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POLYNOMIAL EXTENSIONS OF GENERALIZED QUASI-BAER RINGS

ПОЛІНОМІАЛЬНІ РОЗШИРЕННЯ УЗАГАЛЬНЕНИХ КВАЗІБЕРОВИХ КІЛЕЦЬ

In this paper we consider the behavior of polynomial rings over generalized quasi-Baer rings, and we show that the generalized quasi-Baer condition on a ring R is preserved by many polynomial extensions.

Розглянуто поведінку поліноміальних кілець над узагальненими квазіберовими кільцями і показано, що узагальнена квазіберова умова щодо кільця R зберігається при багатьох поліноміальних розширеннях.

1. Introduction. Throughout this paper all rings are associative with identity. A ring R is called (quasi-)Baer if the right annihilator of every (right ideal) nonempty subset of R is generated as a right ideal by an idempotent. It is easy to see that the Baer and quasi-Baer properties are left-right symmetric for any ring. The study of Baer rings has its roots in functional analysis. In [1] Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$-regular rings. In [2] Clark uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The concepts of Baer and quasi-Baer have been investigated by several authors for rings. Every prime ring is a quasi-Baer ring. Since Baer rings are nonsingular, the prime rings R with $Z_r(R) \neq 0$ are quasi-Baer but not Baer. Another generalization of Baer rings are p.p.-rings. A ring R is called a right (resp. *left*) p.p.-*ring* if every principal right (resp. left) ideal is *projective* (equivalently, if the right (resp. left) annihilator of any element of R is generated by an idempotent of R). A ring R is called a p.p.-*ring* if it is both right and left p.p.-ring. A ring R is said to be *generalized right* p.p.-ring if for any $x \in R$ the right annihilator of x^n is generated by an idempotent for some positive integer n. Von Neumann regular rings are p.p.-*rings,* and π-regular rings are generalized p.p.-rings in the same sense as von Neumann regular rings.

In [3, 4], Birkenmeier, Kim and Park introduced a principally quasi-Baer ring and used them to generalize many results on *reduced* (i.e., it has no nonzero nilpotent elements) p.p.-rings. A ring R is called *right principally quasi-Baer* (or simply *right* p.q.-*Baer*) if the right annihilator of a principal right ideal is generated by an idempotent. Similarly, left p.q.-Baer rings can be defined. In [5] Moussavi, Javadi and Hashemi introduced generalized (principally) quasi-Baer ring. A ring R is *generalized right* (*principally*) *quasi-Baer* if for any (principal) right ideal I of R, the right annihilator of I^n is generated by an idempotent for some positive integer n, depending on I. For example Z_{p^n}, $n > 2$ (p is a prime number), is generalized quasi-Baer but is not quasi-Baer.

In 1974 Armendariz seems to be the first to consider the behavior of polynomial rings over Baer rings [6] (Theorem B). In this paper we consider the behavior of polynomial rings over generalized quasi-Baer rings.

We used $R[x]$, $R[x,\alpha,\delta]$, $R[x,x^{-1}]$, $r_R(X)$, $l_R(X)$ and $\mathrm{Id}(R)$ for the ring of polynomial over R, the skew polynomial ring over R, the laurent polynomial ring over R, the right and left annihilators of X subset of R and the set of all idempotent of R, respectively.

2. Main results. In this section we prove our main result showing that the generalized quasi-Baer condition on R is preserved by many polynomial extensions.

Lemma 1. *Let I be an right ideal of the ring R then we have the following assertions*:

(1) $I^n[x] = (I[x])^n$;

(2) $r_{R[x]}(I[x]) = r_R(I)[x]$.

Proof. The proof is straightforward.

Recall that a ring R is called Armendariz if whenever polynomials

$$f(x) = a_0 + a_1x + \ldots + a_mx^m \quad \text{and} \quad g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x]$$

satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for all i, j.

Let c_f denote the set of all coefficients of $f(x) \in R$.

Proposition 1. *Let R be a generalized right quasi-Baer and Armendariz ring. Then $R[x]$ is a generalized right quasi-Baer ring.*

Proof. Assume R be a generalized right quasi-Baer and Armendariz ring. Let I be a right ideal of $R[x]$ and I_0 denote the set of coefficients of all elements of I in R. It is clear that I_0 is a right ideal of R, thus there exists $e \in \mathrm{Id}(R)$ such that $r_R(I_0^n) = eR$ for some $n \in N$. We claim that $r_{R[x]}(I^n) = eR[x]$. It is clear that $I \subseteq I_0[x]$, then from Lemma 1 $eR[x] = r_{R[x]}(I_0^n[x]) \subseteq r_{R[x]}(I^n)$. Conversely let

$$g(x) = b_0 + b_1x + \ldots + b_nx^n \in r_{R[x]}(I^n) \quad \text{and} \quad a = \sum_{i=1}^{k} a_{i_1} a_{i_2} \ldots a_{i_n} \in I_0^n$$

with $a_{i_j} \in I_0$.

Then there exists $f_{i_j} \in I$ such that $a_{i_j} \in c_{f_{i_j}}$. Therefore $f_{i_1}(x) f_{i_2}(x) \ldots \ldots f_{i_n}(x) g(x) = 0$, then $a_{i_1} a_{i_2} \ldots a_{i_n} b_i = 0$, since R is Armendariz ring. Thus $g(x) \in r_{R[x]}(I_0^n[x]) = eR[x]$.

The proposition is proved.

We know that, if R be quasi-Baer ring then $R[x]$ is quasi-Baer [3] (Theorem 1.2). By Proposition 1 we showed that, if R is an Armendariz generalized right quasi-Baer ring then $R[x]$ is a generalized right quasi-Baer ring. Also in Proposition 2 we will prove the converse of Proposition 1 is correct without Armendariz property. But in fact, we do not know of any example of generalized quasi-Baer polynomial ring such that R is a generalized quasi-Baer but R is not Armendariz.

Question: Let R be a generalized right quasi-Baer ring. Is $R[x]$ generalized right quasi-Baer ring without Armendariz property?

Proposition 2. *Let $R[x]$ be a generalized right quasi-Baer ring then R is a generalized right quasi-Baer ring.*

Proof. Let $R[x]$ be generalized right quasi-Baer ring and I be a right ideal of R. Then there exists an idempotent $e(x) \in R[x]$ such that $r_{R[x]}(I^n[x]) = e(x)R[x]$

for some $n \in N$. Let e_0 be constant term of $e(x)$ then $e_0^2 = e_0$. Since $I^n e(x) = 0$, we have $I^n e_0 = 0$ therefore $e_0 \in r_R(I^n)$. Thus $e_0 R \subseteq r_R(I^n)$.

Conversely, let $b \in r_R(I^n)$ then $b \in r_{R[x]}(I^n[x]) \bigcap R = e(x) R[x] \bigcap R$. Therefore we have $b = e(x)h(x)$ for some $h(x) \in R[x]$. Thus $b = e_0 h_0$ where h_0 is constant term of $h(x)$ so $b \in e_0 R$. Hence $r_R(I^n) = e_0 R$.

The proposition is proved.

Proposition 3. *Let Δ be a multiplicatively closed subset of R consisting of central regular element. Then*:

(1) *If R is generalized right quasi-Baer ring then $\Delta^{-1}R$ is generalized right quasi-Baer ring.*

(2) *Let $\mathrm{Id}(R) = \mathrm{Id}(\Delta^{-1}R)$. If $\Delta^{-1}R$ is generalized right quasi-Baer then R is generalized right quasi-Baer ring.*

(3) *If $R[x]$ is generalized right quasi-Baer ring then $R[x, x^{-1}]$ is generalized right quasi-Baer ring.*

(4) *Let $\mathrm{Id}(R) = \mathrm{Id}([x, x^{-1}])$. If $R[x, x^{-1}]$ is generalized right quasi-Baer then R is generalized right quasi-Baer ring.*

Proof. (1) Assume that R is a generalized right quasi-Baer ring. Let I be a right ideal of $\Delta^{-1}R$ and $I_0 = \{a \in R \mid b^{-1}a \in I, \text{ for some } b \in \Delta\}$. It is clear $I_0 \neq \emptyset$, $I_0 \neq R$, $I_0 \lhd R$ and $(\Delta^{-1}I_0)^n = \Delta^{-1}I_0^n$.

We know $I \subseteq \Delta^{-1}I_0$. Now let $c^{-1}d \in \Delta^{-1}I_0$ such that $c \in \Delta$, $d \in I_0$. Thus there exists $k \in \Delta$ such that $k^{-1}d \in I$. Since $d \in I_0$, therefore $c^{-1}d = k^{-1}dc^{-1}k \in I$ hence $\Delta^{-1}I_0 \subseteq I$.

Now we claim $\Delta^{-1}r_R(I_0) = r_{\Delta^{-1}R}(\Delta^{-1}I_0)$. Let $a^{-1}b \in \Delta^{-1}r_R(I_0)$ then $(c^{-1}d)(a^{-1}b) = 0$ for all $c^{-1}d \in \Delta^{-1}I_0$, since $db = 0$. Thus $a^{-1}b \in r_{\Delta^{-1}R}(\Delta^{-1}I_0)$, therefore $\Delta^{-1}r_R(I_0) \subseteq r_{\Delta^{-1}R}(\Delta^{-1}I_0)$.

Conversely, let $a^{-1}b \in r_{\Delta^{-1}R}(\Delta^{-1}I_0)$ then $c^{-1}d(a^{-1}b) = 0$, for all $c^{-1}d \in \Delta^{-1}I_0$. Thus $db = 0$ then $b \in r_R(I_0)$. Therefore $a^{-1}b \in \Delta^{-1}r_R(I_0)$.

By hypothesis, $r_R(I_0^n) = eR$ for some $e^2 = e \in R$. Thus $I_0^n e = 0$ and so $0 = \Delta^{-1}I_0^n e = (\Delta^{-1}I_0)^n = I^n e$. Hence $e\Delta^{-1}R \subseteq r_{\Delta^{-1}R}(I^n)$. Let $a^{-1}b \in r_{\Delta^{-1}R}(I^n)$. Then $0 = I^n a^{-1}b = (\Delta^{-1}I_0)^n a^{-1}b = \Delta^{-1}I_0^n a^{-1}b$ and so $b \in r_R(I_0^n) = eR$. Hence $a^{-1}b \in e\Delta^{-1}R$. Therefore $r_{\Delta^{-1}R}(I^n) = e\Delta^{-1}R$.

(2) Let $\Delta^{-1}R$ is generalized right quasi-Baer. We prove that R is generalized right quasi-Baer ring. Let I be a right ideal of R then $\Delta^{-1}I$ is right ideal of $\Delta^{-1}R$, thus there exists $e \in R$ such that $e^2 = e$ and $r_{\Delta^{-1}R}((\Delta^{-1}I)^n) = e(\Delta^{-1}R)$ for some $n \in N$. We prove $r_R(I^n) = eR$. We show that $r_R(I^n) \subseteq eR$. Let $b \in r_R(I^n)$ then

$I^n b = 0$, thus $0 = \Delta^{-1} I^n b = (\Delta^{-1} I)^n b$ and so $b \in r_{\Delta^{-1}R}((\Delta^{-1} I)^n) = e(\Delta^{-1} R)$. It follows that $b = eb \in eR$. The other side is similarly.

(3), (4) Let $\Delta = \{1, x, x^2, \ldots\}$, then $\Delta^{-1} R[x] = R[x, x^{-1}]$, and so proof is complete.

Recall that for a ring R with a ring endomorphism $\alpha : R \to R$ and α-derivation $\delta : R \to R$, the Ore extension $R[x, \alpha, \delta]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x + \delta(r)$ for all $r \in R$. If $\delta = 0$, we write $R[x, \alpha]$ for $R[x, \alpha, 0]$ and is called an Ore extension of endomorphism type (also called a skew polynomial ring). In [7] Kerempa defined the rigid rings. Let α be an endomorphism of R, α is called a rigid endomorphism if $r\alpha(r) = 0$ implies $r = 0$ for $r \in R$. A ring R is called to be α-rigid if there exist a rigid endomorphism α of R. If R be a α-rigid then $\mathrm{Id}(R) = \mathrm{Id}(R[x, \alpha, \delta]) = \mathrm{Id}(R[x, \alpha])$ (Corollary 7). Let R be a rigid ring. It is clear that generalized quasi-Baer and quasi-Baer conditions are equivalent. Then if R be α-rigid ring, R is generalized quasi-Baer if and only if $R[x, \alpha]$ is generalized quasi-Baer ring [8] (Corollary 12). In Example 1 we show that rigid condition is not superfluous.

Example 1. Let Z be the ring of integers and consider the ring $Z \oplus Z$ with the usual addition and multiplication. Then the subring $R = \{(a, b) \in Z \oplus Z \mid a \equiv b \pmod 2\}$ of $Z \oplus Z$ is commutative reduced ring. Note that only idempotents of R are $(0, 0)$ and $(1, 1)$. Hence from [5] (Example 2.1) R is not generalized right quasi-Baer. Now let $\alpha : R \to R$ be defined by $\alpha((a, b)) = (b, a)$ Then α is an automorphism of R. Hence $R[x, \alpha]$ is quasi-Baer from [8] (Example 9).

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