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SHARP KOLMOGOROV-TYPE INEQUALITIES FOR NORMS OF FRACTIONAL DERIVATIVES OF MULTIVARIATE FUNCTIONS

ТОЧНІ НЕРІВНОСТІ ТИПУ КОЛМОГОРОВА ДЛЯ НОРМ ДРОБОВИХ ПОХІДНИХ ФУНКЦІЙ БАГАТЬОХ ЗМІННИХ

Let $C(\mathbb{R}^m)$ be spaces of bounded and continuous functions $x\colon \mathbb{R}^m \to \mathbb{R}$, endowed with the norms $\|x\|_C = \|x\|_{C(\mathbb{R}^m)} := \sup\{|x(t)|\colon t \in \mathbb{R}^m\}$. Let e_j, $j = 1, \ldots, m$, be the standard basis in $\mathbb{R}^m$. Given moduli of continuity ω_j, $j = 1, \ldots, m$, denote

$$H^{j,\omega_j} := \left\{ x \in C(\mathbb{R}^m)\colon \ \|x\|_{\omega_j} = \|x\|_{H^{j,\omega_j}} = \sup_{t_j \neq 0} \frac{\|\Delta_{t_j e_j} x(\cdot)\|_C}{\omega_j(|t_j|)} < \infty \right\}.$$

In this paper, new sharp Kolmogorov-type inequalities for norms of mixed fractional derivatives $\|D^\alpha_\varepsilon x\|_C$ of functions $x \in \bigcap\limits_{j=1}^m H^{j,\omega_j}$ are obtained. Some applications of these inequalities are presented.

Нехай $C(\mathbb{R}^m)$ — простори неперервних обмежених функцій $x\colon \mathbb{R}^m \to \mathbb{R}$ з нормами $\|x\|_C = \|x\|_{C(\mathbb{R}^m)} := \sup\{|x(t)|\colon t \in \mathbb{R}^m\}$, e_j, $j = 1, \ldots, m$, — звичайна база в $\mathbb{R}^m$. Для заданих модулів неперервності ω_j, $j = 1, \ldots, m$, позначимо

$$H^{j,\omega_j} := \left\{ x \in C(\mathbb{R}^m)\colon \ \|x\|_{\omega_j} = \|x\|_{H^{j,\omega_j}} = \sup_{t_j \neq 0} \frac{\|\Delta_{t_j e_j} x(\cdot)\|_C}{\omega_j(|t_j|)} < \infty \right\}.$$

У роботі отримано нові точні нерівності типу Колмогорова для норм мішаних частинних похідних $\|D^\alpha_\varepsilon x\|_C$ функцій $x \in \bigcap\limits_{j=1}^m H^{j,\omega_j}$. Наведені деякі застосування цих нерівностей.

1. Introduction. Statements of the problems. Main results. Sharp Kolmogorov-type inequalities for univariate and multivariate functions, estimating the norms of intermediate derivatives through the norms of the function itself and its derivatives of higher order, are of great importance for many branches of mathematics and its applications. After A.N. Kolmogorov obtained his inequality (see [1 – 3]) many inequalities of this type for norms of integer derivatives of univariate functions were obtained (see, for example, [4 – 8]).

In the case of functions of two or more variables very few results of this type are known (see [9]–[15]).

Many questions in Analysis require to consider derivatives and antiderivatives of fractional order (see, for instance, [16]). One of the natural and useful definitions of the fractional derivative for a univariate function $x(u)$, $u \in \mathbb{R}$, is the following definition of Marchaud fractional derivative [17] (see also [16, p. 95 – 97]):

$$(D^\alpha_\pm x)(u) := \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^\infty \frac{x(u) - x(u \mp t)}{t^{1+\alpha}}\, dt, \quad \alpha \in (0, 1).$$

For brevity, we denote $A_\alpha = \dfrac{\alpha}{\Gamma(1-\alpha)}$.

Let $C(\mathbb{R})$ be the space of all bounded and continuous functions $x\colon \mathbb{R} \to \mathbb{R}$ endowed with the norm

$$\|x\|_C := \sup\big\{|x(t)|:\ t \in \mathbb{R}\big\}.$$

Let $\omega(t)$ be a modulus of continuity, i.e., continuous, nondecreasing, subadditive function defined on $[0, +\infty)$ and such that $\omega(0) = 0$. By $H^\omega = H^\omega(\mathbb{R})$ denote the space of functions $x \in C(\mathbb{R})$, for which

$$\|x\|_\omega = \|x\|_{H^\omega} = \sup_{\substack{t_1,\, t_2 \in \mathbb{R} \\ t_1 \neq t_2}} \frac{|x(t_1) - x(t_2)|}{\omega(|t_1 - t_2|)} < \infty.$$

If $\omega(t) = t^\beta$, $\beta \in (0, 1]$, then we write H^β instead of H^ω.

Let $\alpha \in (0, 1)$ and $\omega(t)$ be such that

$$\int_0^1 \frac{\omega(t)}{t^{1+\alpha}}\, dt < \infty,$$

or, equivalently,

$$\int_{\mathbb{R}_+} \frac{\min\{1, \omega(t)\}}{t^{1+\alpha}}\, dt < \infty.$$

It was proved in [18] that for any $h > 0$ the following additive Kolmogorov-type inequality:

$$\|D_\pm^\alpha x\|_C \leq A_\alpha \left[\|x\|_\omega \int_0^h \frac{\omega(t)}{t^{1+\alpha}}\, dt + \frac{2\|x\|_C}{\alpha h^\alpha} \right] \tag{1}$$

holds.

Moreover, this inequality becomes an equality for the function

$$x_h(u) = \begin{cases} \omega(|u|) - \dfrac{\omega(h)}{2}, & |u| \leq h, \\ \dfrac{\omega(h)}{2}, & |u| > h. \end{cases}$$

Note that after minimization over h of the right-hand side of inequality (1), it can be rewritten in the following form:

$$\|D_\pm^\alpha x\|_C \leq A_\alpha \int_0^\infty \frac{\min\{2\|x\|_C, \|x\|_\omega \omega(t)\}}{t^{1+\alpha}}\, dt. \tag{2}$$

In the case $\omega(t) = t^\beta$, $\beta \in (0, 1]$, $\alpha < \beta \leq 1$, inequality (2) becomes

$$\|D_\pm^\alpha x\|_C \leq \frac{1}{\Gamma(1-\alpha)} \frac{2^{1-\alpha/\beta}}{1 - \alpha/\beta} \|x\|_C^{1-\alpha/\beta} \|x\|_{H^\beta}^{\alpha/\beta}. \tag{3}$$

Other known results on sharp Kolmogorov-type inequalities for fractional derivatives can be found in [18–21].

Let $\mathbb{R}^m$ be the space of points $t = (t_1, \ldots, t_m)$ and $\{e_i\}_{i=1}^m$ be the standard basis in $\mathbb{R}^m$.

By $C(\mathbb{R}^m)$ denote the space of all bounded and continuous functions $x\colon \mathbb{R}^m \to \mathbb{R}$ with the norm

$$\|x\|_C = \|x\|_{C(\mathbb{R}^m)} := \sup\big\{|x(t)|\colon\ t \in \mathbb{R}^m\big\}.$$

For a given vector $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$, by $\Delta_{t_j e_j} x(u)$ denote the first difference of the function $x(u)$ along the variable u_j with the step t_j, $j = 1, \ldots, m$,

$$\Delta_{t_j e_j} x(u) := x(u) - x(u + t_j e_j),$$

and define as

$$\Delta_t x(u) := \Delta_{t_1 e_1} \Delta_{t_2 e_2} \ldots \Delta_{t_m e_m} x(u)$$

the mixed difference of the function $x(u)$ with the step t.

Let $\omega_j(t_j)$, $t_j \geq 0$, $j = 1, \ldots, m$, be given moduli of continuity. We will consider the following spaces:

$$H^{j,\omega_j} := \left\{ x \in C(\mathbb{R}^m)\colon\ \|x\|_{\omega_j} = \|x\|_{H^{j,\omega_j}} = \sup_{t_j \neq 0} \frac{\|\Delta_{t_j e_j} x(\cdot)\|_C}{\omega_j(|t_j|)} < \infty \right\}.$$

If $\omega_j(t_j) = t_j^{\beta_j}$, $\beta_j \in (0, 1]$, then we write H^{j,β_j} instead of H^{j,ω_j}, $j = 1, \ldots, m$.

For a given function $x(u)$, $u \in \mathbb{R}^m$, and a vector of smoothness $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_j \in (0, 1)$, $j = 1, \ldots, m$, and a vector of sign distribution $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$, $\varepsilon_j = \pm$, $j = 1, \ldots, m$, the mixed Marchaud derivative of order α is defined in the following way (see [17, p. 347]):

$$(D_\varepsilon^\alpha x)(u) := A_\alpha \int\limits_{\mathbb{R}_+^m} \Delta_{\varepsilon t} x(u) \prod_{j=1}^m t_j^{-\alpha_j - 1}\, dt,$$

where $x \in C(\mathbb{R}^m)$, $A_\alpha = \prod_{j=1}^m A_{\alpha_j}$, $A_{\alpha_j} = \dfrac{\alpha_j}{\Gamma(1 - \alpha_j)}$, $\varepsilon t := (\varepsilon_1 t_1, \ldots, \varepsilon_m t_m)$.

V. F. Babenko and S. A. Pichugov [15] proved that for any function $x \in \bigcap\limits_{j=1}^m H^{j,\beta_j}$, the sharp inequality holds

$$\|D_\varepsilon^\alpha x\|_C \leq \frac{2^{m-1}}{\prod_{j=1}^m \Gamma(1 - \alpha_j)} \frac{2^{1 - \sum_{j=1}^m \frac{\alpha_j}{\beta_j}}}{1 - \sum_{j=1}^m \dfrac{\alpha_j}{\beta_j}} \|x\|_C^{1 - \sum_{j=1}^m \frac{\alpha_j}{\beta_j}} \prod_{j=1}^m \|x\|_{H^{j,\beta_j}}^{\alpha_j/\beta_j}, \tag{4}$$

provided that $\beta_j \in (0, 1]$ and $\alpha_j \in (0, 1)$, $j = 1, \ldots, m$, satisfy the condition $\sum_{j=1}^m \dfrac{\alpha_j}{\beta_j} < 1$.

The inequality (4) is the multivariate analog of (3). In this paper we obtain an inequality, which is a generalization of (4) and represents a multivariate analog of (2).

In what follows, for $\alpha_j \in (0, 1)$, $j = 1, \ldots, m$, and for given moduli of continuity $\omega_1(t_1), \ldots, \omega_m(t_m)$, we will need the following condition:

$$\int\limits_{\mathbb{R}_+^m} \min\Big\{1, \omega_1(t_1), \ldots, \omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j - 1}\, dt < \infty. \tag{5}$$

Theorem 1. *Let moduli of continuity* $\omega_1(t_1), \ldots, \omega_m(t_m)$ *and numbers* $\alpha_j \in (0,1)$, $j = 1, \ldots, m$, *be such that condition* (5) *is satisfied. Then for any function* $x \in \bigcap_{j=1}^{m} H^{j,\omega_j}$ *and any vector* ε *of sign distribution, the following sharp inequality holds*:

$$\left\|D_\varepsilon^\alpha x\right\|_C \le$$

$$\le 2^{m-1} A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2\|x\|_C, \|x\|_{\omega_1}\omega_1(t_1), \ldots, \|x\|_{\omega_m}\omega_m(t_m)\Big\} \prod_{j=1}^{m} t_j^{-\alpha_j-1}\, dt. \qquad (6)$$

For given moduli of continuity $\omega_1, \ldots, \omega_m$, by UH^{j,ω_j}, $j = 1, \ldots, m$, denote the unit ball in the space H^{j,ω_j}.

Consider the function

$$\Omega\left(\delta, \bigcap_{j=1}^{m} UH^{j,\omega_j}\right) := \sup_{\substack{x \in \bigcap_{j=1}^m UH^{j,\omega_j}, \\ \|x\|_C \le \delta}} \|D_\varepsilon^\alpha x\|_C\,, \quad \delta \ge 0. \qquad (7)$$

The function (7) is called the modulus of continuity of the operator D_ε^α on the set $\bigcap_{j=1}^{m} UH^{j,\omega_j}$.

Theorem 1 implies the following statement.

***Corollary* 1.** *Under conditions of Theorem* 1 *for any* $\delta > 0$,

$$\Omega\left(\delta, \bigcap_{j=1}^{m} UH^{j,\omega_j}\right) = 2^{m-1} A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2\delta, \omega_1(t_1), \ldots, \omega_m(t_m)\Big\} \prod_{j=1}^{m} t_j^{-\alpha_j-1}\, dt. \quad (8)$$

In particular, if $\beta_j \in (0,1]$ *and* $\alpha_j \in (0,1)$, $j = 1, \ldots, m$, *are such that* $\sum_{j=1}^{m} \dfrac{\alpha_j}{\beta_j} < 1$, *then*

$$\Omega\left(\delta, \bigcap_{j=1}^{m} UH^{j,\beta_j}\right) = \frac{2^{m-1}}{\prod_{j=1}^{m} \Gamma(1-\alpha_j)} \frac{2^{1-\sum_{j=1}^m \frac{\alpha_j}{\beta_j}}}{1 - \sum_{j=1}^{m} \dfrac{\alpha_j}{\beta_j}} \delta^{1-\sum_{j=1}^m \frac{\alpha_j}{\beta_j}}.$$

The problem of finding of the modulus of continuity for a given operator on a given set is closely related to the problem about approximation of an unbounded operator by bounded ones.

We now consider the general statement of this problem.

Let X and Y be the Banach spaces, let $\mathcal{L}(X,Y)$ be the space of linear bounded operators $S\colon X \to Y$, and let $A\colon X \to Y$ be an operator (not necessarily linear) with the domain $D_A \subset X$. Let also $Q \subset D_A$ be some class of elements.

For $N > 0$, the quantity

$$E_N(A,Q) = \inf_{\substack{S \in \mathcal{L}(X,Y) \\ \|S\| \le N}} \sup_{x \in Q} \|Ax - Sx\|_Y \qquad (9)$$

is called the best approximation of the operator A on the set Q by linear operators $S\colon X \to Y$ such that $\|S\| = \|S\|_{X \to Y} \le N$.

The problem is to compute the quantity (9) and to find the extremal operator, i.e., the operator delivering the infimum on the right-hand side of (9).

This problem appeared in Stechkin's investigations in 1965. The statement of this problem, the first important results and the solution of this problem for low order differentiation operators were presented in [22]. For a survey of further research on this problem see [4, 5].

The function

$$\Omega(\delta, Q) := \sup_{\substack{x\in Q \\ \|x\|_X\le\delta}} \|Ax\|_Y, \quad \delta \ge 0,$$

is called the modulus of continuity of the operator A on the set Q.

Note, that this definition generalizes the above presented definition of the modulus of continuity of the operator D^α_ε on the set $\bigcap\limits_{j=1}^{m} UH^{j,\omega_j}$.

It is easily seen that the problem of computation of the function $\Omega(\delta, Q)$ for a given operator is the abstract version of the problem about the Kolmogorov inequality.

S. B. Stechkin [22] proved that

$$E_N(A, Q) \ge \sup_{\delta\ge 0} \big\{\Omega(\delta, Q) - N\delta\big\}. \tag{10}$$

Namely, the inequality (10) shows the relation between the Stechkin problem and Kolmogorov-type inequalities.

The following theorem gives the solution of the Stechkin problem for the operator D^α_ε on the class $\bigcap\limits_{j=1}^{m} UH^{j,\omega_j}$.

Theorem 2. *Let the strictly increasing moduli of continuity* $\omega_1(t_1), \ldots, \omega_m(t_m)$ *and the numbers* $\alpha_j \in (0, 1)$, $j = 1, \ldots, m$, *be such that the condition* (5) *is satisfied. Given* $N > 0$, *let* $h^N = (h^N_1, \ldots, h^N_m) \in \mathbb{R}^m_+$ *be such that*

$$\omega_1(h^N_1) = \ldots = \omega_m(h^N_m) \quad \textit{and} \quad \frac{2^m A_\alpha}{\alpha_1 \ldots \alpha_m} \prod_{j=1}^{m} (h^N_j)^{-\alpha_j} = N. \tag{11}$$

Let

$$G(h^N) := \Big\{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : \ |u_1| \ge h^N_1, \ldots, |u_m| \ge h^N_m\Big\}.$$

Then

$$E_N\left(D^\alpha_\varepsilon, \bigcap_{j=1}^{m} UH^{j,\omega_j}\right) = 2^{m-1} A_\alpha \int\limits_{\mathbb{R}^m_+\setminus G(h^N)} \min\big\{\omega_1(t_1), \ldots, \omega_m(t_m)\big\} \prod_{j=1}^{m} t_j^{-\alpha_j-1}\, dt.$$

In addition, the operator

$$B_{h^N} x(u) = A_\alpha \int\limits_{G(h^N)} \Delta_{\varepsilon t} x(u) \prod_{j=1}^{m} t_j^{-\alpha_j-1}\, dt$$

is the extremal operator.

Note that the lower estimate for $E_N\Big(D^\alpha_\varepsilon, \bigcap\limits_{j=1}^{m} UH^{j,\omega_j}\Big)$ will be obtained with the help of Corollary 1. In order to obtain the upper bound for $E_N\Big(D^\alpha_\varepsilon, \bigcap\limits_{j=1}^{m} UH^{j,\omega_j}\Big)$ we will estimate from above the quantity $\big\|D^\alpha_\varepsilon x - B_{h^N} x\big\|_C$ on the class $\bigcap\limits_{j=1}^{m} UH^{j,\omega_j}$.

Note also that in the case $\omega(t_j) = t_j^{\beta_j}$, $\beta_j \in (0,1]$, $j = 1,\dots,m$, applying Theorem 2, we immediately obtain the following statement.

Corollary 2. *Suppose that $\beta_j \in (0,1]$ and $\alpha_j \in (0,1)$, $j = 1,\dots,m$, satisfy the inequality $\sum_{j=1}^m \dfrac{\alpha_j}{\beta_j} < 1$. Then for any $N > 0$,*

$$E_N\left(D_\varepsilon^\alpha, \bigcap_{j=1}^m UH^{j,\beta_j}\right) = \left(\frac{2^{m-\sum_{j=1}^m \frac{\alpha_j}{\beta_j}}}{\prod_{j=1}^m \Gamma(1-\alpha_j)}\right)^{\frac{1}{\sum_{j=1}^m \frac{\alpha_j}{\beta_j}}} \frac{\sum_{j=1}^m \dfrac{\alpha_j}{\beta_j}}{1-\sum_{j=1}^m \dfrac{\alpha_j}{\beta_j}} N^{1-\frac{1}{\sum_{j=1}^m \frac{\alpha_j}{\beta_j}}}.$$

The inequalities for intermediate derivatives are also closely related to the Kolmogorov problem about necessary and sufficient conditions of the existence of a function, for which given numbers are the upper bounds of absolute values of its derivatives of corresponding order (see [2, 3]). For some known results in this direction see, for example, [23 – 26] and [6].

We consider the Kolmogorov-type problem in the following setting. It is required to find the necessary and sufficient conditions on the numbers $M_0, M_\alpha, M_{\omega_1},\dots,M_{\omega_m}$ for existence of the function $x \in \bigcap_{j=1}^m H^{j,\omega_j}$ such that

$$\|x\|_C = M_0, \qquad \|D_\varepsilon^\alpha x\|_C = M_\alpha, \qquad \|x\|_{\omega_1} = M_{\omega_1},\dots,\|x\|_{\omega_m} = M_{\omega_m}.$$

Theorem 3. *Let moduli of continuity $\omega_1(t_1),\dots,\omega_m(t_m)$ and numbers $\alpha_j \in (0,1)$, $j = 1,\dots,m$, be such that* (5) *holds, and let numbers $M_0, M_\alpha, M_{\omega_1},\dots,M_{\omega_m}$ be given. There exists a function $x \in \bigcap_{j=1}^m H^{j,\omega_j}$ such that*

$$\|x\|_C = M_0, \qquad \|D_\varepsilon^\alpha x\|_C = M_\alpha, \qquad \|x\|_{\omega_1} = M_{\omega_1},\dots,\|x\|_{\omega_m} = M_{\omega_m},$$

if and only if the inequality

$$M_\alpha \le 2^{m-1} A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2M_0, M_{\omega_1}\omega_1(t_1),\dots,M_{\omega_m}\omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j-1}\,dt \qquad (12)$$

holds.

Corollary 3. *Suppose that $\beta_j \in (0,1]$ and $\alpha_j \in (0,1)$, $j = 1,\dots,m$, satisfy the condition $\sum_{j=1}^m \dfrac{\alpha_j}{\beta_j} < 1$, and let numbers M_0, M_α, $M_{\beta_1},\dots,M_{\beta_m}$ be given. There exists a function $x \in \bigcap_{j=1}^m H^{j,\beta_j}$ such that*

$$\|x\|_C = M_0, \qquad \|D_\varepsilon^\alpha x\|_C = M_\alpha, \qquad \|x\|_{H^{1,\beta_1}} = M_{\beta_1},\dots,\|x\|_{H^{m,\beta_m}} = M_{\beta_m},$$

if and only if the inequality

$$M_\alpha \le \frac{2^{m-1}}{\prod_{j=1}^m \Gamma(1-\alpha_j)} \frac{2^{1-\sum_{j=1}^m \frac{\alpha_j}{\beta_j}}}{1-\sum_{j=1}^m \dfrac{\alpha_j}{\beta_j}} M_0^{1-\sum_{j=1}^m \frac{\alpha_j}{\beta_j}} \prod_{j=1}^m M_{H^{j,\beta_j}}^{\alpha_j/\beta_j}$$

holds.

2. Proofs. ***Proof of Theorem* 1.** We prove the theorem in the case $\varepsilon = (+, \ldots, +)$ only, since for any other ε one can use analogous arguments. Taking into account the definition of the fractional derivative we have

$$\forall u \in \mathbb{R}^m \colon \ |D_\varepsilon^\alpha x(u)| \le A_\alpha \int\limits_{\mathbb{R}_+^m} \|\Delta_t x(\cdot)\|_C \prod_{j=1}^m t_j^{-\alpha_j - 1}\, dt. \tag{13}$$

To estimate the norm $\|\Delta_t x\|_C$ we will use the following inequalities:

$$\|\Delta_t x\|_C \le 2^m \|x\|_C$$

and

$$\|\Delta_t x\|_C \le 2^{m-1} \|\Delta_{t_j e_j} x\|_C \le 2^{m-1} \|x\|_{\omega_j} \omega_j(|t_j|), \quad j = 1, \ldots, m.$$

Combining these estimates we obtain

$$\|\Delta_t x\|_C \le 2^{m-1} \min\Big\{2\|x\|_C, \|x\|_{\omega_1}\omega_1(|t_1|), \ldots, \|x\|_{\omega_m}\omega_m(|t_m|)\Big\}.$$

Applying the last estimate to the right-hand side of (13) we have

$$\forall u \in \mathbb{R}^m \colon \ \big|D_\varepsilon^\alpha x(u)\big| \le$$

$$\le 2^{m-1} A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2\|x\|_C, \|x\|_{\omega_1}\omega_1(t_1), \ldots, \|x\|_{\omega_m}\omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j - 1}\, dt. \tag{14}$$

Let us show that for every function $x \in \bigcap\limits_{j=1}^m H^{j,\omega_j}$, its fractional derivative $D_\varepsilon^\alpha x(u)$ depends on u continuously.

Let

$$\omega(x, \theta) := \sup_{|t| < \theta} \|x(\cdot) - x(\cdot + t)\|_C,$$

where $|t| = \sqrt{t_1^2 + \ldots + t_m^2}$, $t = (t_1, \ldots, t_m)$.

Applying the inequality (14) to the difference $x(u) - x(u + \delta)$, $\delta \in \mathbb{R}^m$, we obtain

$$\big|D_\varepsilon^\alpha x(u) - D_\varepsilon^\alpha x(u + \delta)\big| \le$$

$$\le 2^{m-1} A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2\omega(x, |\delta|), 2\|x\|_{\omega_1}\omega_1(t_1), \ldots, 2\|x\|_{\omega_m}\omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j - 1}\, dt.$$

Note, that the function

$$\min\Big\{2\omega(x, |\delta|), 2\|x\|_{\omega_1}\omega_1(t_1), \ldots, 2\|x\|_{\omega_m}\omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j - 1}$$

uniformly converges to zero (as $\delta \to 0$) on any set of points $(t_1, \ldots, t_m) \in \prod_{j=1}^m [\sigma_j, \infty)$, $\sigma_j > 0$, $j = 1, \ldots, m$, and the integral

$$\int\limits_{\mathbb{R}_+^m} \min\Big\{2\omega(x, |\delta|), 2\|x\|_{\omega_1}\omega_1(t_1), \ldots, 2\|x\|_{\omega_m}\omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j - 1}\, dt$$

uniformly converges on any bounded set of values of the parameter δ.

Therefore

$$|D_\varepsilon^\alpha x(u) - D_\varepsilon^\alpha x(u+\delta)| \to 0, \quad |\delta| \to 0,$$

which proves the continuity of $D_\varepsilon^\alpha x(u)$ for all $u \in \mathbb{R}^m$.

Thus, we obtain from (14):

$$\left\|D_\varepsilon^\alpha x\right\|_C \le$$

$$\le 2^{m-1} A_\alpha \int\limits_{\mathbb{R}_+^m} \min\left\{2\|x\|_C, \|x\|_{\omega_1}\omega_1(t_1), \ldots, \|x\|_{\omega_m}\omega_m(t_m)\right\} \prod_{j=1}^m t_j^{-\alpha_j - 1}\, dt,$$

and inequality (6) is proved. Construct now the function $f(t)$, which turns (6) into equality. To this end, we will use the methods from [15]. Define $f(u)$ for $u \in \mathbb{R}_+^m$, and then extend it to the whole space $\mathbb{R}^m$ evenly with respect to each variable. For $u = (u_1, \ldots, u_m) \in \mathbb{R}_+^m$ and $\delta > 0$, set $\omega_j^\delta(u_j) = \min\{\omega_j(u_j), 2\delta\}$.

Consider the vector $\omega^\delta(u) = \left(\omega_1^\delta(u_1), \ldots, \omega_m^\delta(u_m)\right)$ and denote $v = v(u) :=$ $:= \left(v_1(u), \ldots, v_m(u)\right) = \left(\omega^\delta(u)\right)^*$, where $(\omega^\delta(u))^*$ is the rearrangement of the numbers $\omega_1^\delta(u_1), \ldots, \omega_m^\delta(u_m)$ in nonincreasing order. Now, define the function $f(u)$ by setting for $u \in \mathbb{R}_+^m$,

$$f(u) = v_1(u) - v_2(u) + \ldots + (-1)^{m-1} v_m(u) - \delta.$$

Since $0 \le \sum_{j=1}^m (-1)^{j-1} v_j(u) \le 2\delta$, we have $\|f\|_C \le \delta$. Let us verify, that $f \in$ $\in \bigcap\limits_{j=1}^m H^{j,\omega_j}$, and estimate $\|f\|_{\omega_j}$, $j = 1, \ldots, m$. For this purpose, consider the difference

$$f(u + he_j) - f(u) =$$

$$= v_1(u + he_j) - v_2(u + he_j) + \ldots + (-1)^{m-1} v_m(u + he_j) -$$

$$-(v_1(u) - v_2(u) + \ldots + (-1)^{m-1} v_m(u)), \quad h > 0.$$

The vector $u + he_j$ differs from the vector u in the j-th coordinate only, which is greater then the j-th coordinate of the vector u. Therefore, $v(u + he_j)$ differs from $v(u)$ in the following way. Let the number $\omega_j^\delta(u_j)$ be the ν-th coordinate of vector $v(u)$. Then there exists $\mu \le \nu$ such that $\omega_j^\delta(u_j + he_j)$ is the μ-th coordinate of $v(u + he_j)$. Moreover, coordinates of $v(u + he_j)$ coincide with coordinates of the vector $v(u)$, as soon as they have indices less than μ or greater than ν.

Thus,

$$f(u + he_j) - f(u) =$$

$$= (-1)^{\mu-1} \omega_j^\delta(u_j + h) + (-1)^\mu v_\mu(u) + \ldots + (-1)^{\nu-1} v_{\nu-1}(u) -$$

$$-(-1)^{\mu-1} v_\mu(u) - \ldots - (-1)^{\nu-2} v_{\nu-1}(u) - (-1)^{\nu-1} \omega_j^\delta(u_j) =$$

$$= (-1)^{\mu-1} \omega_j^\delta(u_j + h) + 2(-1)^\mu v_\mu(u) + \ldots$$

$$\ldots + 2(-1)^{\nu-1} v_{\nu-1}(u) - (-1)^{\nu-1} \omega_j^\delta(u_j).$$

Since
$$\omega_j^\delta(u_j+h) \geq v_\mu(u) \geq \ldots \geq v_{\nu-1}(u) \geq \omega_j^\delta(u_j),$$
it is easy to verify that

$$\omega_j^\delta(u_j) - \omega_j^\delta(u_j+h) \leq$$
$$\leq (-1)^{\mu-1}\omega_j^\delta(u_j+h) + 2(-1)^\mu v_\mu(u) + \ldots$$
$$\ldots + 2(-1)^{\nu-1}v_{\nu-1}(u) - (-1)^{\nu-1}\omega_j^\delta(u_j) \leq$$
$$\leq \omega_j^\delta(u_j+h) - \omega_j^\delta(u_j). \tag{15}$$

Taking into account that

$$\omega_j^\delta(u_j+h) - \omega_j^\delta(u_j) \leq \omega_j(u_j+h) - \omega_j(u_j) \leq \omega_j(h),$$

we obtain $f \in H^{j,\omega_j}$ and $\|f\|_{\omega_j} \leq 1$, $j = 1, \ldots, m$.

We now compute $|(D_\varepsilon^\alpha f)(0)|$ when $\varepsilon = (+, \ldots, +)$. In order to do this, we firstly show that for all $t \in \mathbb{R}_+^m$,

$$\Delta_{t_1} \ldots \Delta_{t_m} f(0, \ldots, 0) = -2^{m-1} \min\Big\{2\delta, \omega_1(t_1), \ldots, \omega_m(t_m)\Big\} =$$
$$= -2^{m-1} \min\Big\{\omega_1^\delta(t_1), \ldots, \omega_m^\delta(t_m)\Big\}.$$

The proof is by induction on m. For $m = 2$ (induction basis), this fact can be verified directly.

Since the operators Δ_{t_i} and Δ_{t_j} commute, we can take $\Delta_{t_1}, \ldots, \Delta_{t_m}$ in any convenient order, while computing $\Delta_{t_1} \ldots \Delta_{t_m} f(0, \ldots, 0)$. For definiteness, suppose that $\omega_1^\delta(t_1)$ is the greatest number among $\omega_1^\delta(t_1), \ldots, \omega_m^\delta(t_m)$. We will compute the difference in t_1 in last turn. Represent the difference $\Delta_{t_1} \ldots \Delta_{t_m} f(0, \ldots, 0)$ in the following way:

$$\Delta_{t_1} \ldots \Delta_{t_m} f(0, \ldots, 0) =$$
$$= \sum_{j_1=0}^{1}\sum_{j_2=0}^{1} \ldots \sum_{j_m=0}^{1} (-1)^{j_1+\ldots+j_m} f(j_1t_1, j_2t_2, \ldots, j_mt_m) =$$
$$= \sum_{j_2=0}^{1} \ldots \sum_{j_m=0}^{1} (-1)^{j_2+\ldots+j_m} f(0, j_2t_2, \ldots, j_mt_m) =$$
$$- \sum_{j_2=0}^{1} \ldots \sum_{j_m=0}^{1} (-1)^{j_2+\ldots+j_m} f(t_1, j_2t_2, \ldots, j_mt_m).$$

By the induction assumption,

$$\sum_{j_2=0}^{1} \ldots \sum_{j_m=0}^{1} (-1)^{j_2+\ldots+j_m} f(0, j_2t_2, \ldots, j_mt_m) =$$
$$= -2^{m-2} \min\Big\{\omega_2^\delta(t_2), \ldots, \omega_m^\delta(t_m)\Big\}. \tag{16}$$

Using the fact that $\omega_1^\delta(t_1)$ is the greatest number among $\omega_1^\delta(t_1),\ldots,\omega_m^\delta(t_m)$, and the definition of f, we obtain

$$f(t_1, j_2t_2,\ldots,j_mt_m) = \omega_1^\delta(t_1) - f(0, j_2t_2,\ldots,j_mt_m).$$

Thus,

$$-\sum_{j_2=0}^{1}\ldots\sum_{j_m=0}^{1}(-1)^{j_2+\ldots+j_m}\left(\omega_1^\delta(t_1) - f(0, j_2t_2,\ldots,j_mt_m)\right) =$$

$$= -2^{m-2}\min\left\{\omega_2^\delta(t_2),\ldots,\omega_m^\delta(t_m)\right\}$$

(here we have used the induction hypothesis (16) again). Finaly, we have

$$\Delta_{t_1}\ldots\Delta_{t_m}f(0,\ldots,0) =$$

$$= -2^{m-2}\min\left\{\omega_2^\delta(t_2),\ldots,\omega_m^\delta(t_m)\right\} - 2^{m-2}\min\left\{\omega_2^\delta(t_2),\ldots,\omega_m^\delta(t_m)\right\} =$$

$$= -2^{m-1}\min\left\{\omega_2^\delta(t_2),\ldots,\omega_m^\delta(t_m)\right\} =$$

$$= -2^{m-1}\min\left\{\omega_1^\delta(t_1),\omega_2^\delta(t_2),\ldots,\omega_m^\delta(t_m)\right\}.$$

For $\varepsilon = (+,\ldots,+)$, we estimate $\|D_\varepsilon^\alpha f\|_C$ from below:

$$\left\|D_\varepsilon^\alpha f\right\|_C \geq \left|(D_\varepsilon^\alpha f)(0,\ldots,0)\right| =$$

$$= 2^{m-1}A_\alpha\int\limits_{\mathbb{R}_+^m}\prod_{j=1}^{m}t^{-\alpha_j-1}\Delta_t f(0,\ldots,0)\,dt =$$

$$= 2^{m-1}A_\alpha\int\limits_{\mathbb{R}_+^m}\min\left\{2\delta,\omega_1(t_1),\ldots,\omega_m(t_m)\right\}\prod_{j=1}^{m}t^{-\alpha_j-1}\,dt \geq$$

$$\geq 2^{m-1}A_\alpha\int\limits_{\mathbb{R}_+^m}\min\left\{2\|f\|_C,\|f\|_{\omega_1}\omega_1(t_1),\ldots,\|f\|_{\omega_m}\omega_m(t_m)\right\}\prod_{j=1}^{m}t_j^{-\alpha_j-1}\,dt. \quad (17)$$

Combining (6) (for the function f) with (17), we see that

$$\|D_\varepsilon^\alpha f\|_C = 2^{m-1}A_\alpha\int\limits_{\mathbb{R}_+^m}\min\left\{2\delta,\omega_1(t_1),\ldots,\omega_m(t_m)\right\}\prod_{j=1}^{m}t^{-\alpha_j-1}\,dt =$$

$$= 2^{m-1}A_\alpha\int\limits_{\mathbb{R}_+^m}\min\left\{2\|f\|_C,\|f\|_{\omega_1}\omega_1(t_1),\ldots,\|f\|_{\omega_m}\omega_m(t_m)\right\}\prod_{j=1}^{m}t_j^{-\alpha_j-1}\,dt, \quad (18)$$

i.e., relation (6) turns into equality.

The proof is complete.

***Proof of Corollary* 1.** It follows from equality (6) that for any $\delta > 0$,

$$\Omega\left(\delta, \bigcap_{j=1}^{m} UH^{j,\omega_j}\right) \le 2^{m-1} A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2\delta, \omega_1(t_1), \ldots, \omega_m(t_m)\Big\} \prod_{j=1}^{m} t_j^{-\alpha_j-1}\, dt.$$

For the function f, constructed in proof of Theorem 1,

$$\|f\|_C \le \delta, \;\; f \in \bigcap_{j=1}^{m} H^{j,\omega_j}.$$

Using (18) we obtain

$$\Omega\left(\delta, \bigcap_{j=1}^{m} UH^{j,\omega_j}\right) \ge \|D_\varepsilon^\alpha f\|_C =$$

$$= 2^{m-1} A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2\delta, \omega_1(t_1), \ldots, \omega_m(t_m)\Big\} \prod_{j=1}^{m} t_j^{-\alpha_j-1}\, dt.$$

The corollary is proved.

***Proof of Theorem* 2.** As in the proof of Theorem 1 suppose that $\varepsilon = (+, \ldots, +)$. Remind that for a given $N > 0$, the vector $h^N = (h_1^N, \ldots, h_m^N) \in \mathbb{R}_+^m$ is defined by the following conditions:

$$\omega_1(h_1^N) = \ldots = \omega_m(h_m^N), \qquad \frac{2^m A_\alpha}{\alpha_1 \ldots \alpha_m} \prod_{j=1}^{m} (h_j^N)^{-\alpha_j} = N,$$

and

$$G(h^N) := \Big\{u = (u_1, \ldots, u_m) \in \mathbb{R}^m\colon \; u_1 \ge h_1^N, \ldots, u_m \ge h_m^N\Big\}.$$

Define the operator B_{h^N} as follows

$$B_{h^N} x(u) = A_\alpha \int\limits_{G(h^N)} \Delta_t x(u) \prod_{j=1}^{m} t_j^{-\alpha_j-1}\, dt.$$

Show that B_{h^N} is the bounded operator from $C(\mathbb{R}^m)$ to $C(\mathbb{R}^m)$, and moreover $\|B_{h^N}\| \le N$. Indeed for all $x \in C(\mathbb{R}^m)$,

$$\big\|B_{h^N} x\big\|_C \le 2^m A_\alpha \int\limits_{G(h^N)} \prod_{j=1}^{m} t_j^{-\alpha_j-1}\, dt \|x\|_C =$$

$$= \frac{2^m A_\alpha}{\alpha_1 \ldots \alpha_m} \prod_{j=1}^{m} \left(h_j^N\right)^{-\alpha_j} \|x\|_C = N\|x\|_C.$$

For any $x \in \bigcap\limits_{j=1}^{m} H^{j,\omega_j}$, estimate the deviation $\|D_\varepsilon^\alpha x - B_{h^N} x\|_C$. We have

$$\left\|D^\alpha_\varepsilon x - B_{h^N}x\right\|_C \le \left\|A_\alpha \int\limits_{\mathbb{R}^m_+\setminus G(h^N)} \Delta_t x(u)\prod_{j=1}^m t_j^{-\alpha_j-1}\,dt\right\|_C \le$$

$$\le 2^{m-1}A_\alpha \int\limits_{\mathbb{R}^m_+\setminus G(h^N)} \min\Big\{\omega_1(t_1),\ldots,\omega_m(t_m)\Big\}\prod_{j=1}^m t_j^{-\alpha_j-1}\,dt.$$

We have obtained the estimation of the value $E_N\Big(D^\alpha_\varepsilon,\ \bigcap\limits_{j=1}^m UH^{j,\omega_j}\Big)$ from above.

Let us estimate this value from below. From (10) we obtain

$$E_N\left(D^\alpha_\varepsilon,\ \bigcap_{j=1}^m UH^{j,\omega_j}\right) \ge \sup_{\delta>0}\left\{\Omega\left(\delta,\ \bigcap_{j=1}^m UH^{j,\omega_j}\right) - N\delta\right\}. \tag{19}$$

Using Corollary 1 and condition (11) we have

$$E_N\left(D^\alpha_\varepsilon,\ \bigcap_{j=1}^m UH^{j,\omega_j}\right) \ge$$

$$\ge \sup_{\delta>0}\left\{2^{m-1}A_\alpha\int\limits_{\mathbb{R}^m_+}\min\Big\{2\delta,\omega_1(t_1),\ldots,\omega_m(t_m)\Big\}\prod_{j=1}^m t_j^{-\alpha_j-1}\,dt\ -\right.$$

$$\left. -2^m A_\alpha\delta\int\limits_{G(h^N)}\prod_{j=1}^m t_j^{-\alpha_j-1}\,dt\right\}. \tag{20}$$

Set

$$\delta_N = \omega_1(h_1^N) = \ldots = \omega_m(h_m^N).$$

Note that for $t\in G(h^N)$,

$$\min\Big\{2\delta_N,\omega_1(t_1),\ldots,\omega(t_m)\Big\} = 2\delta_N$$

and for $t\in\mathbb{R}^m_+\setminus G(h^N)$,

$$\min\Big\{2\delta_N,\omega_1(t_1),\ldots,\omega(t_m)\Big\} = \min\Big\{\omega_1(t_1),\ldots,\omega(t_m)\Big\}.$$

From (20) we derive

$$E_N\left(D^\alpha_\varepsilon,\ \bigcap_{j=1}^m UH^{j,\omega_j}\right) \ge$$

$$\ge 2^{m-1}A_\alpha\int\limits_{\mathbb{R}^m_+}\min\Big\{2\delta_N,\omega_1(t_1),\ldots,\omega_m(t_m)\Big\}\prod_{j=1}^m t_j^{-\alpha_j-1}\,dt-$$

$$-2^{m-1}A_\alpha \int\limits_{G(h^N)} \min\Big\{2\delta_N, \omega_1(t_1), \ldots, \omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j-1}\, dt =$$

$$= 2^{m-1}A_\alpha \int\limits_{\mathbb{R}_+^m \setminus G(h^N)} \min\Big\{\omega_1(t_1), \ldots, \omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j-1}\, dt.$$

We have obtained the required estimation from below.

Theorem 2 is proved.

***Proof of Theorem* 3.** Let us consider the case $\varepsilon = (+, \ldots, +)$. For $\delta > 0$ and modulus of continuity $\omega_1, \ldots, \omega_m$, by $f(\,\cdot\,;\ \delta; \omega_1, \ldots, \omega_m)$ denote the function f constructed in the proof of Theorem 1. Suppose the inequality (12) holds true and select $0 < L_0 \le M_0$ such that

$$M_\alpha = 2^{m-1}A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2L_0, M_{\omega_1}\omega_1(t_1), \ldots, M_{\omega_m}\omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j-1}\, dt.$$

For the function $f(\cdot\,; L_0; M_{\omega_1}\omega_1, \ldots, M_{\omega_m}\omega_m)$, we have

$$\big\| f(\cdot\,; L_0; M_{\omega_1}\omega_1, \ldots, M_{\omega_m}\omega_m)\big\|_C \le L_0 \le M_0.$$

In addition, it is easy to verify that

$$\big\| f(\cdot\,; L_0; M_{\omega_1}\omega_1, \ldots, M_{\omega_m}\omega_m)\big\|_{\omega_j} = M_{\omega_j}, \quad j = 1, \ldots, m.$$

As in the proof of Theorem 1, we obtain

$$\big\| D_\varepsilon^\alpha f(\cdot\,; L_0; M_{\omega_1}\omega_1, \ldots, M_{\omega_m}\omega_m)\big\|_C =$$

$$= 2^{m-1}A_\alpha \int\limits_{\mathbb{R}_+^m} \min\Big\{2L_0, M_{\omega_1}\omega_1(t_1), \ldots, M_{\omega_m}\omega_m(t_m)\Big\} \prod_{j=1}^m t_j^{-\alpha_j-1}\, dt = M_\alpha.$$

Now construct the function

$$x(u) = f(u; L_0; M_{\omega_1}\omega_1, \ldots, M_{\omega_m}\omega_m) + M_0 - \big\| f(\cdot\,; L_0; M_{\omega_1}\omega_1, \ldots, M_{\omega_m}\omega_m)\big\|_C.$$

It is obvious that $x \in \bigcap\limits_{j=1}^m H^{j,\omega_j}$, and also

$$\|x\|_C = M_0, \qquad \|D_\varepsilon^\alpha x\|_C = M_\alpha, \qquad \|x\|_{\omega_j} = M_{\omega_j}, \quad j = 1, \ldots, m.$$

Theorem 3 is proved.

1. *Kolmogorov A. N.* Une generalisation de J. Hadamard entre les bornes superieures des derivees succesives d'une fonction // C. r. Acad. sci. – 1938. – **36**. – P. 764 – 765.
2. *Kolmogorov A. N.* On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval // Uch. Zapiski MGU. Mat. – 1939. – **30**, № 3. – S. 3 – 13.
3. *Kolmogorov A. N.* On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval // Selected Words. Math., Mech. – Moskow: Nauka, 1985. – P. 252 – 263 (in Russian).

4. *Arestov A. A.* Approximation of unbounded operators by the bounded ones and relative extremal problems // Uspekhi Mat. Nauk. – 1996. – **51**, № 6. – S. 88 – 124 (in Russian).
5. *Arestov A. A., Gabushin V. N.* The best approximation of unbounded operators by the bounded ones // Izv. Vuzov. Mat. – 1995. – **1**. – S. 44 – 66 (in Russian).
6. *Babenko V. F., Korneichuk N. P., Kofanov V. A., Pichugov S. A.* Inequalities for derivatives and their applicatios. – Kiev: Naukova Dumka, 2003 (in Russian).
7. *Kwong M. K., Zettl A.* Norm inequalities for derivatives and differences // Lect. Notes Math. – Berlin etc.: Springer, 1992. – **1536**.
8. *Mitrinović D. S., Peĉarić J. E., Fink A. M.* Inequalities involving functions and their integrals and derivatives. – Dordrecht etc.: Kluver Acad. Publ., 1991.
9. *Konovalov V. N.* Exact inequalities for norms of the functions, third partial and second mixed derivatives // Mat. Zametki. – 1978. – **23**, № 1. – S. 67 – 78 (in Russian).
10. *Buslaev A. P., Tikhomirov V. M.* On inequalities for derivatives in multivariate case // Mat. Zametki. – 1979. – **25**, № 1. – S. 54 – 74 (in Russian).
11. *Timoshin O. A.* Sharp inequalities between norms of partial derivatives of second and third order // Dokl. RAN. – 1995. – **344**, № 1. – S. 20 – 22 (in Russian).
12. *Babenko V. F., Kofanov V. A., Pichugov S. A.* Multivariate inequalities of Kolmogorov type and their applications // Multivariate Approxim. and Splines / Eds G. Nёrberger, J. W. Schmidt, G. Walz. – Basel: Birkhauser, 1997. – P. 1 – 12.
13. *Babenko V. F.* On sharp Kolmogorov type inequalities for bivariate functions // Dop. NAN Ukrainy. – 2000. – № 5. – S. 7 – 11 (in Russian).
14. *Babenko V. F., Pichugov S. A.* Kolmogorov type inequalities for fractional derivatives of Hölder functions of two variables // E. J. Approxim. – 2007. – **13**, № 3. – P. 321 – 329.
15. *Babenko V. F., Pichugov S. A.* Exact estimates of norms of fractional derivatives of multivariate functions sutisfying Hölder conditions // Mat. Zametki. – 2010. – **87**. – P. 26 – 34.
16. *Samko S. G., Kilbas A. A., Marichev O. I.* Integrals and derivatives of franctional order and their applications. – Minsk, 1987 (in Russian).
17. *Marchaud A.* Sur de derivées et sur les différences des fonctions de variables réelles // J. math. pures et appl. – 1927. – **6**. – P. 337 – 425.
18. *Babenko V. F., Churilova M. S.* On inequalities of Kolmogorov type for derivatives of fractional order // Bull. Dnepropetrovsk Univ. Mat. – 2001. – **6**. – P. 16 – 20 (in Russian).
19. *Geisberg S. P.* Generalization of Hadamard inequality // Sb. Nauch. Tr. Leningr. Mech. Inst. – 1965. – **50**. – S. 42 – 54 (in Russian).
20. *Arestov V. V.* Inequalities for functional derivatives on the half-line // Approxim. Theory. – Warsawa: PWN, 1979. – P. 19 – 34.
21. *Magaril-Il'jaev G. G., Tihomirov V. M.* On the Kolmogorov inequality for fractional derivatives on the half-line // Anal. math. – 1981. – **7**, № 1. – P. 37 – 47.
22. *Stechkin S. B.* Best approximation of linear operators // Mat. Zametki. – 1967. – **1**, № 2. – S. 137 – 148 (in Russian).
23. *Rodov A. M.* Relation between upper bounds of derivatives of real functions, defined on the whole real line // Isv. AN USSR. – 1946. – **10**. – S. 257 – 270 (in Russian).
24. *Rodov A. M.* Relation Sufficient conditions of existence of real values function with given upper bounds of moduluses of the function itself and its five successive derivatives // Uch. Zap. Bel. Gos. Univ. – 1954. – **19**. – S. 65 – 72 (in Russian).
25. *Dzyadyk V. K., Dubovik V. A.* To the Kolmogorov problem on dependence between upper bound of derivatives of real valued functions, defined on the whole real line // Ukr. Mat. Zh. – 1974. – **26**, № 3. – S. 246 – 259 (in Russian).
26. *Dzyadyk V. K., Dubovik V. A.* To Kolmogorov inequalities between upper bound of derivatives of real valued functions, defined on the whole real line // Ukr. Mat. Zh. – 1975. – **27**, № 3. – S. 291–299 (in Russian).

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