UDC 517.9

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TIKHONOV REGULARIZATION METHOD FOR SYSTEM OF EQUILIBRIUM PROBLEMS IN BANACH SPACES*

МЕТОД РЕГУЛЯРИЗАЦІЇ ТІХОНОВА ДЛЯ СИСТЕМИ ЗАДАЧ ПРО РІВНОВАГУ В БАНАХОВИХ ПРОСТОРАХ

The purpose of the paper is to investigate the Tikhonov regularization method for solving a system of ill-posed equilibrium problems in Banach spaces with a posteriori regularization parameter choice. An application to convex minimization problems with coupled constraints is also given.

Метою роботи є дослідження методу регуляризації Тіхонова для розв'язку системи некоректних задач про рівновагу в банахових просторах з апостеріорним вибором параметра регуляризації. Наведено застосування методу до задач опуклої мінімізації із зчепленими обмеженнями.

1. Introduction. Let X be a real reflexive Banach space, X^* be its dual space which both are assumed to be strictly convex, and let K be a nonempty closed (in the strong topology) and convex subset of X. For the sake of simplicity norms of X and X^* are denoted by the symbol $\|.\|$. Assume that the space X possesses the property: weak convergence and convergence in norm for any sequence in X follow its strong convergence. The symbol $\langle x^*, x\rangle$ denotes the value of the linear and continuous functional $x^* \in X^*$ at the point $x \in X$. Let $U^s, s \geq 2$, be the generalized duality mapping of the space X, i.e., U^s is the mapping from X onto X^* satisfying the condition

$$\langle U^s(x), x\rangle = \|U^s(x)\|\|x\|, \qquad \|U^s(x)\| = \|x\|^{s-1}.$$

Concerning U^s, assume that

$$\langle U^s(x) - U^s(y), x - y\rangle \geq m_s\|x - y\|^2,$$

where m_s is some positive number.

Let F_j, $j = 1, \ldots, N$, be a family of bifunctions from $K \times K$ to $(-\infty, +\infty)$, i.e., F_j all satisfy the following set of standard properties.

Condition 1. The bifunction F is such that:

(i) $F(u, u) = 0 \; \forall u \in K$;

(ii) $F(u, v) + F(v, u) \leq 0 \; \forall (u, v) \in K \times K$;

(iii) for every $u \in K$, $F(u, .)\colon K \to (-\infty, +\infty)$ is lower semicontinuous and convex;

(iv) $\overline{\lim}_{t\to+0} F((1-t)u + tz, v) \leq F(u, v) \; \forall (u, z, v) \in K \times K \times K$.

Consider the system of equilibrium problems: find $u_* \in K$ such that

$$F_j(u_*, v) \geq 0 \quad \forall v \in K, \quad j = 1, \ldots, N. \tag{1}$$

In the case of a single equilibrium, i.e., $N = 1$, problem (1) was called equilibrium problem, and shown in [1–3] to cover monotone inclusion problems, saddle point

*This work was supported by the Vietnamese Fundamental Research Program in Natural Sciences № 100506.

problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems (see also [4]). For finding approximative solutions of (1) there exist several approaches: the regularization approach in [5 – 8], the gap-function approach in [8 – 10], and the dynamical system or iterative procedure approach in [1, 2, 7, 11 – 21]. In particular, this problem are considered in Banach spaces in [9, 17].

In the case $N > 1$, we are only aware of result [6] in Hilbert spaces where on base of constructing the resolvant of bifunction, which is a set-valued operator, P. L. Combettes and S. A. Hirstoaga study the block-iterative algorithms, and a regularization method only for the particular case $N = 1$.

In this paper, on the base of the idea in [22] we present the Tikhonov regularization method constructing the regularized solution, the posteriori regularization parameter choice depending on h when F_j are given by the approximations F_j^h, $h > 0$, in the general case $N > 1$, and an application for convex minimization problem with coupled constraints.

Set

$$S_j = \big\{u_* \in K \colon F_j(u_*, v) \geq 0 \ \ \forall v \in H\big\}, \quad j = 1, \ldots, N, \qquad S = \bigcap_{j=1}^{N} S_j.$$

From now on, suppose that $S \neq \varnothing$. In addition, we assume that F_j all are hemicontinuous in the variable u for each fixed $v \in K$ and weakly lower semicontinuous in the variable v for each fixed $u \in K$ instead of (iv) and (iii) in condition 1, respectively.

The strong and weak convergences of any sequence are denoted by $\rightarrow$ and $\rightharpoonup$, respectively.

2. Main results. First, we formulate the following facts in [1, 3] which are necessary in the proof of our results.

Proposition 1. (i) *If $F(.,v)$ is hemicontinuous for each $v \in K$ and F is monotone, i.e., satisfies* (ii) *in condition* 1, *then $U_* = V_*$, where*

U_* is the solution set of $F(u_*, v) \geq 0 \ \forall v \in K$,

V_* is the solution set of $F(u, v_*) \leq 0 \ \forall u \in K$, and it is convex and closed.

(ii) *If $F(.,v)$ is hemicontinuous for each $v \in K$ and F is strongly monotone, i.e., there exists a positive constant τ such that*

$$F(u, v) + F(v, u) \leq -\tau \|u - v\|^2,$$

then U_* contains a unique element.

Each set S_j is closed convex (Proposition 1 (i)). Hence, S is closed convex, too.

We construct the Tikhonov regularization solution u_α by solving the single equilibrium problem

$$F_\alpha(u_\alpha, v) \geq 0 \quad \forall v \in K, \qquad u_\alpha \in K,$$

$$F_\alpha(u, v) := \sum_{j=1}^{N} \alpha^{\mu_j} F_j(u, v) + \alpha \big\langle U^s(u), v - u \big\rangle, \quad \alpha > 0, \tag{2}$$

$$\mu_1 = 0 < \mu_j < \mu_{j+1} < 1, \quad j = 1, 2, \ldots, N - 1,$$

and α is the regularization parameter.

We have the following results.

Theorem 1. (i) *For each* $\alpha > 0$, *problem* (2) *has a unique solution* u_α.

(ii) $\lim_{\alpha\to+0} u_\alpha = u_*$, $u_* \in S$, $\|u_*\| \le \|y\|$ $\forall y \in S$.

(iii) *If* $F_j(u,v)$ *are bounded, i.e., there exists a positive constant* C *such that* $|F_j(u,v)| \le C$ $\forall u,v \in U$ *with* $\|u\|, \|v\| \le \tilde{C}$, *that also is a positive constant, then we have*

$$\|u_\alpha - u_\beta\| \le \frac{|\alpha-\beta|}{2m_s\alpha}\|u_*\|^{s-1} +$$
$$+\frac{\sqrt{|\alpha-\beta|}}{2m_s\alpha}\sqrt{|\alpha-\beta|\|u_*\|^{2(s-1)} + 4m_s\alpha C(N-1)}, \quad \alpha,\beta > 0.$$

Proof. It is easy to verify that $F_\alpha(u,v)$ is a bifunction, i.e., $F_\alpha(u,v)$ satisfies condition 1, and strongly monotone with constant $m_s\alpha > 0$. Therefore, (2) has a unique solution u_α for each $\alpha > 0$.

Now we shall prove that

$$\|u_\alpha\| \le \|y\| \quad \forall y \in S. \tag{3}$$

Since $y \in S$, then $F_j(y,u_\alpha) \ge 0$, $j = 1,\ldots,N$. Consequently,

$$\sum_{j=1}^{N} \alpha^{\mu_j} F_j(y,u_\alpha) \ge 0 \quad \forall y \in S. \tag{4}$$

This fact, u_α is the solution of (2) and property (ii) in condition 1 of F_j give

$$\langle U^s(u_\alpha), y - u_\alpha\rangle \ge 0 \quad \forall y \in S,$$

that implies (3). It means that $\{u_\alpha\}$ is bounded. Let $u_{\alpha_k} \rightharpoonup u_* \in X$, as $k \to +\infty$. First, note that $u_* \in K$, because K also is weakly closed in X. We prove that $u_* \in S_1$. Indeed, from (ii) in condition 1 and (2) we have

$$F_1(v,u_{\alpha_k}) + \sum_{j=2}^{N} \alpha_k^{\mu_j} F_j(v,u_{\alpha_k}) \le \alpha_k\langle U^s(u_{\alpha_k}), v - u_{\alpha_k}\rangle \le$$
$$\le \alpha_k\langle U^s(v), v - u_{\alpha_k}\rangle \quad \forall y \in K.$$

By virtue of weak lower semicontinuous property of the bifunction $F_j(u,v)$ in the variable v we obtain $F_1(v,u_*) \le 0$ $\forall v \in U$, i.e., $u_* \in S_1$. Now, we shall prove that $u_* \in S_j$, $j = 2,\ldots,N$. From (2) and property (ii) in condition 1 of the bifunction F_1 it implies that

$$F_2(y,u_{\alpha_k}) + \sum_{j=3}^{N} \alpha_k^{\mu_j-\mu_2} F_j(y,u_{\alpha_k}) \le {\alpha_k}^{1-\mu_2}\langle U^s(y), y - u_{\alpha_k}\rangle \quad \forall y \in S_1.$$

Tending $k \to \infty$, we have got

$$F_2(y,u_*) \le 0 \quad \forall y \in S_1.$$

Therefore, $F_2(u^*,y) \ge 0$ $\forall v \in S_1$, i.e., u_* is a minimizer of the convex functional $F_2(v,u_*)$ on the set S_1. Since $S_1 \cap S_2 \ne \varnothing$, then

$$u_* \in \arg\min_{v \in K} F_2(u_*, v),$$

i.e., $F_2(u_*, y) \geq 0 \; \forall y \in K$.

Set $\tilde{S}_i = \bigcap_{k=1}^{i} S_k$. Then, $\tilde{S}_i$ is also closed convex, and $\tilde{S}_i \neq \varnothing$.

Now, suppose that we have proved that $u_* \in \tilde{S}_i$, and need to show that u_* belongs to S_{i+1}. Again, by virtue of (2) for $y \in \tilde{S}_i$ we can write

$$F_{i+1}(y, u_{\alpha_k}) + \sum_{j=i+2}^{N} \alpha_k^{\mu_j - \mu_{i+1}} F_j(y, u_{\alpha_k}) \leq {\alpha_k}^{1-\mu_{i+1}} \langle U^s(y), y - u_{\alpha_k} \rangle \quad \forall y \in \tilde{S}_i.$$

After passing $k \to \infty$, we obtain

$$F_{i+1}(y, u_*) \leq 0 \quad \forall y \in \tilde{S}_i.$$

Since $\tilde{S}_i \cap S_{i+1} \neq \varnothing$, then u_* also is an element of S_{i+1}, i.e., $F_{i+1}(u_*, y) \geq 0$ $\forall y \in K$. Inequality (3) and the weak convergence of $\{u_{\alpha_k}\}$ to $u_* \in S$, which is a closed convex subset in the strictly convex space X, give the strong convergence of $\{u_{\alpha_k}\}$ to $u_*\colon \|u_*\| \leq \|y\| \; \forall y \in S$.

Let u_β be a solution of (2) when α is replaced by β. By virtue of (ii) in condition 1 we have $F_j(u_\alpha, u_\beta) + F_j(u_\beta, u_\alpha) \leq 0$. Therefore, from (2) it follows

$$\sum_{j=1}^{N} (\alpha^{\mu_j} - \beta^{\mu_j}) F_j(u_\alpha, u_\beta) + \alpha \langle U^s(u_\alpha), u_\beta - u_\alpha \rangle + \beta \langle U^s(u_\beta), u_\alpha - u_\beta \rangle \geq 0$$

or

$$m_s \alpha \|u_\alpha - u_\beta\|^2 \leq |\alpha - \beta| \|u_\beta\|^{s-1} \|u_\alpha - u_\beta\| + \sum_{j=1}^{N} |\alpha^{\mu_j} - \beta^{\mu_j}| |F_j(u_\alpha, u_\beta)|.$$

Using (3), the boundedness of F_j and the Lagrange's mean-value theorem for the function $\alpha(t) = t^{-\mu}$, $0 < \mu < 1$, $t \in [1, +\infty)$, on $[\alpha, \beta]$ if $\alpha < \beta$ or $[\beta, \alpha]$ if $\beta < \alpha$ we have conclusion (iii).

Theorem is proved.

Remark. Obviously, if $u_{\alpha_k} \to \tilde{u}$ where u_{α_k} is the solution of (2) with $\alpha = \alpha_k \to 0$, as $k \to +\infty$, then $S \neq \varnothing$.

Let F_j^h be the approximation bifunctions for F_j satisfy the condition

$$\|F_j(u, v) - F_j^h(u, v)\| \leq h g(\|u\|) \|u - v\|, \tag{5}$$

whith the bounded (image of bounded set is bounded) nonegative function $g(t)$, $t \geq 0$. Note that condition (5) was used in the regularizing the variational inequality

$$\langle A(x_*), x - x_* \rangle \geq 0 \quad \forall x \in K, \quad x_* \in K,$$

where A is a hemicontinuous monotone from X into X^*, and is given approximatively by the hemicontinuous monotone operators A_h also from X into X^* such that

$$\big\| A_h(x) - A(x) \big\| \leq h g(\|u\|).$$

By setting $\tilde{F}(u,v) = \langle A(u), v-u\rangle$ and $\tilde{F}^h(u,v) = \langle A_h(u), v-u\rangle$ we see that $\tilde{F}(u,v)$ and $\tilde{F}^h(u,v)$ are the bifunctions satisfying condition (5).

Since F_j^h are also the bifunctions, then the following single equilibrium problem:

$$F_\alpha^h(u_\alpha^h, v) \ge 0 \quad \forall v \in K, \quad u_\alpha^h \in K,$$
$$F_\alpha^h(u,v) := \sum_{j=1}^N \alpha^{\mu_j} F_j^h(u,v) + \alpha\langle U^s(u), v-u\rangle, \quad \alpha > 0, \tag{6}$$

has a unique solution denoted by u_α^h for each $\alpha,\ h > 0$. As well as for the variational inequalities [23, 24] or the operator equation of Hammerstein type [25, 26], we have the following conclusion.

Theorem 2. *If* $h/\alpha \to 0$ *as* $h, \alpha \to 0$, *then* $u_\alpha^h \to u_*$.

Proof. From (4) with that u_α is replaced by u_α^h, (5), (6) and the properties of the bifunctions F_j^h it follows

$$\sum_{j=1}^N \alpha^{\mu_j}\left[F_j(y,u_\alpha^h) - F_j^h(y,u_\alpha^h)\right] + \alpha\langle U^s(u_\alpha^h), y - u_\alpha^h\rangle \ge 0 \quad \forall y \in S.$$

Therefore,

$$m_s\|y-u_\alpha^h\|^2 \le \langle U^s(y), y-u_\alpha^h\rangle + \frac{1}{\alpha}\sum_{j=1}^N \alpha^{\mu_j}\left|F_j^h(y,u_\alpha^h) - F_j(y,u_\alpha^h)\right| =$$
$$= \left\langle U^s(y), y-u_\alpha^h\right\rangle + \frac{h}{\alpha}(N-1)g(\|y\|)\|y-u_\alpha^h\|,$$

for $\alpha \le 1$. Thus,

$$\|y-u_\alpha^h\| \le \frac{1}{m_s}\left[\|y\|^{s-1} + \frac{(N-1)h}{\alpha}g(\|y\|)\right]. \tag{7}$$

It means that $\{u_\alpha^h\}$ is bounded, when $h,\ \alpha,\ h/\alpha \to 0$. Since X is reflexive, then there exist a subsequence $\{u_k := u_{\alpha_k}^{h_k}\} \subset \{u_\alpha^h\}$ and an element $\tilde{x} \in X$ such that $u_k \rightharpoonup \tilde{x}$ as $k \to +\infty$, and K is also weak closed. Hence, the element $\tilde{x}$ is an element of K. By repeating the proof in Theorem 1 we obtain that $\tilde{x} \in S$ and $u_\alpha^h \to \tilde{x} = u_*$.

Theorem is proved.

Now, we study the problem of choosing $\alpha = \alpha(h)$. For this purpose, consider the function $\rho(\alpha) := \alpha(a_0 + t(\alpha))$, where $t(\alpha) = \|u_\alpha^h\|$ for each fixed $h > 0$. Obviously, from (5), (6) and property (ii) in condition 1 of F_j^h it implies that

$$m_s\alpha_0\|u_{\alpha_1}^h - u_{\alpha_2}^h\|^2 \le |\alpha_1-\alpha_2|\|u_{\alpha_2}^h\|^{s-1}\|u_{\alpha_1}^h - u_{\alpha_2}^h\| +$$
$$+\sum_{j=1}^N |\alpha_2^{\mu_j} - \alpha_1^{\mu_j}|\,|F_j^h(u_{\alpha_2}^h, u_{\alpha_1}^h)|$$

for $\alpha_i \in [\alpha_0, +\infty)$, $i = 1, 2$, and $\alpha_0 > 0$, where

$$\left|F_j^h(u_{\alpha_2}^h, u_{\alpha_1}^h)\right| \le \left|F_j^h(u_{\alpha_2}^h, u_{\alpha_1}^h) - F_j(u_{\alpha_2}^h, u_{\alpha_1}^h)\right| + \left|F_j(u_{\alpha_2}^h, u_{\alpha_1}^h)\right|.$$

Therefore, if $F_j(u,v)$ all satisfy condition (iii) in Theorem 1, then

$$m_s\alpha_0\|u^h_{\alpha_1}-u^h_{\alpha_2}\|^2 \le \left[|\alpha_1-\alpha_2|\|u^h_{\alpha_2}\|^{s-1}+hg(\|u^h_{\alpha_2}\|)\sum_{j=1}^N|\alpha_2^{\mu_j}-\alpha_1^{\mu_j}|\right]\times$$

$$\times\|u^h_{\alpha_1}-u^h_{\alpha_2}\|+C\sum_{j=1}^N|\alpha_2^{\mu_j}-\alpha_1^{\mu_j}|.$$

Hence,

$$\|u^h_{\alpha_1}-u^h_{\alpha_2}\|\le\tilde{c},$$

$$\tilde{c}=\frac{d}{2m_s\alpha_0}+\frac{1}{2m_s\alpha_0}\sqrt{d^2+4m_s\alpha_0C(N-1)|\alpha_1-\alpha_2|},$$

$$d=\Big[\|u^h_{\alpha_2}\|^{s-1}+h(N-1)g(\|u^h_{\alpha_2}\|)\Big]|\alpha_1-\alpha_2|.$$

Thus, $u^h_{\alpha_1}\to u^h_{\alpha_2}$ as $\alpha_1\to\alpha_2$. It means that $t(\alpha)$ is continuous on $[\alpha_0,+\infty)$. So, is the function $\rho(\alpha)$. We shall choose $\tilde{\alpha}=\alpha(h)$ satisfying the following equation:

$$\rho(\alpha)=h^p\alpha^{-q},\quad p,\,q>0. \tag{8}$$

Theorem 3. *Assume that $F_j(u,v)$ all satisfy condition* (iii) *in Theorem* 1. *Then, we have*:

(i) *for each fixed $h>0$ there exists at least a value $\tilde{\alpha}=\alpha(h)$ satisfying* (8),

(ii) $\lim_{h\to0}\alpha(h)=0$, *and*

(iii) *if $q\ge p$, then* $\lim_{h\to0}h/\alpha(h)=0$.

Proof. First, from (7) we can obtain the following inequality:

$$\alpha^q\rho(\alpha)\le\alpha^{1+q}\left[a_0+\|y\|+\frac{1}{m_s}\|y\|^{s-1}\right]+\alpha^q\frac{(N-1)h}{m_s}g\big(\|y\|\big)$$

for a fixed element $y\in S$. Therefore,

$$\lim_{\alpha\to+0}\alpha^q\rho(\alpha)=0.$$

On the other hand,

$$\lim_{\alpha\to+\infty}\alpha^{1+q}\rho(\alpha)\ge a_0\lim_{\alpha\to+\infty}\alpha^{q+1}=+\infty.$$

The intermidiate value theorem gives (i).

The second conclusion is proved by using the inequality

$$0\le\alpha(h)\le a_0^{-1/(1+q)}h^{p/(1+q)}$$

that is followed from $\alpha^{1+q}(h)\big[a_0+t(\alpha(h))\big]=h^p$.

Since

$$\left[\frac{h}{\alpha(h)}\right]^p=[h^p\alpha^{-q}(h)]\alpha^{q-p}(h)=\rho(\alpha(h))\alpha^{q-p}(h)=$$

$$=\alpha(h)[a_0+t(\alpha(h))]\alpha^{q-p}(h)\le$$

$$\le[a_0+\|y\|+\frac{1}{m_s}\|y\|^{s-1}]\alpha^{1+q-p}(h)+\alpha^{q-p}(h)\frac{(N-1)h}{m_s}g(\|y\|),$$

then

$$\lim_{h\to 0} h/\alpha(h) = 0.$$

Theorem is proved.

3. Application. We consider the following convex minimization problems with coupling constraints: find $u_* \in K$ such that

$$\varphi(u_*) = \min_{u\in S} \quad \varphi(u),$$
$$S = \Big\{\tilde{u} \in K \colon F_j(\tilde{u}, v) \geq 0 \quad \forall v \in K, \quad j = 1, \dots, N\Big\}, \tag{9}$$

where φ is a weak continuous convex functional on X, and F_j all are the bifunctions. In addition, assume that $\varphi(u) \geq 0$ for each $u \in X$ and is Gateau differentiable with the derivative A. Then, u_* solves (9) iff it solves the following variational inequality problem:

$$\langle A(u_*), v - u_*\rangle \geq 0 \quad \forall v \in K, \quad F_j(u_*, v) \geq 0, \quad j = 1, \dots N,$$

that is studied in [27] and [28] in the finite-dimensional Hilbert space $\mathbf{R}^n$. The presence of the functional constraints $F_j(u_*, v)$, which couple the parameters and the variables of the problem, is the basic distintion of this statement from the standard one. Set

$$F_{N+1}(u, v) = \varphi(v) - \varphi(u).$$

It is easy to verify that $F_{N+1}(u, v)$ is a bifunction. The regularized solution of problem (9) can be constructed by solving the single equlibrium problem

$$F_\alpha(u_\alpha, v) \geq 0 \quad \forall v \in K, \quad u_\alpha \in K,$$
$$F_\alpha(u, v) := \sum_{j=1}^{N+1} \alpha^{\mu_j} F_j(u, v) + \alpha\langle U^s(u), v - u\rangle, \quad \alpha > 0,$$
$$\mu_1 = 0 < \mu_j < \mu_{j+1} < 1, \quad j = 2, 3, \dots, N,$$

and α is the regularization parameter.

Note that the nonegative property of φ permits to obtain the estimate (3). From the proof of Theorem 1 it implies that $\varphi(v) \geq \varphi(u_*)\ \forall v \in S = \bigcap_{j=1}^{N} S_j$.

In particular, if the bifunctions F_j all are defined on the whole space X, then we introduce additionally the bifunction $F_0(u, v) := \mathrm{dis}(v, K) - \mathrm{dis}(u, K)$, where

$$\mathrm{dis}(x, K) = \min_{y\in K} \|x - y\|.$$

Then, we have the following single equilibrium:

$$F_\alpha(u_\alpha, v) \geq 0 \quad \forall v \in X, \quad u_\alpha \in X,$$
$$F_\alpha(u, v) := \sum_{j=0}^{N+1} \alpha^{\mu_j} F_j(u, v) + \alpha\langle U^s(u), v - u\rangle, \quad \alpha > 0,$$
$$\mu_0 = 0 < \mu_j < \mu_{j+1} < 1, \quad j = 2, 3, \dots, N,$$

and α is the regularization parameter.

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Received 16.07.07