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ON THE RELATION BETWEEN MEASURES DEFINING THE STIELTJES AND THE INVERTED STIELTJES FUNCTIONS

ПРО ЗВ'ЯЗОК МІЖ МІРАМИ, ЩО ВИЗНАЧАЮТЬ ПОЧАТКОВУ ТА ОБЕРНЕНУ ФУНКЦІЇ СТІЛЬТЬЄСА

A compact formula is found for a measure of the inverted Stieltjes function expressed by the measure of the original Stieltjes function.

Встановлено формулу для міри оберненої функції Стільтьєса, що виражена через міру початкової функції Стільтьєса.

In 1991 Gilewicz [1] posed an open problem of giving an explicit expression for a measure of the inverted Stieltjes function. Peherstorfer [2] gave an answer to this question for a certain class of measures for Stieltjes functions. Here we present a completely different derivation than the one given in [2], actually a quite elementary one, which is also valid for more general types of measures.

Let us recall that if g is a Stieltjes function

$$z \in \mathbb{C} \setminus (-\infty, -R]\colon\ g(z) = \int_0^{1/R} \frac{d\mu(t)}{1+tz} \tag{1}$$

then it is well known that the function h defined by

$$g(z) = \frac{g(0)}{1+zh(z)} \tag{2}$$

is also a Stieltjes function (see, e.g., [3]). The function h is called the *inverted Stieltjes function*. Formula (1) defines the function which is analytic in the whole complex z-plane, except for the cut between $-R$ and $-\infty$. Therefore h has the same analytic properties. However if we want to write h in a similar form, namely

$$h(z) = \int_0^{1/R} \frac{d\nu(t)}{1+tz},$$

then it is not quite obvious what is the relation between the measures $d\mu$ and $d\nu$.

When the relation (2) is reversed, we get

$$h(z) = \frac{g(0)}{zg(z)} - \frac{1}{z}$$

and using (1), it can be written as

$$h(z) = \frac{\int_0^{1/R} \frac{td\mu(t)}{1+tz}}{\int_0^{1/R} \frac{d\mu(t)}{1+tz}}. \tag{3}$$

*M. Pindor passed away in 2003.

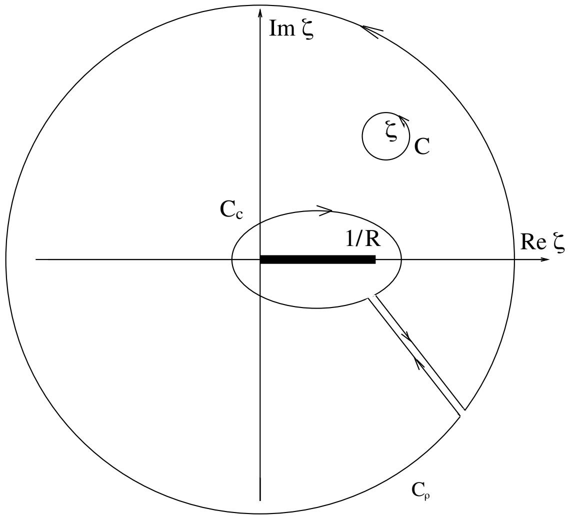

Fig. 1. The contour C for the Cauchy integral and contours C_c and C_ρ into which C is deformed.

In order to find an integral representation of h we use the standard Cauchy formula. However we will work in the ζ-plane where $\zeta = -1/z$, because then both branch points of h lie inside a bounded domain. Therefore, if we set $\tilde{h}(\zeta) = h(-1/\zeta)$, then formula (3) becomes

$$\tilde{h}(\zeta) = \frac{\int_0^{1/R} \frac{td\mu(t)}{\zeta - t}}{\int_0^{1/R} \frac{d\mu(t)}{\zeta - t}}. \tag{4}$$

On the other hand

$$\tilde{h}(\zeta) = \frac{1}{2\pi i} \int\limits_C \frac{\tilde{h}(\xi)d\xi}{\xi - \zeta},$$

where the contour C encircles counterclockwise the point ζ and does not contain the points 0 and $1/R$ (see Fig. 1).

Now (see Fig. 2) we can deform C in such a way that it becomes the contours C_c (encircling both branch points and the cut joining them, clockwise) and C_ρ (being a circle of radius $\rho > 1/R$ where we move counterclockwise). Thus

$$\tilde{h}(\zeta) = \frac{1}{2\pi i} \left(\int\limits_{C_c} \frac{\tilde{h}(\xi)d\xi}{\xi - \zeta} + \int\limits_{C_\rho} \frac{\tilde{h}(\xi)d\xi}{\xi - \zeta} \right).$$

Obviously

$$\int\limits_{C_\rho} \frac{\tilde{h}(\xi)d\xi}{\xi - \zeta} = \int\limits_0^{2\pi} \frac{\rho i e^{i\varphi} \tilde{h}(\rho e^{i\varphi})d\varphi}{\rho e^{i\varphi} - \zeta}.$$

By (4), if $\xi \to \infty$, then

$$\tilde{h}(\xi) \to \frac{\int_0^{1/R} t d\mu(t)}{\int_0^{1/R} d\mu(t)}.$$

Therefore when $\rho \to \infty$

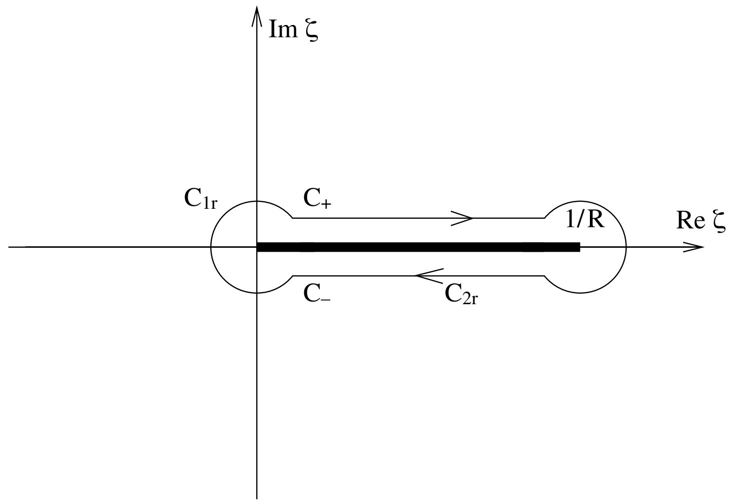

Fig. 2. The final form of the contour C_c.

$$\int\limits_{C_\rho} \frac{\tilde{h}(\xi)d\xi}{\xi-\zeta} \to \frac{\int_0^{1/R} t d\mu(t)}{\int_0^{1/R} d\mu(t)} \int\limits_0^{2\pi} i d\varphi.$$

On the other hand the contour C_c can be split into four paths: C_+, C_{1r}, C_- and C_{2r} as indicated on Fig. 2. The contours C_{1r} and C_{2r} are circles of radius r arround the points 0 and $1/R$, respectively. When $r \to 0$, integrals over C_+ and C_- become integrals between 0 and $1/R$ over the upper and lower lips of the cut joining the branch points, while integrals over C_{1r} and C_{2r} converge to 0. As a consequence, we deduce

$$\tilde{h}(\zeta) = \frac{\int_0^{1/R} t d\mu(t)}{\int_0^{1/R} d\mu(t)} + \frac{1}{2\pi i}\left(\int\limits_{C_+} \frac{\tilde{h}(z)dz}{z-\zeta} + \int\limits_{C_-} \frac{\tilde{h}(z)dz}{z-\zeta}\right).$$

Our task now is to express the difference of integrals over the upper and lower lips of the cut by an integral containing the measure $d\mu$ defining the function g. If the bounded and nondecreasing function μ is differentiable, then $d\mu(t)$ may be written in the form $\mu'(t)dt$ and we may directly use (4). In the following we shall use $d\mu(t) = \mu'(t)dt$ for more general situations meaning that we consider such $d\mu(t)$ that there exists a distribution $\mu'(t)$ with the necessary properties for the existence of the integrals considered. The same for $d\nu(t)$ and $\nu'(t)$.

In particular we consider a case where μ contains a contribution from a Heaviside function, i.e., we take

$$\mu(t) = GH(t) + \sigma(t), \qquad H(t) = \begin{cases} 1, & \text{for } t > 0, \\ 0, & \text{for } t \le 0, \end{cases}$$

where now σ has no jump at $t = 0$. In this case (4) becomes

$$\tilde{h}(\zeta) = \frac{\int_0^{1/R} \frac{t d\sigma(t)}{\zeta - t}}{\frac{G}{\zeta} + \int_0^{1/R} \frac{d\sigma(t)}{\zeta - t}}. \tag{5}$$

Using now the well known Sokhotskij – Plemelj formulae for (5) with $\zeta = x \pm i\varepsilon$ for the sum of integrals over C_+ and C_- we get

$$\int\limits_{C_+} \frac{\tilde{h}(z)dz}{z-\zeta} + \int\limits_{C_-} \frac{\tilde{h}(z)dz}{z-\zeta} =$$

$$= -\int\limits_0^{1/R} \frac{\sigma'(x)\left[G + \int_0^{1/R}\sigma'(t)dt\right]}{\left[\frac{G}{x} - \int_0^{1/R}\frac{\sigma'(t)-\sigma'(x)}{t-x}dt - \sigma'(x)\log\left(\frac{1}{Rx}-1\right)\right]^2 + \pi^2(\sigma'(x))^2}\frac{dx}{x-\zeta}.$$

Going back to z variable we get

$$h(z) = \frac{\int_0^{1/R} t\sigma'(t)dt}{G + \int_0^{1/R}\sigma'(t)dt} + \left(G + \int\limits_0^{1/R}\sigma'(t)dt\right)\times$$

$$\times\left(-\int\limits_0^{1/R}\frac{\sigma'(x)dx}{x\left[\left(\frac{G}{x} - \int_0^{1/R}\frac{\sigma'(t)-\sigma'(x)}{t-x}dt - \sigma'(x)\log\left(\frac{1}{Rx}-1\right)\right)^2 + \pi^2(\sigma'(x))^2\right]} + \right.$$

$$\left. + \int\limits_0^{1/R}\frac{\sigma'(x)dx}{x(1+zx)\left[\left(\frac{G}{x} - \int_0^{1/R}\frac{\sigma'(t)-\sigma'(x)}{t-x}dt - \sigma'(x)\log\left(\frac{1}{Rx}-1\right)\right)^2 + \pi^2(\sigma'(x))^2\right]}\right).$$

This formula can be stated in a more compact form using the fact that

$$\frac{G}{x} - \int\limits_0^{1/R}\frac{\sigma'(t)-\sigma'(x)}{t-x}dt - \sigma'(x)\log\left(\frac{1}{Rx}-1\right) \pm \pi i\sigma'(x) = \int\limits_0^{1/R}\frac{d\mu(t)}{x \pm i\varepsilon - t}, \quad (6)$$

where, by (1), we immediately see that the right-hand side is just $g(-1/z)/z$ for $z = x \pm i\varepsilon$. Another simplification comes from the following observation:

$$\int\limits_0^{1/R}\frac{\sigma'(x)dx}{x\left[\left(\frac{G}{x} - \int_0^{1/R}\frac{\sigma'(t)-\sigma'(x)}{t-x}dt - \sigma'(x)\log\left(\frac{1}{Rx}-1\right)\right)^2 + \pi^2(\sigma'(x))^2\right]} =$$

$$-\frac{1}{2\pi i}\left[\int\limits_0^{1/R}\frac{dx}{x\left(\frac{G}{x} - PV\int_0^{1/R}\frac{\sigma'(t)dt}{t-x} + \pi i\sigma'(x)\right)} - \right.$$

$$\left. - \int\limits_0^{1/R}\frac{dx}{x\left(\frac{G}{x} - PV\int_0^{1/R}\frac{\sigma'(t)dt}{t-x} - \pi i\sigma'(x)\right)}\right]. \quad (7)$$

The denominators of the two integrals above are

$$x\left[\frac{G}{x} - \int\limits_0^{1/R}\frac{\sigma'(t)dt}{t-x}\right] = x\int\limits_0^{1/R}\frac{d\mu(t)}{x-t} = g\left(-\frac{1}{x}\right)$$

for x just below and just above the interval $[0, 1/R]$ (i.e., over C_- and C_+ (see (6))). Therefore the integrals on the right-hand side of (7) may be written as

$$\frac{1}{2\pi i}\int\limits_{C_\rho}\frac{dz}{z\left(\frac{G}{z}-\int_0^{1/R}\frac{\sigma'(t)dt}{t-z}\right)}-\frac{1}{2\pi i}\lim_{r\to 0}\int\limits_{C_{1r}}\frac{dz}{z\left(\frac{G}{z}-\int_0^{1/R}\frac{\sigma'(t)dt}{t-z}\right)} \tag{8}$$

(see Fig. 2) because an integral over the small circle around $1/R$ converges to 0 with the radius r of that circle. Now, the integral over C_ρ can be deformed to a large circle of radius ρ going to ∞. In this limit the integral is

$$\frac{2\pi i\int_0^{1/R}t\sigma'(t)dt}{\left(G+\int_0^{1/R}\sigma'(t)dt\right)^2}.$$

To understand what is the limit of the integral over C_{1r} when $r\to 0$, we recall that $z=re^{i\varphi}$ on C_{1r}. Looking now at the explicit form of the denominator of the integrand in (8) and using

$$\lim_{r\to 0} re^{i\varphi}\int\limits_0^{1/R}\frac{\sigma'(t)dt}{t-re^{i\varphi}}=0,$$

it follows that if $G\neq 0$ then the limit of the integral over C_{1r} is 0. The same is true if $G=0$, and

$$\lim_{r\to 0}\int\limits_0^{1/R}\frac{\sigma'(t)dt}{t-re^{i\varphi}}=\infty.$$

Finally, if the above limit is finite, then

$$\lim_{r\to 0}\int\limits_{C_{1r}}\frac{dz}{g\left(-1/z\right)}=\frac{2\pi i}{\int_0^{1/R}\frac{\sigma'(t)dt}{t}}.$$

Therefore, since

$$G+\int\limits_0^{1/R}\sigma'(t)dt=g(0),$$

then our final formulae are either

$$h(z)=g(0)\left[\frac{1}{2\pi i}\lim_{r\to 0}\int\limits_{C_{1r}}\frac{d\xi}{G+\xi\int_0^{1/R}\frac{d\sigma(t)}{\xi-t}}+\int\limits_0^{1/R}\frac{td\sigma(t)}{(1+tz)\left|g\left(-1/t\right)\right|^2}\right]$$

or

$$\nu'(t)=g(0)\left[\delta(t)\lim_{r\to 0}\frac{1}{2\pi i}\int\limits_{C_{1r}}\frac{d\xi}{G+\xi\int_0^{1/R}\frac{d\sigma(u)}{\xi-u}}+\frac{t\sigma'(t)}{\left|g\left(-1/t\right)\right|^2}\right],$$

where $\delta(t)=H'(t)$ is the Dirac distribution. In particular this result shows that if the measure $d\mu$ defining g contains a δ (Dirac) at the origin (that is, if $G\neq 0$), then the measure $d\nu$ defining h does not contain δ at the origin, and vice-versa.

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