r_v)}$, that are finitely E-$K[t]_h$-copresented and form an almost parametrising family for the family $\mathrm{ind}_v(C\text{-Comod}_{fc}^{E})$ of all indecomposable C-comodules M with $\mathbf{cdn}M = v$. It follows that all $L^{(j)}$ lie in $C\text{-Comod}_{fc}^{E\otimes K[t]_h}$. Then, for each $j \in \{1, \ldots, r_v\}$, there is an exact sequence

$$0 \longrightarrow {}_C L^{(j)}_{K[t]_h} \longrightarrow E_0^{(j)} \otimes K[t]_h \xrightarrow{\psi^{(j)}} E_1^{(j)} \otimes K[t]_h$$

in $C\text{-Comod}_{fc}^{E\otimes K[t]_h}$, with $E_0^{(j)}, E_1^{(j)}$ in add(E), such that

$$\widehat{L}^{(j)} = (E_0^{(j)} \otimes K[t]_h, E_1^{(j)} \otimes K[t]_h, \psi^{(j)})$$

is an object of $\mathcal{M}ap_1(E\otimes K[t]_h)$, see (3.12). By applying Theorem 3.1, one can show that the objects $\widehat{L}^{(1)}, \ldots, \widehat{L}^{(r_v)}$ form a finitely E-copresented almost parametrising family for $\mathrm{ind}_v(\mathcal{M}ap_1(E))$, that is, all but finitely many indecomposable objects (E', E'', g) in $\mathcal{M}ap_1(E)$, with $\mathbf{cdn}(E', E'', g) = v$, are of the form

$$(E', E'', g) \cong \widehat{L}^{(s)} \otimes K[t]_h := (E_0^{(s)} \otimes K[t]_h, E_1^{(s)} \otimes K[t]_h, \psi^{(j)} \otimes K^1_\lambda),$$

where $s \le r_v$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$. This shows that the category $\mathcal{M}ap_1(E)$ is tame. The functor $G_S^{-1} \circ H_{E,S}$ in the diagram (3.12), with $S = K[t]_h$, carries each of the objects $\widehat{L}^{(s)}$ to some object $U^{(s)} \in \mathrm{rep}(\mathbf{B}_E, K[t]_h))$ such that all but finitely many indecomposable objects X in $\mathrm{rep}_K(\mathbf{B}_E)$, with $\underline{\dim}(X) = \sigma(v)$, are of the form $X \cong U^{(j)} \otimes K^1_\lambda$, where $s \le r_v$. This shows that the category $\mathrm{rep}_K(\mathbf{B}_E)$ is tame and, by [3] (Section 6) and [7], the algebra R_E^{op} and R_E are tame. Since, by Proposition 4.9 (b) and Theorem 6.5 in [20], the category $\mathrm{rep}_K(\mathbf{B}_E)$ is tame if and only if $\widehat{R}_E\text{-mod}_{pr}^{pr}$ is tame then the proof of (a) is complete.

Moreover, it follows that, given a proper vector $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U$, any almost parametrising family for $\mathrm{ind}_v(C\text{-Comod}_{fc}^{E})$ consisting of finitely E-copresented bicomodules $L^{(1)}, \ldots, L^{(r_v)}$ leads to an almost parametrising family $\widehat{L}^{(1)}, \ldots, \widehat{L}^{(r_v)} \in$ $\in \mathcal{M}ap_1(E\otimes S)$, with $S = K[t]_h$, for $\mathrm{ind}_v(\mathcal{M}ap_1(E))$. By applying the functor $H_{E,S}$ in (3.12) and then the functor $(G'_{E,S})^{-1}$ in (3.11), to $\widehat{L}^{(1)}, \ldots, \widehat{L}^{(r_v)}$, we get an almost parametrising family $\widehat{\widehat{L}^{(1)}}, \ldots, \widehat{\widehat{L}^{(r_v)}} \in (\widehat{R}_E \otimes S)\text{-mod}_{pr}^{pr}$, for $\mathrm{ind}_v(\widehat{R}_E\text{-mod}_{pr}^{pr})$. Since the vector $v = (v'|v'')$ is proper then, up to a localisation of $S = K[t]_h$, by applying the functor $\mathbf{cok}$ in (3.12) we get an almost parametrising family $\mathbf{cok}(\widehat{\widehat{L}^{(1)}}), \ldots, \mathbf{cok}(\widehat{\widehat{L}^{(r_v)}})$ for $\mathrm{ind}_{\sigma(v)}(\mathrm{mod}(R_E^{op}))$.

By Lemma 3.1, any finitely copresented family for $\mathrm{ind}_v(C\text{-Comod}_{fc})$ can be corrected to a finitely E-copresented almost parametrising family for $\mathrm{ind}_v(C\text{-Comod}_{fc}) = \mathrm{ind}_v(C\text{-Comod}_{fc}^{E})$, for any $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U$. Hence (b) follows and the proof is complete.

Now we are able to give an alternative proof of the fc-tame-wild dichotomy for computable coalgebras established in [28].

Theorem 3.2. *Assume that C is a basic coalgebra over an algebraically closed field K such that* $\dim_K \mathrm{Hom}_K(E', E'')$ *is finite, for each pair E', E'' of indecomposable direct summands of ${}_CC$. Then C is either of tame fc-comodule type or of wild fc-comodule type, and these two types are mutually exclusive.*

Proof. Since C is basic, ${}_CC$ has a decomposition (1.1). Assume that C is not of fc-wild comodule type. To show that C is of fc-tame comodule type, fix a non-negative bipartite vector $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} \cong K_0(C) \times K_0(C)$. Since the support $U_v = \mathbf{supp}(v)$ of v is a finite subset of I_C then the injective C-comodule $E = = E_{U_v} = \bigoplus_{j\in U_v} E(j)$ is socle-finite and, according to our assumption the algebra $R_E = = \mathrm{End}_C E$ is finite dimensional. Moreover, every left C-comodule N, with $\mathbf{cdn}(N) = = v$ lies in the subcategory $C\text{-Comod}^E_{fc}$ of $C\text{-Comod}_{fc}$. Then $\mathrm{ind}_v(C\text{-Comod}_{fc}) = = \mathrm{ind}_v(C\text{-Comod}^E_{fc})$ and, by our assumption, the category $C\text{-Comod}^E_{fc}$ is not of K-wild comodule type. Then, by Theorem 3.1, there exist minimal bocses $\mathbf{B}_1, \ldots, \mathbf{B}_n$, with $\mathbf{B}_i = (B_i, W_i)$, finitely $E \otimes R_i$-copresented C-B_i-bicomodules T_i and full functors $F_i(-) = T_i \otimes_{B_i} (-) \colon \mathrm{rep}_K(\mathbf{B}_i) \longrightarrow C\text{-Comod}^E_{fc}$ which reflect isomorphisms such that the conditions (c_1) – (c_3) in Theorem 3.1 are satisfied. In particular, every indecomposable comodule N in $C\text{-Comod}^E_{fc}$ with $\mathbf{cdn}(N) = v$ is isomorphic to $F_i(X)$, for some i and some representation X in $\mathrm{rep}_K(\mathbf{B}_i)$. Hence we conclude, as in the proof of [3] (Corollary C), that there is a finite set of pairs $(R_i, L^{(i)})$, where each $R_i = K[t]_h$ is a localisation of $K[t]$ and $L^{(i)}$ is a finitely E-copresented C-R_i-bicomodule such that

$$L^{(i)} \in (C \otimes R_i^{op})\text{-Comod}_{fc}^{E\otimes R_i^{op}} \tag{3.14}$$

and all but finitely many indecomposable left C-comodules N in $C\text{-Comod}_{fc}$, with $\mathbf{cdn}(N) = v$, are of the form $N \cong L^{(s)} \otimes Y$, for some i and some indecomposable R_i-module Y. Hence we conclude, as in the proof of Theorem 14.18 in [18, p. 297], that there exist finitely E-copresented C-$K[t]_h$-bicomodules $\widehat{L}^{(1)}, \ldots, \widehat{L}^{(r_v)}$ such that all but finitely many indecomposable left C-comodules N in $C\text{-Comod}_{fc}$, with $\mathbf{cdn}(N) = v$, are of the form $N \cong \widehat{L}^{(s)} \otimes K^1_\lambda$, where $s \le r_v$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$. Consequently, the coalgebra is of fc-tame comodule type.

It remains to prove that the coalgebra C can not be both of fc-tame and of fc-wild comodule type. Assume to the contrary, that C is of fc-tame and of fc-wild comodule type. Let $T \colon \mathrm{mod}\Gamma_3(K) \longrightarrow C\text{-Comod}_{fc}$ be an exact K-linear representation embedding, where $\Gamma_3(K) = \begin{bmatrix} K & K^3 \\ 0 & K \end{bmatrix}$. Let S_1 be the unique simple injective right $\Gamma_3(K)$-module, and let S_2 be the unique simple projective right $\Gamma_3(K)$-module, up to isomorphism. Since $T(S_1)$ and $T(S_2)$ lie in $C\text{-Comod}_{fc}$, then there are exact sequences $0 \to T(S_1) \to E_0^{(1)} \longrightarrow E_1^{(1)}$ and $0 \to T(S_2) \to E_0^{(2)} \longrightarrow E_1^{(2)}$, where $E_0^{(1)}$, $E_1^{(1)}$, $E_0^{(2)}$, $E_1^{(2)}$ are socle-finite injective C-modules.

Let E be a socle-finite direct summand of C such that the comodules $E_0^{(1)}$, $E_1^{(1)}$, $E_0^{(2)}$, $E_1^{(2)}$ lies in $\mathrm{add}(E)$. We show that $\mathrm{Im}\,T \subseteq C\text{-Comod}^E_{fc}$. Indeed, if $N = T(X)$ lies in $\mathrm{Im}\,F$, where X is a module in $\mathrm{mod}\Gamma_3(K)$, then there is an exact sequence $0 \to S_2^n \to X \to S_1^m \to 0$, with $n, m \ge 0$. Since T is exact, we get the exact sequence $0 \to T(S_2)^n \to N \to T(S_1)^m \to 0$ in C-Comod. The comodules $T(S_1)^m$ and $T(S_2)^n$ obviously lie in $C\text{-Comod}^E_{fc}$ and, hence, also N lies in $C\text{-Comod}^E_{fc}$. This shows that $\mathrm{Im}\,T \subseteq C\text{-Comod}^E_{fc}$ and, hence, the category $C\text{-Comod}^E_{fc}$ is fc-wild and, according to Corollary 3.1, the finite dimensional algebra R_E is wild.

On the other hand, in view of the fc-parametrisation correction lemma (Lemma 3.1), the assumption that C is of fc-tame comodule type implies that $C\text{-Comod}^E_{fc}$ is fc-tame. Hence, by Lemma 3.2, the finite dimensional algebra R_E is tame and we get a contradiction with the tame-wild dichotomy [7] for finite dimensional K-algebras.

Now we can complete [28] (Proposition 2.8 (a)) as follows.

***Corollary* 3.2.** *Under the assumption made in Theorem* 3.1, *for a given socle-finite injective direct summand* $E = E_U$ *of* ${}_CC$ *such that the algebra* $R_E = \mathrm{End}_C E$ *is of finite dimension, the following conditions are equivalent.*

(a) *The category* $C\text{-}\mathrm{Comod}^E_{fc}$ *is* fc*-tame.*

(b) *The finite dimensional* K*-algebra* R_E *is tame.*

(c) *The additive* K*-categories* $\mathcal{M}ap_1(E) \cong \mathrm{rep}_K(\mathbf{B}_E)$ *are tame, where* $\mathbf{B}_E$ *is the additive Roiter bocs of* R_E^{op}, *see* (3.5).

(d) *The additive* K*-category* $\widehat{R}_E\text{-}\mathrm{mod}^{pr}_{pr}$ *is tame, where* $\widehat{R}_E$ *is the bipartite algebra* (3.9).

Moreover, if $C\text{-}\mathrm{Comod}^E_{fc}$ *is* fc*-tame then, given a proper bipartite vector* $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U \subseteq K_0(C) \times K_0(C)$, *we have* $\widehat{\boldsymbol{\mu}}^1_C(v) = \widehat{\boldsymbol{\mu}}^1_{\widehat{R}_E}(\sigma(v)) = \widehat{\boldsymbol{\mu}}^1_{R_E^{op}}(\sigma(v))$. *In particular,* $C\text{-}\mathrm{Comod}^E_{fc}$ *is of polynomial growth if and only if* $\widehat{R}_E\text{-}\mathrm{mod}^{pr}_{pr}$ *is of polynomial growth.*

Proof. The equivalence (b) ⇔ (c) follows from the theorem of Drozd [7] (see also [3], [28] (Proposition 2.8) and from the proof of Theorem 3.1. The equivalence (c) ⇔ (d) follows from [20] (Theorem 6.5) (or from the proof of Theorem 3.1). To prove (c) ⇒ (a), note that, according to [7], if $\mathrm{rep}_K(\mathbf{B}_E)$ is tame, it is not wild. Then, by Theorem 3.2 and its proof, the category $C\text{-}\mathrm{Comod}^E_{fc}$ is fc-tame. Since (a) ⇒ (c) follows from Lemma 3.2 (a), the conditions (a) – (d) are equivalent. The remaining statement follows from Lemma 3.2 (b).

***Corollary* 3.3.** *Let* C *be a basic coalgebra over an algebraically closed field* K *such that* $\dim_K \mathrm{Hom}_K(E', E'')$ *is finite, for each pair* E', E'' *of indecomposable direct summands of* ${}_CC$. *The following conditions are equivalent.*

(a) *The coalgebra* C *is of tame* fc*-comodule type.*

(b) *For any proper bipartite vector* $v = (v'|v'') \in K_0(C) \times K_0(C)$, *there is a finitely* E_{U_v}*-copresented almost parametrising family for* $\mathrm{ind}_v(C\text{-}\mathrm{Comod}_{fc}) = \mathrm{ind}_v(C\text{-}\mathrm{Comod}^{E_{U_v}}_{fc})$, *where* $U_v = \mathbf{supp}(v) \subseteq \mathbb{Z}^{(I_C)}$ *is the support of* v *and* $E_{U_v} = \bigoplus_{j \in U_v} E(j)$.

(c) *For any socle-finite direct summand* E *of* ${}_CC$, $C\text{-}\mathrm{Comod}^E_{fc}$ *is* fc*-tame.*

(d) *For any socle-finite direct summand* E *of* ${}_CC$, $C\text{-}\mathrm{Comod}^E_{fc}$ *is not* fc*-wild.*

(e) *For any socle-finite direct summand* E *of* ${}_CC$, *the finite dimensional* K*-algebra* $R_E = \mathrm{End}_C E$ *is tame.*

(f) *For any socle-finite direct summand* E *of* ${}_CC$, *the category* $\widehat{R}_E\text{-}\mathrm{mod}^{pr}_{pr}$ *is tame, where* $\widehat{R}_E$ *is the bipartite algebra* (3.9).

The coalgebra C *is of* fc*-discrete comodule type if and only if, for any proper bipartite vector* $v = (v'|v'') \in K_0(C) \times K_0(C)$, *the family* $\mathrm{ind}_v(C\text{-}\mathrm{Comod}^{E_{U_v}}_{fc})$, *with* $U_v = \mathbf{supp}(v) \subseteq \mathbb{Z}^{(I_C)}$, *is finite up to isomorphism, or equivalently, the family* $\mathrm{ind}_v(\widehat{R}_{E_{U_v}}\text{–}\mathrm{mod}^{pr}_{pr})$ *is finite up to isomorphism.*

Proof. The implication (b) ⇒ (a) is obvious. The implication (c) ⇒ (b) and the equivalence of the statements (c) – (f) is an immediate consequence of previous results.

To prove (a) ⇔ (b), we fix a proper bipartite vector $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$ and set $U_v = \mathbf{supp}(v)$, $E_{U_v} = \bigoplus_{j \in U_v} E(j)$. It is clear that $\mathrm{ind}_v(C\text{-}\mathrm{Comod}_{fc}) = \mathrm{ind}_v(C\text{-}\mathrm{Comod}^{E_{U_v}}_{fc})$. Since C is fc-tame then there are finitely copresented $C\text{-}K[t]_h$-bimodules $L^{(1)}, \ldots, L^{(r_v)}$ forming an almost parametrising family of for

$\mathrm{ind}_v(C\text{-}\mathrm{Comod}_{fc})$. By Lemma 3.1, the family corrects to an almost parametrising family $\widetilde{L}^{(1)}, \ldots, \widetilde{L}^{(r_v)}$ for $\mathrm{ind}_v(C\text{-}\mathrm{Comod}_{fc}) = \mathrm{ind}_v(C\text{-}\mathrm{Comod}_{fc}^{E_{U_v}})$ consisting of finitely E_{U_v}-copresented bicomodules. Hence (b) follows and the conditions (a) – (f) are equivalent. Since the remaining statement of corollary is a consequence of Lemma 3.2 (b), the proof is complete.

4. A geometry context for computable coalgebras. Througouht we assume that K is an algebraically closed field and C a basic computable K-coalgebra with a fixed decomposition ${}_CC = \bigoplus_{j \in I_C} E(j)$ (1.1). Following [7, 17, 19, 20], we introduce in Definitions 4.1 and 4.2 a geometry context for a coalgebra C, compare with [15]. We use it in the study of comodules over a K-coalgebra C by applying the geometry of orbits. In particular, we give a geometric characterisation of fc-tame coalgebras.

Definition 4.1. *Given a computable K-coalgebra C* (1.1) *and a bipartite non-negative vector $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$, we define an action*

$$* : \mathbf{G}_v^C \times \mathbf{Map}_v^C \longrightarrow \mathbf{Map}_v^C \tag{4.1}$$

of an algebraic (parabolic) group $\mathbf{G}_v^C$ on an affine K-variety $\mathbf{Map}_v^C$ as follows.

(a) $\mathbf{G}_v^C = \mathrm{Aut}_C\mathbf{E}(v') \times \mathrm{Aut}_C\mathbf{E}(v'')$ *viewed as an algebraic group with respect to Zariski topology, where $\mathbf{E}(v') = \bigoplus_{i \in I_C} E(j)^{v'_i}$ and $\mathbf{E}(v'') = \bigoplus_{j \in I_C} E(j)^{v''_j}$ are the standard injective C-comodules* (2.2) *with $\mathbf{lgth}\mathbf{E}(v') = (v'|0)$ and $\mathbf{lgth}\mathbf{E}(v'') = (v''|0)$.*

(b) $\mathbf{Map}_v^C = \{\psi \in \mathrm{Hom}_C(\mathbf{E}(v'), \mathbf{E}(v'')); \psi(\mathrm{soc}\,\mathbf{E}(v')) = 0\} \subseteq \mathrm{Hom}_C(\mathbf{E}(v'), \mathbf{E}(v''))$ *is viewed as an affine K-variety* (*Zariski closed subset of the affine space* $\mathrm{Hom}_C(\mathbf{E}(v'), \mathbf{E}(v''))$ *of finite K-dimension*).

(c) *The algebraic group* (*left*) *action* (4.1) *of $\mathbf{G}_v^C$ on $\mathbf{Map}_v^C$ is defined by the conjugation $(f', f'') * \psi = f'' \circ g \circ (f')^{-1}$, where $\psi \in \mathbf{Map}_v^C$, $f' \in \mathrm{Aut}_C\mathbf{E}(v')$ and $f'' \in \mathrm{Aut}_C\mathbf{E}(v'')$.*

Definition 4.2. *Given a computable K-coalgebra C and a bipartite non-negative vector $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} = K_0(C) \times K_0(C)$, the open subset*

$$\mathbf{Comod}_v^C = \{\psi \in \mathbf{Map}_v^C; \mathrm{soc}\,\mathbf{E}(v'') \subseteq \mathrm{Im}\,\psi\} \tag{4.2}$$

of the variety $\mathbf{Map}_v^C$ is called a **variety of C-comodules** *N with $\mathbf{cdn}(N) = v$.*

We start with the following useful facts.

Lemma 4.1. *Let C be a computable K-coalgebra and $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} = K_0(C) \times K_0(C)$ a non-negative bipartite vector.*

(a) *$\mathbf{Comod}_v^C$ is a $\mathbf{G}_v^C$-invariant and Zariski open subset of the affine variety $\mathbf{Map}_v^C$.*

(b) *The map $\psi \mapsto \mathrm{Ker}\psi$ defines a bijection between the $\mathbf{G}_v^C$-orbits of $\mathbf{Comod}_v^C$ and the isomorphism classes of comodules N in C-Comod_{fc} such that $\mathbf{cdn}(N) = v$.*

Proof. (a) To see that $\mathbf{Comod}_v^C$ is a Zariski open subset of $\mathbf{Map}_v^C$, note that, given $a \in \mathbf{supp}(v'') \subseteq I_C$, the subset $\mathfrak{D}_a$ of $\mathbf{Map}_v^C$ consisting of all $\psi \in \mathbf{Map}_v^C$ such that $\psi\colon \mathbf{E}(v') \longrightarrow \mathbf{E}(v'')$ has a factorisation through the subcomodule $\mathbf{E}(v'')_a = \bigoplus_{j \neq a} E(j)^{v''_j}$ of $E(v'')$ is Zariski closed. Since the set $\mathbf{supp}(v'')$ is finite then $\mathfrak{D} = \bigcup_{a \in \mathbf{supp}(v'')} \mathfrak{D}_a$ is closed and therefore $\mathbf{Comod}_v^C = \mathbf{Map}_v^C \setminus \mathfrak{D}$ is open. The fact that $\mathbf{Comod}_v^C$ is a $\mathbf{G}_v^C$-invariant subset of $\mathbf{Map}_v^C$ follows by applying the definitions.

(b) Note that a C-comodule homomorphism $\psi\colon \mathbf{E}(v') \longrightarrow \mathbf{E}(v'')$ is an element of $\mathbf{Comod}_v^C$ if and only if $0 \longrightarrow \mathrm{Ker}\psi \longrightarrow \mathbf{E}(v') \xrightarrow{\psi} \mathbf{E}(v'')$ is a minimal injective copresentation of Kerψ in $C\text{-Comod}_{fc}$. Hence every comodule N in $C\text{-Comod}_{fc}$, with $\mathbf{cdn}(N) = v$, is isomorphic to Kerψ, for some $\psi\colon \mathbf{E}(v') \longrightarrow \mathbf{E}(v'')$ in $\mathbf{Comod}_v^C$. Obviously, two elements $\psi\colon \mathbf{E}(v') \longrightarrow \mathbf{E}(v'')$ and $\psi'\colon \mathbf{E}(v') \longrightarrow \mathbf{E}(v'')$ of $\mathbf{Comod}_v^C$ lie in the same $\mathbf{G}_v^C$-orbits if and only if the comodules Kerψ and Kerψ' are isomorphic. Hence (b) follows.

The lemma is proved.

Now we characterise computable K-colagebras of fc-discrete comodule type in terms of the $\mathbf{G}_v^C$-orbits of $\mathbf{Comod}_v^C$ as follows.

Proposition 4.1. *Let K be an algebraically closed field and C a computable K-coalgebra. The following four conditions are equivalent.*

(a) *The coalgebra C is fc-tame of discrete comodule type.*

(b) *For every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$, there is only a finite number of indecomposable objects (E_0, E_1, ψ) in $\mathcal{M}ap_1(E_{U_v})$ with $\mathbf{cdn}(E_0, E_1, \psi) = v$, up to isomorphism, where $U_v = \mathbf{supp}(v)$.*

(c) *The number of $\mathbf{G}_v^C$-orbits in $\mathbf{Comod}_v^C$ is finite, for every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$.*

(d) *The number of $\mathbf{G}_v^C$-orbits in $\mathbf{Map}_v^C$ is finite, for every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$.*

Proof. (a) $\Rightarrow$ (b) Assume that C is fc-tame of discrete comodule type. Let $v = (v'|v'')$ be a bipartite vector in $K_0(C) \times K_0(C)$ and let (E_0, E_1, ψ) be an indecomposable object of $\mathcal{M}ap_1(E_U))$ such that $\mathbf{cdn}(E_0, E_1, \psi) = (v'|v'')$, where we set $U = U_v = \mathbf{supp}(v)$.

If $v' = 0$ then $E_0 = 0$, $E_1 \cong E(a)$, with $a \in U$, and therefore the number of the indecomposable objects (E_0, E_1, ψ) of $\mathcal{M}ap_1(E_U))$ with $\mathbf{cdn}(E_0, E_1, \psi) = (0|v'')$ equals the cardinality of the finite subset $U = \mathbf{supp}(v)$ of I_C.

Assume that $v' \neq 0$, that is, the vector v is proper. Since (E_0, E_1, ψ) is indecomposable, it lies in $\mathcal{M}ap_2(E_U)$, because it has no non-zero direct summand of the form $(0, Z, 0)$, By Proposition 4.1 (a), with E and E_U interchanged, the functor $\mathbf{ker}_{E_U}$ in the diagram (3.5) restrict to the representation equivalence $\mathbf{ker}_{E_U}\colon \mathcal{M}ap_2(E_U) \longrightarrow C\text{-Comod}_{fc}^{E_U}$. Then Ker$\psi = \mathbf{ker}_{E_U}(E_0, E_1, \psi)$ is an indecomposable comodule in $C\text{-Comod}_{fc}^{E_U}$ such that $\mathbf{cdn}(\text{Ker}\psi) = \mathbf{cdn}(E_0, E_1, \psi) = v$, see Proposition 4.1 (b). Since C is fc-tame of discrete comodule type then the number of the isomorphism classes of such comodules is finite and, hence, the number of the isomorphism classes of indecomposable objects (E_0, E_1, ψ) in $\mathcal{M}ap_1(E_U)$ with $\mathbf{cdn}(E_0, E_1, \psi) = v$ is also finite.

(b) $\Rightarrow$ (d) Let $v = (v'|v'') \in K_0(C) \times K_0(C)$ be a vector with non-negative coordinates and let (E_0, E_1, ψ) be an object in $\mathcal{M}ap_1(E_U)$. Since the coalgebra C is assumed to be computable then the endomorphism ring End(ψ) of (E_0, E_1, ψ) is a finite dimension K-algebra, and End(ψ) is a local algebra if (E_0, E_1, ψ) is indecomposable. It follows that $\mathcal{M}ap_1(E_U)$, with $U = \mathbf{supp}(v) \subseteq I_C$, is a Krull – Schmidt category such that each of its objects is a finite direct sum of indecomposable objects, and every such a decomposition is unique up to isomorphism and a permutation of the indecomposables.

By our assumption, there is only a finite number of indecomposable objects (E_0', E_1', ψ') in $\mathcal{M}ap_1(E_{U_v})$ with $\mathbf{cdn}(E_0', E_1', \psi') \leq v$, up to isomorphism. Let $\mathbb{E}_1, \ldots, \mathbb{E}_{s_v}$ be a complete set of such indecomposable objects. Then, up to isomorphism, any object (E_0, E_1, ψ) in $\mathcal{M}ap_1(E_{U_v})$, with $\mathbf{cdn}(E_0, E_1, \psi) = v$, has the form

$$(\mathbf{E}(v'), \mathbf{E}(v''), \psi) \cong \mathbb{E}_1^{\ell_1} \oplus \ldots \oplus \mathbb{E}_{s_v}^{\ell_{s_v}}$$

where $\ell(\mathbf{E}(v'), \mathbf{E}(v''), \psi) = (\ell_1, \ldots, \ell_{s_v}) \in \mathbb{N}^{s_v}$ is a vector with non-negative coordinates such that

$$\ell_1 \cdot \mathbf{cdn}(\mathbb{E}_1) + \ldots + \ell_{s_v} \cdot \mathbf{cdn}(\mathbb{E}_{s_v}) = v.$$

Obviously, the number of such vectors $(\ell_1, \ldots, \ell_{s_v})$ is finite. The unique decomposition property in $\mathcal{M}ap_1(E_{U_v})$ yields

$$\ell(\mathbf{E}(v'), \mathbf{E}(v''), \psi) = \ell(\mathbf{E}(v'), \mathbf{E}(v''), \psi')$$

$$\text{if and only if} \quad (\mathbf{E}(v'), \mathbf{E}(v''), \psi) \cong (\mathbf{E}(v'), \mathbf{E}(v''), \psi'),$$

or equivalently, if and only if the elements ψ and ψ' of $\mathbf{Map}_v^C$ lie in the same $\mathbf{G}_v^C$-orbit. Hence the number of $\mathbf{G}_v^C$-orbits in $\mathbf{Map}_v^C$ is finite and (d) follows.

Since the implication (d) $\Rightarrow$ (c) is obvious and the implication (c) $\Rightarrow$ (a) follows from Lemma 4.1 (b), the proof is complete.

Now we present a characterisation of computable fc-tame colagebras in terms of geometry of the $\mathbf{G}_v^C$-orbits of $\mathbf{Comod}_v^C$.

Theorem 4.1. *Let K be an algebraically closed field and C a computable K-coalgebra.*

(a) *C is fc-tame.*

(b) *For every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$, the category $\mathcal{M}ap_1(E_{U_v})$, with $U_v = \mathbf{supp}(v)$, is tame.*

(c) *For every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$, the subset $\mathbf{indComod}_v^C$ of $\mathbf{Comod}_v^C$ defined by the indecomposable C-comodules is constructible and there exists a constructible subset $\mathcal{C}(v)$ of $\mathbf{indComod}_v^C$ such that*

$$\mathbf{G}_v^C * \mathcal{C}(v) = \mathbf{indComod}_v^C \quad \text{and} \quad \dim \mathcal{C}(v) \leq 1.$$

(d) *For every bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$, the subset $\mathbf{indMap}_v^C$ of $\mathbf{Map}_v^C$ defined by the indecomposable C-comodules is constructible and there exists a constructible subset $\widehat{\mathcal{C}}(v)$ of $\mathbf{indMap}_v^C$ such that*

$$\mathbf{G}_v^C * \widehat{\mathcal{C}}(v) = \mathbf{indMap}_v^C \quad \text{and} \quad \dim \widehat{\mathcal{C}}(v) \leq 1.$$

Proof. (a) $\Rightarrow$ (b) Apply Lemma 3.2 (a) to $E = E_U = \bigoplus_{j \in U} E(j)$, where $U = \mathbf{supp}(v) \subseteq I_C$.

(b) $\Rightarrow$ (a) Apply Corollary 3.3.

We prove the equivalence of (b), (c) and (d) by applying the arguments used by Drozd [7], see also [3], [18] (Section 15.2) and [20] (Theorem 6.5).

(b) $\Rightarrow$ (d) Fix a bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$ and assume that the category $\mathcal{M}ap_1(E_{U_v})$, with $U_v = \mathbf{supp}(v)$, is tame. Then there is a parametrising family of functors

$$\widehat{L}^{(1)}, \ldots, \widehat{L}^{(r)} \colon \mathrm{ind}_1(K[t]_h) \longrightarrow \mathcal{M}ap_1(E_{U_v})$$

for the family $\mathrm{ind}_v(\mathcal{M}ap_1(E_{U_v})$, where $h \in K[t]$ and $U_v = \mathbf{supp}(v)$. Here $\mathrm{ind}_1(K[t]_h)$ is the category of one-dimensional $K[t]_h$-modules. Hence we conclude, as in [18] (Lemma 14.30, Remark 14.27) that the functors $\widehat{L}^{(1)}, \ldots, \widehat{L}^{(r)}$ induce regular maps

$$\ell_1, \ldots, \ell_r \colon \mathbf{mod}^{K[t]_h}(1) \longrightarrow \mathbf{Map}_v^C$$

such that every point of $\mathbf{indMap}_v^C$ belongs to an $\mathbf{G}_v^C$–orbit of the set

$$\widehat{\mathcal{C}}(v) = \operatorname{Im} \ell_1 \cup \ldots \cup \operatorname{Im} \ell_r.$$

Here $\mathbf{mod}^{K[t]_h}(1)$ is the variety of one-dimensional $K[t]_h$-modules. Since we have $\dim \mathbf{mod}^{K[t]_h}(1) = 1$ then, according to the Chevalley Theorem, the subsets $\operatorname{Im} \ell_1, \ldots, \operatorname{Im} \ell_r$ of $\mathbf{indMap}_v^C$ are constructible and therefore $\widehat{\mathcal{C}}(v)$ is a constructible subset of $\mathbf{indMap}_v^C$. Moreover, it follows that $\dim(\operatorname{Im} \ell_j) \leq 1$, for $j = 1, \ldots, r$, and therefore $\dim \widehat{\mathcal{C}}(v) \leq 1$, compare with [15] and [18, p. 317].

The equivalence (d)⇔(c) easily follows from the fact that $\mathbf{indMap}_v^C \setminus \mathbf{indComod}_v^C$ is a finite set and $\mathbf{Comod}_v^C$ is an open subset of $\mathbf{Map}_v^C$, by Lemma 4.1.

(d) ⇒ (b) Assume to the contrary that there is a bipartite vector $v = (v'|v'') \in K_0(C) \times K_0(C)$ such that the category $\mathcal{M}ap_1(E_{U_v})$, with $U_v = \mathbf{supp}(v)$, is not tame. By Corollary 3.2, the finite dimensional algebra R_{U_v} is not tame. Then R_{U_v} is wild [7] and therefore the category $\mathcal{M}ap_1(E_{U_v})$ is wild, by [3] (Section 6) and the proof of Theorem 3.1.

Let $\mathcal{W} = K\langle t_1, t_2 \rangle$ be the free polynomial K-algebra in two non-commuting indeterminates t_1 and t_2. Since the category $\mathcal{M}ap_1(E_{U_v})$ is wild then there exists an object ${}_C N_{\mathcal{W}} = (E' \otimes \mathcal{W}, E'' \otimes \mathcal{W}, \psi)$ in $\mathcal{M}ap_1(E_{U_v} \otimes \mathcal{W})$, with E', E'' in $\operatorname{add}(E_{U_v})$, such that the functor

$$\widehat{N} = {}_C N \otimes_{\mathcal{W}} (-) \colon \operatorname{fin}(\mathcal{W}) \longrightarrow \mathcal{M}ap_1(E_{U_v})$$

preserves the indecomposability and respects the isomorphism classes.

Let $w = (w'|w'')$, where $w' = \mathbf{lgth}(\operatorname{soc} E')$ and $w'' = \mathbf{lgth}(\operatorname{soc} E'')$. It is well known that $\mathbf{indMap}_w^C$ is a constructible subset of $\mathbf{Map}_w^C$, compare with [18] (Lemma 14.32).

Note that $U_w = \mathbf{supp}(w) \subseteq U_v$, $\mathbf{cdn}(\widehat{N}(X)) = w$, and $\widehat{N}(X)) \cong (\mathbf{E}(w'), \mathbf{E}(w''), \psi)$, for some $\psi \in \mathbf{indMap}_w^C \subseteq \mathbf{Map}_w^C$, if $X \in \operatorname{fin}(\mathcal{W})$ and $\dim_K X = 1$. It follows that the restriction $\widehat{N} \colon \operatorname{ind}_1(\mathcal{W}) \longrightarrow \mathcal{M}ap_1(E_{U_v})$ of $\widehat{N}$ to $\operatorname{ind}_1(\mathcal{W})$ induces a regular map (see [18], Lemma 14.30)

$$\ell_N \colon \mathbf{mod}^{\mathcal{W}}(1) \longrightarrow \mathbf{indMap}_w^C \subseteq \mathbf{Map}_w^C.$$

Since $\mathbf{mod}^{\mathcal{W}}(1) \cong K^2$, the map ℓ_N is injective, and according to the Chevalley Theorem the set $\operatorname{Im} \ell_N$ is constructible then the variety dimension $\dim(\operatorname{Im} \ell_N)$ of $\operatorname{Im} \ell_N$ equals two. Hence, in view of (d) with v and w interchanged, we get the contradiction $2 = \dim(\operatorname{Im} \ell_N) \leq \dim \mathcal{C}(v) \leq 1$ (apply [12] (Lemma 3.16) or [18] (Lemma 15.15)). This completes the proof.

5. On fc-tameness for arbitrary coalgebras. The fc-tame-wild dichotomy for an arbitrary basic coalgebra C over an algebraically closed field K remains an open problem. Some suggestions for the proof in case C is not computable is given in the following proposition that collects important consequences of the technique described in Section 3. In particular, it shows that the coalgebra C is fc-tame if and only if every socle-finite colocalisation $C_E \cong R_E^{\circ}$ of C (in the sense of [11, 25]) is fc-tame.

Proposition 5.1. *Assume that K is an algebraically closed field and C is an arbitrary basic coalgebra with a decomposition ${}_C C = \bigoplus_{j \in I_C} E(j)$ (1.1).*

(a) *Given a socle-finite injective direct summand* $E = E_U = \bigoplus_{u \in U} E(u)$ (3.1) *of* ${}_CC$, *with a finite subset* U *of* I_C, *the* K-*algebra* $R_E = \mathrm{End}_C E$ *is semi-perfect and pseudocompact with respect to the topology defined by* (5.2) *below. There is a commutative diagram*

$$\begin{tikzcd}
 & \mathcal{M}ap_1(E) \arrow[r, "H_E", "\simeq"'] \arrow[d, "\mathbf{ker}_E"'] & \mathcal{P}_1(R_E^{op}) \arrow[d, "\mathbf{cok}_E"'] & \widehat{R}_E\text{-mod}^{pr}_{pr} \arrow[l, "G'"', "\simeq"] \\
C_E\text{-Comod}_{fc} \arrow[r, phantom, "\cong"] & C\text{-Comod}^E_{fc} \arrow[r, "h^{\bullet}_E", "\simeq"'] & \mathrm{mod}_{fp}(R_E^{op}), &
\end{tikzcd} \tag{5.1}$$

where $C_E \cong R_E^{\circ}$ *is the colocalisation of* C *at* E *in the sense of* [11, 25], $\mathrm{mod}_{fp}(R_E^{op})$ *is the category of finitely presented left* R_E-*modules,* $\widehat{R}_E$-mod^{pr}_{pr} *is the category of finitely generated propartite left modules over the bipartite* K-*algebra* $\widehat{R}_E$ (3.9), H_E *and* $h^{\bullet}_E = \mathrm{Hom}_C(\bullet, E)$ *are* K-*linear contravariant equivalences of categories defined as in* (3.5), G' *is the covariant* K-*linear equivalence of categories defined in* (3.9), $h^{\bullet}_E$ *is an exact functor,* $\mathbf{ker}_E(E_0, E_1, \psi) = \mathrm{Ker}\psi$, $\mathbf{cok}_E(P_1, P_0, \phi) = \mathrm{Coker}\phi$.

(b) *For any socle-finite comodule* $E = E_U$ *as in* (a), *the* fc-*tameness of the coalgebra* C *implies that the category* C-$\mathrm{Comod}^{E_U}_{fc}$ *is* fc-*tame, that is, the coalgebra* C_{E_U} *is* fc-*tame.*

(c) *Conversely, if the category* C_{E_U}-$\mathrm{Comod}_{fc} \cong C$-$\mathrm{Comod}^{E_U}_{fc}$ *is* fc-*tame, for all socle-finite injective direct summands* $E = E_U$, *then the coalgebra* C *is* fc-*tame.*

Proof. (a) Let $E = E_U$ be a socle-finite direct summand of C as in (a). The K-algebra $R_E = \mathrm{End}_C E$ has the decomposition $R_E = \bigoplus_{u \in U} e_u R_E$, where $e_u R_E = \mathrm{Hom}_C(E, E(u))$ is an indecomposable projective right ideal of R_E and e_u is the primitive idempotent of R_E defined by the summand $E(u)$ of E. Since the set U is finite then $\sum_{u \in U} e_u$ is the identity of R_E, see [25, 26, 28]. It is easy to see that the Jacobson radical $J(R_E)$ of R_E has the form $J(R_E) = \{h \in \mathrm{End}_C E; h(\mathrm{soc}\, E) = 0\}$. It follows that the algebra R_E is semiperfect and pseudocompact with respect to the K-linear topology defined by the left ideals $\mathfrak{a}_\beta = \mathrm{Hom}_C(E/V_\beta, E) \subseteq R_E$, where $\{V_\beta\}_\beta$ is the directed set of all finite dimensional subcomodules of E. Since $E = \bigcup_\beta V_\beta$, then there are isomorphisms

$$R_E = \mathrm{End}_C E \cong \varprojlim_\beta \mathrm{Hom}_C(V_\beta, E) \cong \varprojlim_\beta R_E/\mathfrak{a}_\beta. \tag{5.2}$$

The remaining statements in (a) follow from the proof of Theorem 3.1.

For the proof of (b) and (c), apply Lemma 3.1 and the arguments used in the proof of Theorem 3.1.

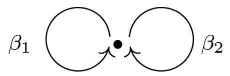
It follows from [28] (Corollaries 2.12 and 2.13) and the results of Section 3 that the fc-tameness and fc-wildness of a computable coalgebra C is equivalent, respectively, to the K-tameness and the K-wildness of the finite dimensional algebra R_E, for every socle-finite direct summand of C. Proposition 5.1 shows that the fc-tameness and fc-wildness of a basic coalgebra C (that is not necessarily computable) can be studied by means of the tameness and wildness of the categories $\widehat{R}_E$-mod^{pr}_{pr} and $\mathrm{mod}_{fp}(R_E^{op})$ over the semiperfect algebras $\widehat{R}_E$ and R_E that are not finite dimensional, in general.

We recall from [26] (Corollary 2.10) that a socle-finite coalgebra C is computable if and only if $\dim_K C$ is finite. Hence, if C is a cocommutative noncomputable coalgebra with simple socle then C is infinite dimensional and, in view of Proposition 5.1, we have the following consequence of Drozd [6].

***Corollary* 5.1.** *Assume that K is an algebraically closed field and C is a basic infinite dimensional cocommutative K-coalgebra with a unique simple subcoalgebra S. If S is finitely copresented and C is not fc-wild then*

(i) *C is a subcoalgebra of the path K-coalgebra $K^{\Box}(\mathcal{L}_2, \Omega)$* (*see* [21] (*Example* 6.18), [22], [24]), *where $\mathcal{L}_2$ is the two loop quiver*

$\mathcal{L}_2 :$



and $\Omega \subseteq K\mathcal{L}_2$ is the ideal of the path algebra $K\mathcal{L}_2$ generated by the two zero-relations $\beta_1\beta_2$ and $\beta_2\beta_1$, and

(ii) *$K^{\Box}(\mathcal{L}_2, \Omega)$ is a string coalgebra in the sense of* [22] (*Section* 6),

(iii) *the colagebras $K^{\Box}(\mathcal{L}_2, \Omega)$ and C are of tame comodule type, and $K^{\Box}(\mathcal{L}_2, \Omega)$ is of non-polynomial growth.*

Proof. By our assumption, C has a simple socle S and $C = E(S)$ is the injective envelope of S, that is, the set I_C in the decomposition (1.1) has one element and Proposition 5.1 applies to $E = E(S) = C$. It follows that the K-algebra R_E is pseudocompact, infinite dimensional, commutative, local, and complete. Since C is not fc-wild, the category $\mathrm{mod}_{fp}(R_E)$ is not K-wild, by Proposition 5.1. Since S is finitely copresented then $C\text{-comod} \subseteq C\text{-Comod}_{fc}$ and therefore $\mathrm{fin}(R_E) \subseteq \mathrm{mod}_{fp}(R_E)$. It follows that the category $\mathrm{fin}(R_E)$ is not K-wild. Hence, by [6], the unique maximal ideal $J(R_E)$ of R_E is generated by at most two elements and R_E is isomorphic to a quotient of the K-algebra $K[[t_1, t_2]]/(t_1t_2)$, where $K[[t_1, t_2]]$ is the power series K-algebra in two commuting indeterminates t_1, t_2 and (t_1t_2) is the ideal of $K[[t_1, t_2]]$ generated by t_1t_2.

It is easy to see that the path coalgebra $K^{\Box}(\mathcal{L}_2, \Omega) = \Omega^{\perp} \subseteq K^{\Box}\mathcal{L}_2$ is isomorphic with the coalgebra

$$K[t_1, t_2]^{\diamond} = K \oplus \bigoplus_{n=1}^{\infty} K\bar{t}_1^{\,n} \oplus \bigoplus_{m=1}^{\infty} K\bar{t}_2^{\,m},$$

where the comultiplication $\Delta\colon K[t_1, t_2]^{\diamond} \longrightarrow K[t_1, t_2]^{\diamond} \otimes K[t_1, t_2]^{\diamond}$ and the counity $\varepsilon\colon K[t_1, t_2]^{\diamond} \longrightarrow K$ are defined by the formulae $\Delta(\bar{t}_j^{\,m}) = \sum\limits_{r+s=m} \bar{t}_j^{\,r} \otimes \bar{t}_j^{\,s}$ for $j = 1, 2$, $\varepsilon(1) = 1$ and $\varepsilon(\bar{t}_j^{\,s}) = 0$ for $s \geq 1$ and $j = 1, 2$, see [21] (Example 6.18).

Moreover, it follows from [24] that C is isomorphic to a subcoalgebra of $K^{\Box}(\mathcal{L}_2, \Omega)$. Since $K^{\Box}(\mathcal{L}_2, \Omega)$ is a string coalgebra then, according to [21] (Example 6.18) and [22] (Theorem 6.2) $K^{\Box}(\mathcal{L}_2, \Omega) \cong K[t_1, t_2]^{\diamond}$ is of tame comodule type and, hence, the coalgebra C is of tame comodule type, too. It is shown in [21] (Example 6.18) that $K^{\Box}(\mathcal{L}_2, \Omega) \cong K[t_1, t_2]^{\diamond}$ is tame of non-polynomial growth.

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