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ON GENERALIZATION OF $\oplus$-COFINITELY SUPPLEMENTED MODULES

ПРО УЗАГАЛЬНЕННЯ $\oplus$-КОФІНІТНО ПОПОВНЕНИХ МОДУЛІВ

We study the properties of $\oplus$-cofinitely radical supplemented modules or briefly $cgs^{\oplus}$-modules. It is shown that: a module with Summand Sum Property (SSP) is $cgs^{\oplus}$ if and only if $M/w\operatorname{Loc}^{\oplus} M$ ($w\operatorname{Loc}^{\oplus} M$ is the sum of all w-local direct summands of a module M) does not contain any maximal submodule; every cofinite direct summand of a UC-extending $cgs^{\oplus}$-module is $cgs^{\oplus}$; for any ring R, every free R-module is $cgs^{\oplus}$ if and only if R is semiperfect.

Досліджено властивості $\oplus$-кофінітно радикальних поповнених модулів або скорочено $cgs^{\oplus}$-модулів. Показано, що модуль із властивістю суми доданків SSP є $cgs^{\oplus}$-модулем тоді і тільки тоді, коли $M/w\operatorname{Loc}^{\oplus} M$ ($w\operatorname{Loc}^{\oplus} M$ — сума всіх w-локальних прямих доданків модуля M) не містить жодного максимального субмодуля; кожний прямий доданок UC-розширюваного $cgs^{\oplus}$-модуля є $cgs^{\oplus}$-модулем; для будь-якого кільця R кожний вільний R-модуль є $cgs^{\oplus}$-модулем тоді і тільки тоді, коли R є напівперфектним.

1. Introduction. In this note R will be an associative ring with identity and all modules are unital left R-modules. Let M be an R-module. The notation $N \subseteq M$ means that N is a submodule of M. $\operatorname{Rad} M$ will indicate Jacobson radical of M. A submodule N of an R-module M is called *small* in M (notation $N \ll M$), if $N + L \neq M$ for every proper submodule L of M. Let M be an R-module and let N and K be any submodules of M. K is called a *supplement* of N in M if $M = N + K$ and $N \cap K \ll K$ (see [1]). Following [1], M is called *supplemented* if every submodule of M has a supplement in M. A submodule N of a module M is called *cofinite* in M if the factor module $\dfrac{M}{N}$ is finitely generated. A module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M (see [2]). Clearly supplemented modules are cofinitely supplemented. A module M is called $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M (see [3]). As a proper generalization of $\oplus$-supplemented modules, the notation of $\oplus$-cofinitely supplemented modules was introduced by Calisici and Pancar [4]. A module M is called $\oplus$-*cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M. Also, finitely generated $\oplus$-cofinitely supplemented modules are $\oplus$-supplemented.

In [5] (Theorem 10.14), another generalization of supplement submodule was called as *radical supplement* or briefly Rad-*supplement* (according to [6], generalized supplement). For a module M and a submodule N of M, a submodule K of M is called a Rad-*supplement* of N in M if $N + K = M$ and $N \cap K \subseteq \operatorname{Rad} K$. An R-module M is called *radical supplemented* or briefly Rad-*supplemented* if every submodule of M has a Rad-*supplement* in M (in [6], generalized supplemented or GS-module). Since the Jacobson radical of a module is sum of all small submodules, every supplement is a Rad-supplement. Therefore every supplemented module is Rad-supplemented. In [7], M is called *cofinitely radical supplemented* or briefly *cofinitely* Rad-*supplemented* if every cofinite submodule of M has a Rad-supplement in M. Clearly Rad-supplemented modules are cofinitely Rad-supplemented.

Let M be an R-module. M is called *$\oplus$-radical supplemented* or briefly $\oplus$-Rad-*supplemented* or *generalized $\oplus$ -supplemented* if every submodule of M has a Rad-supplement that is a direct summand of M. Clearly $\oplus$-Rad-supplemented modules are Rad-supplemented. A module M is called $\oplus$-cofinitely radical supplemented (according to [8], generalized $\oplus$-cofinitely supplemented) if every cofinite submodule of M has a Rad-supplement that is a direct summand of M. Instead of a $\oplus$-cofinitely radical supplemented module, we will use a $cgs^{\oplus}$-module.

In this paper we study the properties of $cgs^{\oplus}$-modules as both a proper generalization of $\oplus$-Rad-supplemented modules and a generalization of $\oplus$-cofinitely supplemented modules. We prove that a module M with SSP is $cgs^{\oplus}$ if and only if $M/w\operatorname{Loc}^{\oplus} M$ does not contain any maximal submodule, where $w\operatorname{Loc}^{\oplus} M$ is the sum of all w-local direct summands of M. Also we show that any direct sum of $cgs^{\oplus}$-modules is also a $cgs^{\oplus}$-module. Using the mentioned fact we give a characterization of semiperfect rings.

2. Some properties of $\oplus$-cofinitely radical supplemented modules. It is clear that every $\oplus$-cofinitely supplemented module is $cgs^{\oplus}$, but it is not generally true that every $cgs^{\oplus}$-module is $\oplus$-cofinitely supplemented. Later we shall give an example of such modules (see Example 2.1). Now we give an analogue of these modules.

Proposition 2.1. *Let M be a $cgs^{\oplus}$-module with small radical. Then M is $\oplus$-cofinitely supplemented.*

Proof. Let U be any cofinite submodule of M. By the hypothesis, there exist submodules V, V' of M such that $M = U + V$, $U \cap V \subseteq \operatorname{Rad} V$ and $M = V \oplus V'$. Since $U \cap V \subseteq \operatorname{Rad} V \subseteq \operatorname{Rad} M \ll M$ and V is a direct summand of M, then $U \cap V \ll V$ by [1] (19.3.(5)). Hence M is $\oplus$-cofinitely supplemented.

Let M be an R-module. If every proper submodule of M is contained a maximal submodule of M, M is called *coatomic*. Note that every coatomic module has small radical.

***Corollary* 2.1.** *Let M be a coatomic$R$-module. Then M is a $cgs^{\oplus}$-module if and only if it is $\oplus$-cofinitely supplemented.*

Every $cgs^{\oplus}$-module is cofinitely Rad-supplemented but the converse is not true. For example, a left (cofinitely) Rad-supplemented ring which is not supplemented (i.e., semiperfect) is cofinitely Rad-supplemented over itself, but not a $cgs^{\oplus}$-module.

Therefore we have the following implications on modules:

$\oplus$-supplemented
↙ ↘
$\oplus$-cofinitely supplemented $\oplus$- Rad-supplemented
↘ ↙ ↘
$\oplus$-cofinitely radical supplemented Rad-supplemented
↘ ↙
cofinitely Rad-supplemented

We begin by some general properties of $cgs^{\oplus}$-modules. To prove that any direct sum of $cgs^{\oplus}$-modules is $cgs^{\oplus}$, we use the following standart Lemma ([7], 3.4).

Lemma 2.1. *Let M be an R-module and N, U be submodules of M such that N is cofinitely* Rad-*supplemented, U cofinite and $N + U$ has a* Rad *-supplement A in M. Then $N \cap (U + A)$ has a* Rad *-supplement B in N and $A + B$ is a* Rad-*supplement of U in M.*

Proof. Let A be a Rad-supplement of $N+U$ in M. Then

$$\frac{N}{N\cap(U+A)}\cong\frac{N+U+A}{U+A}\cong\frac{M/U}{(U+A)/U}.$$

Since U is a cofinite submodule of N, $N\cap(U+A)$ is cofinite. By hypothesis, N is cofinitely Rad-supplemented, $N\cap(U+A)$ has a Rad-supplement B in N. Then $M=(N+U)+A=U+A+B$ and by [1] (19.3), $U\cap(A+B)\subseteq A\cap(U+B)+$ $+\ B\cap(U+A)\subseteq A\cap(N+U)+B\cap(U+A)\subseteq\mathrm{Rad}\,(A+B)$. Therefore $A+B$ is a Rad-supplement of U in M.

Theorem 2.1. *For any ring R, any direct sum of $cgs^{\oplus}$-modules is a $cgs^{\oplus}$-module.*

Proof. Let R be any ring and $\{M_i\}_{i\in I}$ be any family of $cgs^{\oplus}$-modules. Let $M=$ $=\ \oplus_{i\in I}M_i$ and N be a cofinite submodule of M. Then $M=\oplus_{j=1}^{n}M_{i_j}+N$ and it is clear that $\{0\}$ is Rad-supplement of $M=M_{i_1}+(\oplus_{j=2}^{n}M_{i_j}+N)$. Since M_{i_1} is a $cgs^{\oplus}$-module, $M_{i_1}\cap(\oplus_{j=2}^{n}M_{i_j}+N)$ has a Rad-supplement V_{i_1} in M_{i_1} such that V_{i_1} is a direct summand of M_{i_1}. By Lemma 2.1, V_{i_1} is a Rad-supplement of $\oplus_{j=2}^{n}M_{i_j}+N$ in M. Note that since M_{i_1} is a direct summand of M, V_{i_1} is also a direct summand of M. By repeated use of Lemma 2.1, since the set J is finite at the end we will obtain that N has a Rad-supplement $V_{i_1}+V_{i_2}+\ldots+V_{i_r}$ in M such that every V_{i_j}, $1\leq j\leq n$, is a direct summand of M_{i_j}. Since every M_{i_j} is a direct summand of M, $\sum_{j=1}^{n}V_{i_j}=\oplus_{j=1}^{n}V_{i_j}$ is a direct summand of M. Hence M is a $cgs^{\oplus}$-module.

Recall from [7] that a module M is called *w-local* if it has a unique maximal submodule. It is clear that a module is w-local if and only if its radical is maximal.

Local modules are w-local. But it is not generally true that every w-local module is local. For example, p any prime, the $\mathbb{Z}$-module $\mathbb{Q}\oplus\mathbb{Z}_p$ is w-local but it is not local. It is trivial that w-local modules are a generalization of local modules. This fact plays a key role in our working.

Proposition 2.2. *The following statements are equivalent for a w-local module M.*

(i) $\mathrm{Rad}\,M\ll M$.

(ii) *M is finitely generated.*

Proof. Suppose that M is a w-local module. Then $\mathrm{Rad}\,M$ is a maximal submodule of M. Thus $\mathrm{Rad}\,M+Rm=M$ for every $m\in M\setminus\mathrm{Rad}\,M$. Since $\mathrm{Rad}\,M\ll M$, then $Rm=M$. Hence M is finitely generated. The converse is clear.

Proposition 2.3. *Let M be a w-local R-module. Then M is a $cgs^{\oplus}$-module.*

Proof. It follows from [7] (Lemma 3.2).

Proposition 2.4. *Let M be a $cgs^{\oplus}$-module. If M has a maximal submodule, then M contains a w-local direct summand.*

Proof. Let L be a maximal submodule of M. Then L is cofinite and it follows that there exist K,K' submodules of M such that $L+K=M$, $L\cap K\subseteq\mathrm{Rad}\,K$ and $M=K\oplus K'$. By Lemma 3.3 in [7], K is w-local. Hence K is a w-local direct summand of M.

Let M be an R-module. $w\,\mathrm{Loc}^{\oplus}\,M$ will denote the sum of all w-local direct summands of M.

Recall from [1] that an R-module M has *Summand Sum Property* (*SSP*) if the sum of two direct summands of M is again a direct summand of M.

We give a characterization of $cgs^{\oplus}$-modules. Firstly we need the following lemma which is a generalization of [2] (Lemma 2.9).

Lemma 2.2. *Let M be an R-module and N be a cofinite submodule of M. Let $\{L_i\}_{i=1}^n$ be the family of w-local submodules such that K is a* Rad*-supplement of $N + L_1 + \ldots + L_n$ in M. Then $K + \sum_{i \in I} L_i$ is a* Rad*-supplement of N in M such that I is a subset of $\{1, 2, \ldots, n\}$.*

Proof. Suppose that $n = 1$. Consider the submodule $H = (N + K) \cap L_1$ of L_1. K is a Rad-supplement of $N + L_1$, so that $M = N + L_1 + K$ and $(N + L_1) \cap K \subseteq \mathrm{Rad}\, K$. Then H is a cofinite submodule of L_1. Since L_1 is w-local, then $\mathrm{Rad}\, L_1$ is a unique maximal submodule of L_1. Note that $H \subseteq \mathrm{Rad}\, L_1$. By [9] (19.3), $N \cap (K + L_1) \subseteq$ $\subseteq K \cap (N + L_1) + H \subseteq \mathrm{Rad}\, K + \mathrm{Rad}\, L_1 \subseteq \mathrm{Rad}(K + L_1)$. Therefore $K + L_1$ is a Rad-supplement of N. This proves the result when $n = 1$. Suppose that $n \geq 2$. By induction on n, there exists a subset I' of $\{2, 3, \ldots, n\}$ such that $K + \sum_{i \in I'} L_i$ is a Rad-supplement of $N + L_1$ in M. Now the case $n = 1$ shows that $K + L_1 + \sum_{i \in I'} L_i$ is a Rad-supplement of N in M.

Theorem 2.2. *Let R be any ring and M be an R-module with SSP. Then the following statements are equivalent.*

(i) *M is a $cgs^{\oplus}$-module.*

(ii) *Every maximal submodule of M has a* Rad*-supplement that is a direct summand of M.*

(iii) *$M/w\,\mathrm{Loc}^{\oplus} M$ does not contain a maximal submodule.*

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii). Suppose that $M/w\,\mathrm{Loc}^{\oplus} M$ contains a maximal submodule $U/w\,\mathrm{Loc}^{\oplus} M$. Then U is a maximal submodule of M. By assumption, U has a Rad-supplement V that is a direct summand of M. Then V is w-local and it follows that $V \subseteq w\,\mathrm{Loc}^{\oplus} M$. Since $M = U + V$ and $w\,\mathrm{Loc}^{\oplus} M \subseteq U$, we get $M = U$ which is a contradiction.

(iii) $\Rightarrow$ (i). Let N be any cofinite submodule of M. Then $N + w\,\mathrm{Loc}^{\oplus} M$ is a cofinite submodule of M. By (iii), $M = N + w\,\mathrm{Loc}^{\oplus} M$. Because M/N is finitely generated, there exist w-local submodules L_i, $1 \leq i \leq n$, for some positive integer n, such that each of them is a direct summand of M and $M = N + \sum_{i=1}^{n} L_i$ has a Rad-supplement $\{0\}$ in M. By Lemma 2.2, $\sum_{i \in I'} L_i$ is a Rad-supplement of N in M such that I' is a subset of $\{1, 2, \ldots, n\}$. Moreover $\sum_{i \in I'} L_i$ is a direct summand of M. Thus M is a $cgs^{\oplus}$-module.

***Example* 2.1.** Let R be a commutative local ring which is not a valuation ring. Let x and y be elements of R, neither of them divides the other. By taking a suitable quotient ring, we may assume that $(x) \cap (y) = 0$ and $xI = yI = 0$, where I is the maximal ideal of R. Let F be a free module with generators a_1, a_2, a_3. Let N be the submodule generated by $xa_1 - ya_2$ and $M = F/N$. R is local, so ${}_RR$ is a $cgs^{\oplus}$-module. By Theorem 2.1, F is a $cgs^{\oplus}$-module. Suppose that M is a $cgs^{\oplus}$-module. Since F is finitely generated, M is finitely generated and it follows that M has a small radical. By Proposition 2.1, M is $\oplus$-(cofinitely) supplemented. This is a contradiction by [10] (Example 2.3).

This example shows that the factor module of a $cgs^{\oplus}$-module is not in general $cgs^{\oplus}$.

Let R be a ring and M be an R-module. We consider the following condition.

(D_3) If K and N are direct summands of M with $M = K + N$, then $K \cap N$ is also a direct summand of M (see [11]).

Proposition 2.5. *Let M be a $cgs^{\oplus}$-module with (D_3). Then every cofinite direct summand of M is a $cgs^{\oplus}$-module.*

Proof. Let N be any cofinite direct summand of M. Then there exists a submodule N' of M such that $M = N \oplus N'$ and N' is finitely generated. Let U be any cofinite submodule of N. Note that $M/U \cong N/U \oplus N'$ is finitely generated so that U is also cofinite submodule of M. Since M is a $cgs^{\oplus}$-module, then there exists a direct summand V of M such that $M = U + V$ and $U \cap V \subseteq \operatorname{Rad} V$. Hence $N = U + (N \cap V)$. Since M has (D_3), $N \cap V$ is a direct summand of M. Furthermore $N \cap V$ is a direct summand of N because N is a direct summand of M. Then $U \cap (N \cap V) = U \cap V \subseteq \operatorname{Rad} M$. Note that $U \cap (N \cap V) \subseteq \operatorname{Rad}(N \cap V)$ by [1] (19.3). Hence N is a $cgs^{\oplus}$-module.

***Corollary* 2.2.** *Let M be a UC-extending module. If M is a $cgs^{\oplus}$-module, then every cofinite direct summand of M is a $cgs^{\oplus}$-module.*

Recall from [1] that a submodule U of an R-module M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. Let M be an R-module and τ be a preradical for the category of R-modules. Then, $\operatorname{Rad} M$ and $\tau(M)$ are fully invariant submodule of M. An R-module M is called a (*weak) duo module* if every (direct summand) submodule of M is fully invariant. Note that weak duo modules has SSP (see [9]).

***Corollary* 2.3.** *Let R be a ring and M be a weak duo R-module. Then M is a $cgs^{\oplus}$-module if and only if every maximal submodule of M has a* Rad*-supplement that is a direct summand of M.*

Proposition 2.6. *Let M be a $cgs^{\oplus}$-module and U be a fully invariant submodule of M. Then M/U is a $cgs^{\oplus}$-module.*

Proof. Let K/U be a cofinite submodule of M/U. Then K is a cofinite submodule of M. Since M is a $cgs^{\oplus}$-module, then $(N+U)/U$ is a Rad-supplement of K/U in M/U by [6] (Proposition 2.6) and $M = N \oplus N'$ for N' is a submodule of M. By hypothesis, U is a fully invariant submodule of M. Note that $U = (U \cap N) \oplus (U \cap N')$ by [9] (Lemma 2.1). Then $M/U = (N+U)/U \oplus (N'+U)/U$. $(N+U)/U$ is a Rad-supplement of K/U such that $(N+U)/U$ is a direct summand of M/U. Hence M/U is a $cgs^{\oplus}$-module.

***Corollary* 2.4.** *Let M be a $cgs^{\oplus}$-module. Then $M/\operatorname{Rad} M$ and $M/\tau(M)$ is a $cgs^{\oplus}$-module.*

Proposition 2.7. *Let M be a $cgs^{\oplus}$-module and U be a fully invariant submodule of M. If U is a cofinite direct summand of M, then U is a $cgs^{\oplus}$-module.*

Proof. Let U be a cofinite submodule of M. Since U is a cofinite direct summand of M, it follows that $U \oplus U' = M$ for $U' \subseteq M$. Let V be a cofinite submodule of U. Then U/V and U' is finitely generated. Therefore V is a cofinite submodule of M. By hypothesis, $V + K = M$, $V \cap K \subseteq \operatorname{Rad} K$ and $M = K \oplus K'$ such that K, $K' \subseteq M$. Note that $U = (U \cap K) \oplus (U \cap K')$ by [9] (Lemma 2.1). Then $U = V \oplus (U \cap K)$ and $V \cap (U \cap K) \subseteq \operatorname{Rad} M$. Since $U \cap K$ is a direct summand of M, then $V \cap (U \cap K) \subseteq \operatorname{Rad}(U \cap K)$. $U \cap K$ is a Rad-supplement of V in U that is a direct summand of U. It follows that U is a $cgs^{\oplus}$-module.

Let $\{L_i\}_{i \in I}$ be the family of $cgs^{\oplus}$-submodules of M. $Cgs^{\oplus}M$ will denote the sum of L_is for all $i \in I$. That is $Cgs^{\oplus}M = \sum_{i \in I} L_i$. It is clear that $w\operatorname{Loc}^{\oplus} M \subseteq Cgs^{\oplus}M$.

Proposition 2.8. *Let R be a ring, M be an R-module and every $cgs^{\oplus}$-submodule of M be a direct summand of M. Then every maximal submodule of M has a* Rad-

supplement that is a direct summand of M *if and only if* $M/Cgs^{\oplus}M$ *does not contain a maximal submodule.*

Proof. $(\Rightarrow)$ Suppose that $M/Cgs^{\oplus}M$ contains a maximal submodule $U/Cgs^{\oplus}M$. Then U is a maximal submodule of M. By assumption, there exist V, V' submodules of M such that $U + V = M$, $U \cap V \subseteq \operatorname{Rad} V$ and $M = V \oplus V'$. By [7] (Lemma 3.3) V is w-local. Then V is a $cgs^{\oplus}$-module by Proposition 2.3. It follows that $V \subseteq Cgs^{\oplus}M$. $M/Cgs^{\oplus}M = U/Cgs^{\oplus}M$, so that $M = U$ which is a contradiction.

$(\Leftarrow)$ Let P be a maximal submodule of M. By assumption, P does not contain $Cgs^{\oplus}M$. Hence there exists a $cgs^{\oplus}$-module L of M such that it is not a submodule of P is a maximal submodule of M and $L \nsubseteq P$, then $M = P + L$. Note that $M/P \cong \cong L/(P \cap L)$. It follows that $P \cap L$ is a maximal submodule of L. Then $P \cap L$ is a cofinite submodule of L. By assumption, there exist X, X' submodules of M such that $L = (P \cap L) + X$, $(P \cap L) \cap X \subseteq \operatorname{Rad} X$ and $L = X \oplus X'$. It follows that $M = P + X$ and $P \cap X \subseteq \operatorname{Rad} X$. Moreover by hypothesis, X is a direct summand of M. Therefore P has a Rad-supplement that is a direct summand of M.

Theorem 2.3. *Let* M *be an* R*-module such that* $M = M_1 \oplus M_2$ *is a direct sum of submodules* M_1, M_2. *Then* M_2 *is a* $cgs^{\oplus}$*-module if and only if there exists a submodule* K *of* M_2 *such that* K *is a direct summand of* M, $M = K + N$ *and* $N \cap K \subseteq \operatorname{Rad} K$ *for every cofinite submodule* N/M_1 *of* M/M_1.

Proof. $(\Rightarrow)$ Let N/M_1 be any cofinite submodule of M/M_1. Then N is a cofinite submodule of M and it follows that $N \cap M_2$ is a cofinite submodule of M_2. By hypothesis, there exist K, K' submodules of M_2 such that $M_2 = (N \cap M_2) + K$, $(N \cap M_2) \cap K \subseteq \operatorname{Rad} K$ and $M_2 = K \oplus K'$. Note that $M = N + K$ and $N \cap K \subseteq \operatorname{Rad} K$. Since K is a direct summand of M_2, then K is a direct summand of M.

$(\Leftarrow)$ Let U be any cofinite submodule of M_2. Then M_2/U is finitely generated. It follows that $(U + M_1)/M_1$ is a cofinite submodule of M/M_1. By hypothesis, there exists a submodule K of M_2 such that K is a direct summand of M, $M = K + U + M_1$ and $(U + M_1) \cap K \subseteq \operatorname{Rad} K$. It follows that $M_2 = U + K$ and $U \cap K \subseteq \operatorname{Rad} K$. Therefore M_2 is a $cgs^{\oplus}$-module.

A ring R is *semiperfect* if $R/\operatorname{Rad} R$ is semisimple and idempotents can be lifted modulo $\operatorname{Rad} R$. It is shown [4] (Theorem 2.9) that R is semiperfect if and only if ${}_RR$ is $\oplus$-supplemented if and only if every free R-module is $\oplus$-cofinitely supplemented. Now we generalize this fact.

Theorem 2.4. *Let* R *be any ring. Then* R *is semiperfect if and only if every free* R*-module is a* $cgs^{\oplus}$*-module.*

Proof. Let F be any free R-module. Since R is semiperfect, then ${}_RR$ is $\oplus$-cofinitely supplemented and it follows that ${}_RR$ is a $cgs^{\oplus}$-module. By Theorem 2.1, F is a $cgs^{\oplus}$-module. Conversely, suppose that every free R-module is $cgs^{\oplus}$. Then ${}_RR$ is a $cgs^{\oplus}$-module. By Proposition 2.1, ${}_RR$ is (cofinitely) $\oplus$-supplemented, i.e., R is semiperfect.

Finally, we give an example of module, which is $cgs^{\oplus}$ but not $\oplus$-cofinitely supplemented.

***Example* 2.2** (see [12], Theorem 4.3 and Remark 4.4)**.** Let M be a biuniform module and $S = \operatorname{End}(M)$. Suppose that P is the projective S-module with $\dim(P) = = (1, 0)$. Then P is a indecomposable w-local module. Since $\dim(P) = (1, 0)$, P is not finitely generated. Hence P is a $cgs^{\oplus}$-module but not $\oplus$-cofinitely supplemented.

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