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DIRECT AND INVERSE PROBLEMS FOR THE DIRAC OPERATOR WITH SPECTRAL PARAMETER LINEARLY CONTAINED IN BOUNDARY CONDITION

ПРЯМІ ТА ОБЕРНЕНІ ЗАДАЧІ ДЛЯ ОПЕРАТОРА ДІРАКА ІЗ СПЕКТРАЛЬНИМ ПАРАМЕТРОМ, ЩО ЛІНІЙНО ВХОДИТЬ ДО ГРАНИЧНОЇ УМОВИ

We investigate a problem for the Dirac differential operators in the case where an eigenparameter not only appears in the differential equation but is also linearly contained in the boundary condition. We prove uniqueness theorems for the inverse spectral problem with a known collection of eigenvalues and normalizing constants or two spectra.

Досліджено задачу для диференціальних операторів Дірака у випадку, коли власний параметр не тільки присутній у диференціальному рівнянні, але й лінійно входить до граничної умови. Доведено теореми єдиності в оберненій спектральній задачі з відомим набором власних значень і нормуючих сталих або двох спектрів.

Let us consider the canonical system of Dirac differential equations

$$ly \;:=\; By' + \Omega(x)y \;=\; \lambda y, \quad x \in (0,\pi), \tag{1}$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $p(x)$ and $q(x)$ are real valued functions in $L_2(0,\pi)$, λ is a spectral parameter.

By L we denote the boundary-value problem generated by equation (1) with the boundary conditions

$$U(y) \;:=\; y_1(0) \;=\; 0, \tag{2}$$

$$V(y) \;:=\; \lambda\big(y_2(\pi) + Hy_1(\pi)\big) - H_1 y_1(\pi) - H_2 y_2(\pi) \;=\; 0, \tag{3}$$

where H, H_1 and H_2 are real numbers. We assume that $\rho := HH_1 - H_2 > 0$.

Boundary-value problems often appear in mathematics, mechanics, physics, geophysics and other branches of natural properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is central importance in disciplins ranging from engineering to the geosciences.

Eigenvalue dependent boundary conditions were examined even before the time of Sturm and Liouville [1]. Linear conditions like (2) and (3) were investigated in [2, 3]. Direct and inverse problems for Dirac operators are fairly well studied (see [4 – 6] and references therein).

The inner product in the Hilbert space $\mathcal{H} = L_2(0,\pi) \oplus L_2(0,\pi) \oplus \mathbb{C}$ is defined by

$$\langle Y, Z \rangle \;:=\; \int_0^\pi \big(y_1(x)\overline{z}_1(x) + y_2(x)\overline{z}_2(x)\big)\,dx + \frac{1}{\rho}\, y_3 \overline{z}_3$$

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for

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3 \end{pmatrix} \in \mathcal{H}.$$

Define an operator T (see [3]) acting in $\mathcal{H}$ such that

$$T(Y) := \begin{pmatrix} By'(x) + \Omega(x)y(x) \\ H_1 y_1'(\pi) + H_2 y_1(\pi) \end{pmatrix}$$

with

$$D(T) = \Big\{ Y \in \mathcal{H} : Y = (y_1, y_2, y_3)^T,\ y_1, y_2 \in AC[0,\pi],$$

$$lY \in \mathcal{H},\ y_1(0) = 0,\ y_3(0) = y_2(\pi) + Hy_1(\pi) \Big\}.$$

It is clear that T is a closed operator in $\mathcal{H}$ and the eigenvalue problem of operator T is adequate problem of (1) – (3).

1. Properties of spectrum. In this section, we investigate some properties of operator T and its spectrum. We assume that $q(x) = 0$ (see [4]).

Lemma 1. (i) *Two eigenfunctions* $y(x,\lambda_1) = [y_1(x,\lambda_1),\ y_2(x,\lambda_1)]^T$, $z(x,\lambda_2) = [z_1(x,\lambda_2),\ z_2(x,\lambda_2)]^T$ *corresponding to different eigenvalues* λ_1 *and* λ_2 *are orthogonal in the sense of*

$$\int_0^\pi [y_1(x,\lambda_1)z_1(x,\lambda_2) + y_2(x,\lambda_1)z_2(x,\lambda_2)]dx + \frac{1}{\rho} y_3 z_3 = 0. \tag{4}$$

(ii) *All eigenvalues of the operator* T (*or problem* L) *are real numbers and all eigenfunctions are real valued.*

Proof. (i) Since the eigenfunctions $y(x,\lambda_1)$ and $z(x,\lambda_2)$ are the solutions of the system (1), the following equalities hold:

$$y_2'(x,\lambda_1) + \{p(x) - \lambda_1\}\, y_1(x,\lambda_1) = 0,$$

$$y_1'(x,\lambda_1) + \{p(x) + \lambda_1\}\, y_2(x,\lambda_1) = 0,$$

$$z_2'(x,\lambda_2) + \{p(x) - \lambda_2\}\, z_1(x,\lambda_2) = 0,$$

$$z_1'(x,\lambda_2) + \{p(x) + \lambda_2\}\, z_2(x,\lambda_2) = 0.$$

If the multiply these equalities by $z_1(x,\lambda_2)$, $-z_2(x,\lambda_2)$, $-y_1(x,\lambda_1)$ and $y_2(x,\lambda_1)$, respectively, to get

$$\frac{d}{dx}\Big\{ y_2(x,\lambda_1)\, z_1(x,\lambda_2) - y_1(x,\lambda_1)\, z_2(x,\lambda_2) \Big\} =$$

$$= (\lambda_1 - \lambda_2)\Big\{ y_1(x,\lambda_1)\, z_1(x,\lambda_2) + y_2(x,\lambda_1)\, z_2(x,\lambda_2) \Big\}.$$

Integrate last equality from 0 to π with respect to x to obtain

$$(\lambda_1-\lambda_2)\int_0^\pi \left\{ y_1(x,\lambda_1)\,z_1(x,\lambda_2) + y_2(x,\lambda_1)\,z_2(x,\lambda_2) \right\} dx =$$

$$= \left\{ y_1(x,\lambda_1)\,z_2(x,\lambda_2) - y_2(x,\lambda_1)\,z_1(x,\lambda_2) \right\}.$$

Since the functions $y(x,\lambda_1)$ and $z(x,\lambda_2)$ are eigenfunctions, the equality

$$(\lambda_1-\lambda_2)\int_0^\pi \left\{ y_1(x,\lambda_1)\,z_1(x,\lambda_2) + y_2(x,\lambda_1)\,z_2(x,\lambda_2) \right\} dx =$$

$$= y_1(\pi,\lambda_1)\,z_2(\pi,\lambda_2) - y_2(\pi,\lambda_1)\,z_1(\pi,\lambda_2) \tag{5}$$

is valid. On the other hand, from (3) we have

$$\lambda_1\left[y_2(\pi,\lambda_1) + Hy_1(\pi,\lambda_1) \right] = H_1 y_1(\pi,\lambda_1) + H_2 y_2(\pi,\lambda_1),$$

$$\lambda_2\left[z_2(\pi,\lambda_2) + Hz_1(\pi,\lambda_2) \right] = H_1 z_1(\pi,\lambda_2) + H_2 z_2(\pi,\lambda_2).$$

Let us multiply these equalities by $z_2(\pi,\lambda_2)+Hz_1(\pi,\lambda_2)$ and $y_2(\pi,\lambda_1)+Hy_1(\pi,\lambda_1)$, respectively, and subtract side by side to get

$$(\lambda_1-\lambda_2)\left[y_2(\pi,\lambda_1) + Hy_1(\pi,\lambda_1) \right]\left[z_2(\pi,\lambda_2) + Hz_1(\pi,\lambda_2) \right] =$$

$$= -\rho\left(y_1(\pi,\lambda_1)\,z_2(\pi,\lambda_2) - y_2(\pi,\lambda_1)\,z_1(\pi,\lambda_2) \right). \tag{6}$$

The proof is completed by using (5) and (6).

(ii) Let $\lambda \neq \bar{\lambda}$, then $y(x,\lambda)$ and $\bar{y}(x,\lambda)$ are the different eigenfunctions of operator T associated with eigenvalues λ and $\bar{\lambda}$, respectively. From (i) we have

$$\int_0^\pi \left(y_1(x,\lambda)\,\bar{y}_1(x,\lambda) + y_2(x,\lambda)\,\bar{y}_2(x,\lambda) \right) dx + \frac{1}{\rho}\, y_3\bar{y}_3 = 0,$$

then

$$\int_0^\pi \left\{ \left| y_1(x,\lambda) \right|^2 + \left| y_2(x,\lambda) \right|^2 \right\} dx + \frac{1}{\rho}\left| y_3 \right|^2 = 0,$$

hence, $y_1(x,\lambda) = y_2(x,\lambda) = y_3 = 0$. This contradiction gives the proof of (ii).

The lemma is proved.

Let us denote the solutions of (1) by $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ satisfying the initial conditions

$$\varphi(0,\lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad \psi(\pi,\lambda) = \begin{pmatrix} H_2-\lambda \\ \lambda H - H_1 \end{pmatrix}, \tag{7}$$

respectively.

It is shown in [7] that every solution of equations (1) satisfying the above initial conditions, has a representation as follows:

$$\varphi(x,\lambda) = \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix} + \int_0^\pi K(x,t)\begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix} dt, \tag{8}$$

where $K(x,t) = \begin{pmatrix} K_{11}(x,t) & -K_{12}(x,t) \\ K_{21}(x,t) & -K_{22}(x,t) \end{pmatrix}$ and $K_{ij}(x,.) \in L_2(-x,x)$, $i,j = 1,2$, for every fixed $x \in [0,\pi]$.

One can easily check that the following asymptotic formulae hold for sufficiently large $|\lambda|$:

$$\varphi_j(x,\lambda) = O\left(\exp|\tau|x\right), \quad j = 1,2, \quad 0 < x < \pi, \tag{9}$$

$$\varphi'_j(x,\lambda) = O\left(|\lambda|\exp|\tau|x\right), \quad j = 1,2, \quad 0 < x < \pi, \tag{10}$$

$$\psi_j(x,\lambda) = O\left(|\lambda|\exp|\tau|(\pi - x)\right), \quad j = 1,2, \quad 0 < x < \pi, \tag{11}$$

where $\tau = \operatorname{Im}\lambda$.

The characteristic function $\Delta(\lambda)$ of the problem L is defined as follows:

$$\Delta(\lambda) = \lambda\left(\varphi_2(\pi,\lambda) + H\varphi_1(\pi,\lambda)\right) - H_1\varphi_1(\pi,\lambda) + H_2\varphi_2(\pi,\lambda) \tag{12}$$

and zeros of $\Delta(\lambda)$ coincide with the eigenvalues of problem L.

We define norming constants by

$$\alpha_n := \int_0^\pi \left(\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right)dx + \frac{1}{\rho}\varphi_3^2. \tag{13}$$

Lemma 2. *The eigenvalues of the problem L are simple and separated.*

Proof. Let us write the following equations:

$$\psi'_2(x,\lambda) + \{p(x) - \lambda\}\psi_1(x,\lambda) = 0,$$

$$\psi'_1(x,\lambda) + \{p(x) + \lambda\}\psi_2(x,\lambda) = 0,$$

$$\varphi'_2(x,\lambda_n) + \{p(x) - \lambda_n\}\varphi_1(x,\lambda_n) = 0,$$

$$\varphi'_1(x,\lambda_n) + \{p(x) + \lambda_n\}\varphi_2(x,\lambda_n) = 0.$$

Multiply these equalities by $\varphi_1(x,\lambda_n)$, $-\varphi_2(x,\lambda_n)$, $-\psi_1(x,\lambda)$ and $\psi_2(x,\lambda)$, respectively, to get

$$\frac{d}{dx}\left\{\varphi_1(x,\lambda_n)\psi_2(x,\lambda) - \varphi_2(x,\lambda_n)\psi_1(x,\lambda)\right\} =$$

$$= (\lambda_n - \lambda)\left\{\varphi_1(x,\lambda_n)\psi_1(x,\lambda) + \varphi_2(x,\lambda_n)\psi_2(x,\lambda)\right\}.$$

After integrating last equalities from 0 to π with respect to x, we obtain

$$\left[\varphi_1(x,\lambda_n)\psi_2(x,\lambda) - \varphi_2(x,\lambda_n)\psi_1(x,\lambda)\right] =$$

$$= (\lambda_n - \lambda)\int_0^\pi \left\{\varphi_1(x,\lambda_n)\psi_1(x,\lambda) + \varphi_2(x,\lambda_n)\psi_2(x,\lambda)\right\}dx.$$

Let the functions $\varphi(x,\lambda_n)$ be an eigenfunction. Use (2) to get

$$\int_0^\pi \left\{\varphi_1(x,\lambda_n)\psi_1(x,\lambda) + \varphi_2(x,\lambda_n)\psi_2(x,\lambda)\right\}dx =$$

$$= \frac{1}{\lambda_n - \lambda}\left\{ \varphi_1(\pi,\lambda_n)\,\psi_2(\pi,\lambda) - \varphi_2(\pi,\lambda_n)\,\psi_1(\pi,\lambda) - \psi_1(0,\lambda)\right\} =$$

$$= -\frac{1}{\rho}\left\{ \varphi_2(\pi,\lambda_n) + H\varphi_1(\pi,\lambda_n)\right\}\left\{ \psi_2(\pi,\lambda) + H\psi_1(\pi,\lambda)\right\} + \frac{\Delta(\lambda)}{\lambda-\lambda_n}.$$

If we pass throuth the limit as $\lambda \to \lambda_n$ and use the equality $\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n)$, then

$$\beta_n\left\{ \int_0^\pi \left\{ \varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right\} dx + \frac{1}{\rho}\varphi_3^2 \right\} = \Delta'(\lambda_n).$$

Hence, $\Delta'(\lambda_n) = \beta_n\alpha_n$, where $\beta_n = -\psi_2(0,\lambda_n)$. It is obvious that $\Delta'(\lambda_n) \neq 0$. So, eigenvalues of the problem L is simple.

Since the function $\Delta(\lambda)$ is an entire function of λ, the zeros of $\Delta(\lambda)$ are separated.

The lemma is proved.

Theorem 1. *For the eigenvalues* λ_n *and the normalizing numbers* α_n *of the problem* (1) – (3), *the following asymptotic formulae hold*:

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \tag{14}$$

$$\alpha_n = \pi + \gamma_n, \tag{15}$$

where ε_n, $\gamma_n \in l_2$ *and* λ_n^0 *are the zeros of* $\Delta_0(\lambda) := -\cos\lambda\pi + H\sin\lambda\pi$, *i. e.,* $\lambda_n^0 = n + \frac{1}{\pi}\arctan\frac{1}{H}$.

Proof. Using (8), we get

$$\Delta(\lambda) = \lambda(-\cos\lambda\pi + H\sin\lambda\pi) - H_1\sin\lambda\pi + H_2\cos\lambda\pi +$$

$$+ \lambda\int_0^\pi \left(K_{21}(\pi,t) + HK_{11}(\pi,t)\right)\sin\lambda t\,dt + \lambda\int_0^\pi \left(K_{22}(\pi,t) + HK_{12}(\pi,t)\right)\cos\lambda t\,dt -$$

$$- \int_0^\pi \left(H_1K_{11}(\pi,t) + H_2K_{21}(\pi,t)\right)\sin\lambda t\,dt - \int_0^\pi \left(H_1K_{12}(\pi,t) + H_2K_{22}(\pi,t)\right)\cos\lambda t\,dt.$$

Since the eigenvalues are the zeros of $\Delta(\lambda)$, we can write the following equation for them:

$$-\cos\lambda\pi + H\sin\lambda\pi - \frac{H_1}{\lambda}\sin\lambda\pi + \frac{H_2}{\lambda}\cos\lambda\pi +$$

$$+ \int_0^\pi \left(K_{21}(\pi,t) + HK_{11}(\pi,t)\right)\sin\lambda t\,dt + \int_0^\pi \left(K_{22}(\pi,t) + HK_{12}(\pi,t)\right)\cos\lambda t\,dt -$$

$$- \frac{1}{\lambda}\int_0^\pi \left(H_1K_{11}(\pi,t) + H_2K_{21}(\pi,t)\right)\sin\lambda t\,dt -$$

$$-\frac{1}{\lambda}\int_0^{\pi}\left(H_1K_{12}(\pi,t)+H_2K_{22}(\pi,t)\right)\cos\lambda t\,dt = 0.$$

Denote

$$G_n = \left\{\lambda\in C: |\lambda| = \left|\lambda_n^0\right|+\frac{\beta}{2}\right\}, \quad n = 0,\pm1,\pm2,\ldots,$$

$$G_\delta = \left\{\lambda: \left|\lambda-\lambda_n^0\right|\geq\delta,\ n = 0,\pm1,\pm2,\ldots\right\}, \qquad \hat{\Delta}(\lambda) = \frac{\Delta(\lambda)}{\lambda},$$

where δ is a sufficiently small number.

Since $\left|\Delta_0(\lambda)\right|\geq C_\delta\exp\left(|\tau|\pi\right)$ for $\lambda\in\bar{G}_\delta$ and $\left|\hat{\Delta}(\lambda)-\Delta_0(\lambda)\right| < C_\delta\exp|\tau|\pi$ for sufficiently large values n and $\lambda\in G_n$, we have

$$\left|\Delta_0(\lambda)\right| \geq C_\delta\exp\left(|\tau|\pi\right) > \left|\hat{\Delta}(\lambda)-\Delta_0(\lambda)\right|.$$

Using the Rouche theorem, we establish that, for sufficiently large n, the functions $\Delta_0(\lambda)$ and $\Delta_0(\lambda)+\left\{\hat{\Delta}(\lambda)-\Delta_0(\lambda)\right\} = \tilde{\Delta}(\lambda)$ have the same number of zeros inside the contour G_n. That is, they have $(2n+1)$ numbers of zeros: $\lambda_{-n},\ldots,\lambda_0,\ldots,\lambda_n$. Thus, the eigenvalues $\{\lambda_n\}_{n\geq0}$ are of the form $\lambda_n = \lambda_n^0+\varepsilon_n$, where $\lim\limits_{n\to\infty}\varepsilon_n = 0$.

In last equality, if we put $\lambda_n^0+\varepsilon_n$ instead of λ_n and use $\Delta_0(\lambda_n^0+\varepsilon_n) = \Delta_0'(\lambda_n^0)\,\varepsilon_n+o(\varepsilon_n)$, we get $\varepsilon_n\in l_2$.

Hence, the asymptotic formula (14) for the eigenvalues λ_n of the problem (1) – (3) is true.

Finally, to prove (15), we can write the following equalities from (8):

$$\varphi_1(x,\lambda_n) = \sin\lambda_n x+f_n(x), \qquad \varphi_2(x,\lambda_n) = -\cos\lambda_n x+g_n(x),$$

where

$$f_n(x) = \int_0^x K_{11}(x,t)\sin\lambda_n t\,dt+\int_0^x K_{12}(x,t)\cos\lambda_n t\,dt,$$

$$g_n(x) = \int_0^x K_{21}(x,t)\sin\lambda_n t\,dt+\int_0^x K_{22}(x,t)\cos\lambda_n t\,dt,$$

and $f_n(x)$, $g_n(x)\in l_2$ for all $x\in(0,\pi)$. Using (3) and (13), we get

$$\alpha_n := \int_0^{\pi}\left(\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)\right)dx+\frac{1}{\rho\lambda_n^2}\left(H_1\varphi_1(\pi,\lambda_n)+H_2\varphi_2(\pi,\lambda_n)\right)^2 =$$

$$= \int_0^{\pi}dx+\gamma_n.$$

Hence, $\alpha_n = \pi+\gamma_n$, $\gamma_n\in l_2$.

The theorem is proved.

***Remark* 1.** If the function $\Omega(x)$ is differentiable, eigenvalues and normalizing numbers of the problem (1) – (3) satisfy the following asymptotic estimates:

$$\gamma_n = \lambda_n^0 + \frac{\delta_n}{n} + \frac{\zeta_n}{n},$$

$$\alpha_n = n + \frac{\gamma_n}{n} + \frac{\xi_n}{n},$$

where $\zeta_n, \xi_n \in l_2$, $\delta_n, \gamma_n \in l_\infty$.

2. Weyl solution, Weyl function. Assume that a vector function $\Phi(x,\lambda) = \begin{pmatrix}\Phi_1(x,\lambda)\\ \Phi_2(x,\lambda)\end{pmatrix}$ is a solution of the system (4) that satisfies the conditions $\Phi_1(0,\lambda) = 1$ and $\lambda(\Phi_2(\pi) + H\Phi_1(\pi)) = H_1\Phi_1(\pi) + H_2\Phi_2(\pi)$. The function $\Phi(x,\lambda)$ is called the Weyl solution of the boundary-value problem L.

Let $C(x,\lambda) = \begin{pmatrix}C_1(x,\lambda)\\ C_2(x,\lambda)\end{pmatrix}$ denote solutions of system (1) that satisfy the initial conditions $C(0,\lambda) = \begin{pmatrix}1\\0\end{pmatrix}$. It is clear that the functions $\psi(x,\lambda)$ and $C(x,\lambda)$ are entire with respect to λ. Then the function $\psi(x,\lambda)$ can be represented as follows:

$$\psi(x,\lambda) = \psi_2(0,\lambda)\varphi(x,\lambda) + \Delta(\lambda)C(x,\lambda)$$

or

$$\frac{\psi(x,\lambda)}{\Delta(\lambda)} = C(x,\lambda) - \frac{\psi_2(0,\lambda)}{\Delta(\lambda)}\varphi(x,\lambda). \tag{16}$$

Denote

$$M(\lambda) = -\frac{\psi_2(0,\lambda)}{\Delta(\lambda)}. \tag{17}$$

It is clear that

$$\Phi(x,\lambda) = C(x,\lambda) + M(\lambda)\varphi(x,\lambda). \tag{18}$$

The function $M(\lambda) = -\Phi_2(0,\lambda)$ is called the Weyl function for the problem L. The Weyl solution and Weyl function are meromorphic functions with respect to λ having poles in the spectrum of the problem L. Relations (17) and (18) yield

$$\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)}. \tag{19}$$

The lemma is proved.

Theorem 2. *The following representation is true*:

$$M(\lambda) = \frac{1}{\alpha_0(\lambda-\lambda_0)} + \sum_{n=1}^{\infty}\left\{\frac{1}{\alpha_n(\lambda-\lambda_n)} + \frac{1}{\alpha_n^0\lambda_n^0}\right\}. \tag{20}$$

Proof. Note that $\psi_1(x,\lambda) = \psi_{01}(x,\lambda) + f_1$, $\psi_2(x,\lambda) = \psi_{02}(x,\lambda) + f_2$, where

$$f_1(x,\lambda) = \int_x^\pi \sin\lambda(t-x)\psi_1(t,\lambda)p(t)\,dt + \int_x^\pi \cos\lambda(t-x)\psi_2(t,\lambda)p(t)\,dt,$$

$$f_2(x,\lambda) = \int_x^\pi \cos\lambda(t-x)\psi_1(t,\lambda)p(t)\,dt - \int_x^\pi \sin\lambda(t-x)\psi_2(t,\lambda)p(t)\,dt.$$

If we use these equalities, we get

$$M(\lambda) - M_0(\lambda) = \frac{\psi_2(0,\lambda)}{\psi_1(0,\lambda)} - \frac{\psi_{02}(0,\lambda)}{\psi_{01}(0,\lambda)} = \frac{\psi_{02}(0,\lambda)+f_2}{\psi_{01}(0,\lambda)+f_1} - \frac{\psi_{02}(0,\lambda)}{\psi_{01}(0,\lambda)} =$$

$$= \frac{(\psi_{02}(0,\lambda)+f_2)\psi_{01}(0,\lambda) - (\psi_{01}(0,\lambda)+f_1)\psi_{02}(0,\lambda)}{(\psi_{01}(0,\lambda)+f_1)\psi_{01}(0,\lambda)} =$$

$$= \frac{f_2}{\Delta(\lambda)} - \frac{f_1}{\Delta(\lambda)} M_0(\lambda).$$

Since $|\Delta(\lambda)| > C_\delta |\lambda| e^{|\operatorname{Im}\lambda|\pi}$, we have $\lim\limits_{|\lambda|\to\infty} e^{-|\operatorname{Im}\lambda|\pi} \dfrac{|f_i(\lambda)|}{|\Delta(\lambda)|} = 0$ and, for $\lambda \in G_\delta$, the equality

$$M(\lambda) - M_0(\lambda) = \frac{f_2}{\Delta(\lambda)} - \frac{f_1}{\Delta(\lambda)} M_0(\lambda)$$

yields

$$\lim_{|\lambda|\to\infty} \varepsilon_n \sup_{\lambda\in G_\delta} |M(\lambda) - M_0(\lambda)| = 0. \tag{21}$$

The vector functions $\varphi(x,\lambda_n)(\varphi_0(x,\lambda_n^0))$ and $\psi(x,\lambda_n)(\psi_0(x,\lambda_n^0))$ are eigenfunctions of the problem $L(L_0)$. Therefore, there exists constants $\beta_n(\beta_n^0)$ such that

$$\psi(x,\lambda_n) = \beta_n \varphi(x,\lambda_n) \quad (\psi_0(x,\lambda_n^0) = \beta_n^0 \varphi_0(x,\lambda_n^0)).$$

Since $\psi_2(0,\lambda_n) = \beta_n \varphi_2(0,\lambda_n)$, we get the following equalities:

$$\beta_n = \psi_2(0,\lambda_n), \qquad \beta_n^0 = \psi_{02}(0,\lambda_n^0).$$

Since $\psi_2(0,\lambda)$ and $\Delta(\lambda)$ are the analytic functions at the point $\lambda = \lambda_n$ and $\psi_2(0,\lambda_n) \neq 0$, $\Delta(\lambda_n) = 0$, $\Delta'(\lambda_n) \neq 0$, we have that the functions $M(\lambda)$ and $M_0(\lambda)$ have simple poles at these points. Hence, using the equalities $\alpha_n\varphi_2(0,\lambda_n) = \Delta'(\lambda_n)$ and $\varphi_{02}(0,\lambda_n^0)\alpha_n^0 = -\Delta_0'(\lambda_n^0)$, we get

$$\operatorname{Res}(M(\lambda), \lambda=\lambda_n) = \frac{\psi_2(0,\lambda_n)}{\Delta'(\lambda_n)} = \frac{1}{\alpha_n},$$
$$\operatorname{Res}(M_0(\lambda), \lambda=\lambda_n) = \frac{\psi_2(0,\lambda_n)}{\Delta_0'(\lambda_n)} = \frac{1}{\alpha_n^0}. \tag{22}$$

Denote $\Gamma_n = \{\lambda : |\lambda| = |\lambda_n| + \varepsilon\}$, where ε is a sufficiently small number. Consider the contour integral

$$I_n(x) = \frac{1}{2\pi i}\int_{\Gamma_n}\frac{M(\mu)-M_0(\mu)}{\lambda-\mu}\,d\mu, \qquad \lambda\in\operatorname{int}\Gamma_n.$$

By virtue of (21), we have $\lim\limits_{n\to\infty} I_n(x) = 0$. On the other hand, according to the theorem on residues, relation (22) yields

$$I_n(x) = -M(\lambda) + M_0(\lambda) - \sum_{\lambda_n\in\operatorname{int}\Gamma_n}\frac{1}{\alpha_n(\lambda_n-\lambda)} + \sum_{\lambda_n^0\in\operatorname{int}\Gamma_n}\frac{1}{\alpha_n^0(\lambda_n^0-\lambda)}.$$

Hence, as $n\to\infty$, $\lim\limits_{n\to\infty} I_n(x) = 0$ imply

$$M(\lambda) = M_0(\lambda) + \sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n(\lambda-\lambda_n)} - \frac{1}{\alpha_n^0(\lambda-\lambda_n^0)}\right). \tag{23}$$

It follows from the function $M_0(\lambda)$ that

$$M_0(\lambda) = \frac{1}{\alpha_0^0\lambda} + \sum_{n=1}^{\infty}\frac{1}{\alpha_n^0}\left(\frac{1}{\lambda-\lambda_n^0}+\frac{1}{\lambda_n^0}\right).$$

The comparison of the last two equalities yields

$$M(\lambda) = \frac{1}{\alpha_0^0\lambda} + \sum_{n=1}^{\infty}\frac{1}{\alpha_n^0}\left(\frac{1}{\lambda-\lambda_n^0}+\frac{1}{\lambda_n^0}\right) + \sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n(\lambda-\lambda_n)} - \frac{1}{\alpha_n^0(\lambda-\lambda_n^0)}\right) =$$

$$= \frac{1}{\alpha_0^0\lambda} + \frac{1}{\alpha_0(\lambda-\lambda_0)} - \frac{1}{\alpha_0^0\lambda} +$$

$$+ \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n^0(\lambda-\lambda_n^0)} + \frac{1}{\alpha_n^0\lambda_n^0} + \frac{1}{\alpha_n(\lambda-\lambda_n)} - \frac{1}{\alpha_n^0(\lambda-\lambda_n^0)}\right).$$

Hence,

$$M(\lambda) = \frac{1}{\alpha_0(\lambda-\lambda_0)} + \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n^0\lambda_n^0} + \frac{1}{\alpha_n(\lambda-\lambda_n)}\right).$$

The theorem is proved.

3. Inverse problem. In this section, we investigate the inverse problem of the reconstruction of a boundary-value problem L from its spectral characteristics. We consider three statements of the inverse problem of the reconstruction of the boundary-value problem L from the Weyl function, from the spectral data $\{\lambda_n,\alpha_n\}_{n\geq 0}$, and from two spectra $\{\lambda_n,\mu_n\}_{n\geq 0}$.

Let us formulate a theorem on the uniqueness of a solution of the inverse problem with the use of the Weyl function. For this purpose, together with L we consider the boundary-value problem $\tilde{L}$ of the same form but with potential $\tilde{\Omega}(x)$. It is assumed in what follows that if a certain symbol α denotes an object related to the problem L, then $\tilde{\alpha}$ denotes the coresponding object related to te problem $\tilde{L}$.

Theorem 3. *If* $M(\lambda) = \tilde{M}(\lambda)$, *then* $L = \tilde{L}$. *Thus, the boundary-value problem* L *is uniquely defined by a Weyl function.*

Proof. We introduce a matrix $P(x,\lambda) = [P_{j\lambda}(x,\lambda)]_{j,\lambda=1,2}$, by the formula

$$P(x,\lambda)\begin{pmatrix}\tilde{\varphi}_1 & \tilde{\Phi}_1\\ \tilde{\varphi}_2 & \tilde{\Phi}_2\end{pmatrix} = \begin{pmatrix}\varphi_1 & \Phi_1\\ \varphi_2 & \Phi_2\end{pmatrix}.$$

For the Wronskian of the solutions

$$\tilde{\varphi} = \begin{pmatrix}\tilde{\varphi}_1\\ \tilde{\varphi}_2\end{pmatrix}, \qquad \tilde{\Phi}_1 = \begin{pmatrix}\tilde{\Phi}_1\\ \tilde{\Phi}_2\end{pmatrix},$$

we have

$$W\left\{\tilde{\varphi}(x,\lambda),\tilde{\Phi}(x,\lambda)\right\} \equiv \tilde{\varphi}_1(x,\lambda)\tilde{\Phi}_2(x,\lambda) - \tilde{\varphi}_2(x,\lambda)\tilde{\Phi}_1(x,\lambda) = 1.$$

Take this into account and multiply both sides of the following equation from left by the matrix $\begin{pmatrix}\tilde{\Phi}_2 & -\tilde{\Phi}_1\\ -\tilde{\varphi}_2 & \tilde{\varphi}_1\end{pmatrix}$,

$$\begin{pmatrix}P_{11}(x,\lambda) & P_{12}(x,\lambda)\\ P_{21}(x,\lambda) & P_{22}(x,\lambda)\end{pmatrix}\begin{pmatrix}\tilde{\varphi}_1 & \tilde{\Phi}_1\\ \tilde{\varphi}_2 & \tilde{\Phi}_2\end{pmatrix} = \begin{pmatrix}\varphi_1 & \Phi_1\\ \varphi_2 & \Phi_2\end{pmatrix},$$

in order to get

$$\begin{pmatrix}P_{11}(x,\lambda) & P_{12}(x,\lambda)\\ P_{21}(x,\lambda) & P_{22}(x,\lambda)\end{pmatrix} = \begin{pmatrix}\varphi_1\tilde{\Phi}_2 - \Phi_1\tilde{\varphi}_2 & -\varphi_1\tilde{\Phi}_1 + \Phi_1\tilde{\varphi}_1\\ \varphi_2\tilde{\Phi}_2 - \Phi_2\tilde{\varphi}_2 & -\varphi_2\tilde{\Phi}_1 + \Phi_2\tilde{\varphi}_1\end{pmatrix}$$

or

$$\begin{aligned}
P_{11}(x,\lambda) &= \varphi_1(x,\lambda)\tilde{\Phi}_2(x,\lambda) - \Phi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda),\\
P_{12}(x,\lambda) &= -\varphi_1(x,\lambda)\tilde{\Phi}_1(x,\lambda) + \Phi_1(x,\lambda)\tilde{\varphi}_1(x,\lambda),\\
P_{21}(x,\lambda) &= \varphi_2(x,\lambda)\tilde{\Phi}_2(x,\lambda) - \Phi_2(x,\lambda)\tilde{\varphi}_2(x,\lambda),\\
P_{22}(x,\lambda) &= -\varphi_2(x,\lambda)\tilde{\Phi}_1(x,\lambda) + \Phi_2(x,\lambda)\tilde{\varphi}_1(x,\lambda),
\end{aligned} \tag{24}$$

$$\begin{aligned}
\varphi_1(x,\lambda) &= P_{11}\tilde{\varphi}_1(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}_2(x,\lambda),\\
\varphi_2(x,\lambda) &= P_{21}\tilde{\varphi}_1(x,\lambda) + P_{22}(x,\lambda)\tilde{\varphi}_2(x,\lambda),\\
\Phi_1(x,\lambda) &= P_{11}(x,\lambda)\tilde{\Phi}_1(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}_2(x,\lambda),\\
\Phi_2(x,\lambda) &= P_{21}(x,\lambda)\tilde{\Phi}_1(x,\lambda) + P_{22}(x,\lambda)\tilde{\Phi}_2(x,\lambda).
\end{aligned} \tag{25}$$

Relations (16) and (18) yield

$$P_{11}(x,\lambda) - 1 = \frac{\tilde{\psi}_2(x,\lambda)\left[\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)\right]}{\tilde{\Delta}(\lambda)} - \tilde{\varphi}_2(x,\lambda)\left[\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right],$$

$$P_{12}(x,\lambda) = \frac{\psi_1(x,\lambda)\left[\tilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda)\right]}{\Delta(\lambda)} + \varphi_1(x,\lambda)\left[\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right],$$

$$P_{21}(x,\lambda) = \frac{\tilde{\psi}_2(x,\lambda)\left[\varphi_2(x,\lambda)-\tilde{\varphi}_2(x,\lambda)\right]}{\tilde{\Delta}(\lambda)} - \tilde{\varphi}_2(x,\lambda)\left[\frac{\psi_2(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)}\right],$$

$$P_{22}(x,\lambda) - 1 = \frac{\psi_2(x,\lambda)\left[\tilde{\varphi}_1(x,\lambda)-\varphi_1(x,\lambda)\right]}{\Delta(\lambda)} + \varphi_2(x,\lambda)\left[\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right].$$

It follows from the representations of the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, the inequality $\left|\Delta(\lambda)\right| > C_\delta\left|\lambda\right|e^{\left|\operatorname{Im}\lambda\right|\pi}$, and the Lebesque lemma that

$$\lim_{\substack{\lambda\to\infty\\ \lambda\in R}} \max_{0\le x\le\pi}\left|P_{11}(x,\lambda)-1\right| = \lim_{\substack{\lambda\to\infty\\ \lambda\in R}} \max_{0\le x\le\pi}\left|P_{22}(x,\lambda)-1\right| =$$

$$= \lim_{\substack{\lambda\to\infty\\ \lambda\in R}} \max_{0\le x\le\pi}\left|P_{12}(x,\lambda)\right| = \lim_{\substack{\lambda\to\infty\\ \lambda\in R}} \max_{0\le x\le\pi}\left|P_{21}(x,\lambda)\right| = 0. \tag{26}$$

According to (18) and (25), we have

$$P_{11}(x,\lambda) = \varphi_1(x,\lambda)\tilde{C}_2(x,\lambda) - C_1(x,\lambda)\tilde{\varphi}_2(x,\lambda) + (\tilde{M}(\lambda)-M(\lambda))\varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda),$$

$$P_{12}(x,\lambda) = C_1(x,\lambda)\tilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda)\tilde{C}_1(x,\lambda) + (M(\lambda)-\tilde{M}(\lambda))\varphi_1(x,\lambda)\tilde{\varphi}_1(x,\lambda),$$

$$P_{21}(x,\lambda) = \varphi_2(x,\lambda)\tilde{C}_2(x,\lambda) - C_2(x,\lambda)\tilde{\varphi}_2(x,\lambda) + (\tilde{M}(\lambda)-M(\lambda))\varphi_2(x,\lambda)\tilde{\varphi}_2(x,\lambda),$$

$$P_{22}(x,\lambda) = C_2(x,\lambda)\tilde{\varphi}_1(x,\lambda) - \tilde{C}_1(x,\lambda)\varphi_2(x,\lambda) + (M(\lambda)-\tilde{M}(\lambda))\varphi_2(x,\lambda)\tilde{\varphi}_2(x,\lambda).$$

Thus, the functions $P_{j\lambda}(x,\lambda)$ are entire with respect to λ for every fixed x as $M(\lambda) = \tilde{M}(\lambda)$. From (26) we have

$$P_{11}(x,\lambda) \equiv 1, \quad P_{12}(x,\lambda) \equiv 0, \quad P_{22}(x,\lambda) \equiv 1, \quad P_{21}(x,\lambda) \equiv 0.$$

Substituting these relations in (25), we get

$$\varphi_1(x,\lambda) \equiv \tilde{\varphi}_1(x,\lambda), \qquad \varphi_2(x,\lambda) \equiv \tilde{\varphi}_2(x,\lambda),$$

$$\Phi_1(x,\lambda) \equiv \tilde{\Phi}_1(x,\lambda), \qquad \Phi_2(x,\lambda) \equiv \tilde{\Phi}_2(x,\lambda)$$

for all x and λ. Hence, $L = \tilde{L}$.

Theorem 4. *If* $\lambda_n = \tilde{\lambda}_n$ *and* $\alpha_n = \tilde{\alpha}_n$ *for all* $n \in Z$, *then* $L = \tilde{L}$. *Thus, the problem* L *is uniquelly defined by spectral data.*

Proof. Since $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$ for all $n\in Z$ and

$$M(\lambda) = \frac{1}{\alpha_0(\lambda-\lambda_0)} + \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n^0\lambda_n^0} + \frac{1}{\alpha_n(\lambda-\lambda_n)}\right),$$

$$\tilde{M}(\lambda) = \frac{1}{\tilde{\alpha}_0(\tilde{\lambda}-\tilde{\lambda}_0)} + \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{\alpha}_n^0\tilde{\lambda}_n^0} + \frac{1}{\tilde{\alpha}_n(\tilde{\lambda}-\tilde{\lambda}_n)}\right),$$

we get

$$M(\lambda) = \tilde{M}(\lambda).$$

From Theorem 4 we prove that $L = \tilde{L}$.

Let us consider the boundary-value problem L_1 in which we take the condition $y_2(0,\lambda) = 0$ instead of the boundary condition (2) of the problem L. Let $\{\mu_n\}_{n\geq 0}$ be the eigenvalues of the problem L_1.

Theorem 5. *If* $\lambda_n = \tilde{\lambda}_n$ *and* $\mu_n = \tilde{\mu}_n$ *for all* $n \in \mathbb{N}$, *then* $L = \tilde{L}$, *i.e., the problem* L *is uniquelly determined by the sequences* $\{\lambda_n\}$ *and* $\{\mu_n\}$.

Proof. Since $\lambda_n = \tilde{\lambda}_n$, $\dfrac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)}$ is an entire function in λ. Moreover, $\Delta(\lambda) = \tilde{\Delta}(\lambda)$, since $\lim\limits_{\lambda\to\infty}\dfrac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)} = 1$. On the other hand, it is easy to see that $\psi_2(0,\lambda_n) = \tilde{\psi}_2(0,\lambda_n)$ as $\mu_n = \tilde{\mu}_n$. So, from the equality $\alpha_n = -\dfrac{\Delta'(\lambda_n)}{\psi_2(0,\lambda_n)}$, $\alpha_n = \tilde{\alpha}_n$ is obtained. Thus, the proof is completed by Theorem 5.

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