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ON ASYMPTOTIC EXTENSION DIMENSION*

ПРО АСИМПТОТИЧНУ РОЗШИРЕНУ ВИМІРНІСТЬ

The aim of this paper is to introduce an asymptotic counterpart of the extension dimension defined by Dranishnikov. The main result establishes a relation between the asymptotic extensional dimension of a proper metric space and extension dimension of its Higson corona.

Метою статті є введення асимптотичного аналога розширеної вимірності, що визначена Дранішніковим. Основний результат полягає у встановленні співвідношення між асимптотичною розширеною вимірністю власного метричного простору та розширеною вимірністю його корони Хігсона.

1. Introduction. Asymptotic dimension of metric spaces was first defined by Gromov [1] for finitely generated groups. Since then, this dimension is an object of study in numerous publications (see an expository paper [2]).

A metric space (X, d) is of asymptotic dimension $\leq n$ (written $\operatorname{asdim} X \leq n$) if for every $D > 0$ there exists a uniformly bounded cover $\mathcal{U}$ of X such that $\mathcal{U} = \mathcal{U}^0 \cup \ldots \cup \mathcal{U}^n$, where every family $\mathcal{U}^i$ is D-disjoint, $i = 0, 1, \ldots, n$. Recall that a family $\mathcal{A}$ of subsets of X is *uniformly bounded* if

$$\operatorname{mesh} \mathcal{A} = \sup\{\operatorname{diam} A \mid A \in \mathcal{A}\} < \infty$$

(as usual, $\operatorname{diam} A = \sup\{d(x, y) \mid x, y \in A\}$ is the *diameter* of a subset A in a metric space (X, d)) and is called *D-disjoint* if $\inf\{d(a, a') \mid a \in A,\ a' \in A'\} > D$, for every distinct $A, A' \in \mathcal{A}$.

The asymptotic dimension can be characterized in different terms; in particular, in terms of extension of maps into Euclidean spaces [3]: a proper metric space X is of asymptotic dimension $\leq n$ if and only if any proper asymptotically Lipschitz map $f\colon A \to \mathbb{R}^{n+1}$ (see the definition below) defined on a closed subset A of X admits a proper asymptotically Lipschitz extension over X. In the classical dimension theory, to this result there corresponds the Aleksandrov theorem: for any metric space X, $\dim X \leq n$, where $\dim$ stands for the covering dimension, if and only if any continuous map $f\colon A \to S^n$ defined on a closed subset A of X admits a continuous extension over X.

In [3, 4] Dranishnikov introduced the notion of extension dimension. This dimension takes its values in the so called dimension types of CW-complexes. The aim of this paper is to develop an asymptotic counterpart of the extension dimension. Our main results is a generalization of the well-known result due to Dranishnikov [3] on the equality, for the spaces of finite asymptotic dimension, of the asymptotic dimension of a proper metric space and the dimension of the Higson corona of this space.

2. Preliminaries. A typical metric is denoted by d. By $N_r(x)$ we denote the open ball of radius r centered at a point x of a metric space.

2.1. Asymptotic category. A map $f\colon X \to Y$ between metric spaces is called (λ, ε)-*Lipschitz* for $\lambda > 0$, $\varepsilon \geq 0$ if $d(f(x), f(x')) \leq \lambda d(x, x') + \varepsilon$ for every $x, x' \in X$. A map is called *asymptotically Lipschitz* if it is (λ, ε)-Lipschitz for some $\lambda, \varepsilon > 0$.

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The $(\lambda, 0)$-Lipschitz maps are also called λ-*Lipschitz*, $(1, 0)$-Lipschitz maps are also called *short*.

A metric space X is called *proper* if every closed ball in X is compact.

The *asymptotic category* $\mathcal{A}$ is introduced by A. Dranishnikov [3]. The objects of $\mathcal{A}$ are proper metric spaces and the morphisms are proper asymptotically Lipschitz maps. Recall that a map is called *proper* if the preimage of every compact set is compact.

We also need the notion of a *coarse map*. A map between proper metric spaces is called *coarse uniform* if for every $C > 0$ there is $K > 0$ such that for every $x,\ x' \in X$ with $d(x, x') < C$ we have $d(f(x), f(x')) < K$. A map $f\colon X \to Y$ is called *metric proper* if the preimage $f^{-1}(B)$ is bounded for every bounded set $B \subset Y$. A map is *coarse* if it is metric proper and coarse uniform.

2.2. Higson compactification and Higson corona. Let $\varphi\colon X \to \mathbb{R}$ be a function defined on a metric space X. For every $x \in X$ and every $r > 0$ let

$$\mathrm{Var}_r\varphi(x) = \sup\{|\varphi(y) - \varphi(x)| \mid y \in N_r(x)\}.$$

A function φ is called *slowly oscillating* whenever for every $r > 0$ we have $\mathrm{Var}_r\varphi(x) \to \to 0$ as $x \to \infty$ (the latter means that for every $\varepsilon > 0$ there exists a compact subspace $K \subset X$ such that $|\mathrm{Var}_r\varphi(x)| < \varepsilon$ for all $x \in X \setminus K$. Let $\bar{X}$ be the compactification of X that corresponds to the family of all continuous bounded slowly oscillation functions. The *Higson corona* of X is the remainder $\nu X = \bar{X} \setminus X$ of this compactification.

It is known that the Higson corona is a functor from the category of proper metric spaces and coarse maps into the category of compact Hausdorff spaces. In particular, if $X \subset Y$, then $\nu X \subset \nu Y$.

For any subset A of X we denote by A' its trace on νX, i.e., the intersection of the closure of A in $\bar{X}$ with νX. Obviously, the set A' coincides with the Higson corona νA.

2.3. Cone. Let X be a metric space of diameter ≤ 1. The *open cone* of X is the set $\mathcal{O}X = (X \times \mathbb{R}_+)/(X \times \{0\})$ endowed with the metric (by $[x, t]$ we denote the equivalence class of $(x, t) \in X \times \mathbb{R}_+$):

$$d([x_1, t_1], [x_2, t_2]) = |t_1 - t_2| + \min\{t_1, t_2\}d(x_1, x_2).$$

For a map $f\colon X \to Y$ of metric spaces we denote by $\mathcal{O}f\colon \mathcal{O}X \to \mathcal{O}Y$ the map defined as $\mathcal{O}f([x, t]) = [f(x), t]$.

Proposition 2.1. *If $f\colon X \to Y$ is a Lipschitz map than $\mathcal{O}f$ is an asymptotically Lipschitz map.*

Proof. Suppose a map $f\colon X \to Y$ is λ-Lipschitz. Then for any $[x_1, t_1], [x_2, t_2] \in \in \mathcal{O}X$ we have

$$d(\mathcal{O}f([x_1, t_1]), \mathcal{O}f([x_2, t_2])) = d([f(x_1), t_1], [f(x_2), t_2]) =$$

$$= |t_1 - t_2| + \min\{t_1, t_2\}d(f(x_1), f(x_2)) \leq$$

$$\leq \lambda'(|t_1 - t_2| + \min\{t_1, t_2\}d(x_1, x_2)),$$

where $\lambda' = \max\{\lambda, 1\}$.

Proposition 2.1 is proved.

The open cone of a finite CW-complex is a coarse CW-complex in the sense of [5].

Denote by $\alpha_L\colon \mathcal{O}L \to \mathbb{R}$ the function defined by $\alpha_L([x,t]) = t$. Obviously, α_L is a short function.

Let $\tilde{\mathcal{O}}L = \{[x,t] \in \mathcal{O}L \mid t \geq 1\}$. Denote by $\beta_L\colon \tilde{\mathcal{O}}L \to L$ the map $\beta_L([x,t]) = x$.

Lemma 2.1. *The map β_L is slowly oscillating.*

Proof. For $R > 0$, the R-ball centered at $[x,0]$ is $\{[x,t] \mid t < R$. If $d([x,t],[x_1,t_1]) < K < R$, then $|t - t_1| + \min\{t,t_1\} d(x,x_1) < K$, i.e., $(t-R)d(x,x_1) < R$ and $d(x,x_1) < K/(t-K)$. Therefore, $d(\beta_L(x),\beta_L(x_1)) < K/(R-K) \to 0$ as $R \to \infty$.

Lemma 2.1 is proved.

Let $\bar{\beta}_L\colon \tilde{\mathcal{O}}L \to L$ be the (unique) extension of the map β_L. Denote by $\eta_L\colon \nu\tilde{\mathcal{O}}L \to L$ the restriction of β_L.

Proposition 2.2. *Let $f\colon A \to \mathcal{O}L$ be a proper asymptotically Lipschitz map defined on a proper closed subset A of a proper metric space X. There exists a neighborhood W of A in X, a proper asymptotically Lipschitz map $g\colon W \to \mathcal{O}L$ with the following property: there exist constants $\lambda, s > 0$ such that $\alpha_L(g(a)) \leq \lambda d(a, X\setminus W) + s$.*

Proof. We may assume that L is a subset of I^n for some n and there exists a Lipschitz retraction $r\colon U \to L$ of a neighborhood U of L in I^n. Since $\mathcal{O}I^n$ is Lipschitz equivalent to $\mathbb{R}^{n+1}_+$, there exists a (λ', s')-Lipschitz extension $\tilde{g}\colon X \to \mathcal{O}I^n$ of g.

Put $W = \tilde{g}^{-1}(\mathcal{O}U)$ and $\bar{g} = \tilde{g}|W$. For every $a \in A$ and $w \in X \setminus W$ we have

$$d(g(a), \tilde{g}(w) \leq \lambda' d(a,w) + s' \leq \lambda' d(a, X\setminus W) + s.$$

Suppose that $d(L, I^n \setminus U) = c > 0$, then, since $\tilde{g}(w) \notin CU$,

$$d(g(a),\tilde{g}(w)) = |\alpha_L(g(a)) - \alpha_L(\tilde{g}(w))| +$$

$$+ \min\{\alpha_L(g(a)), \alpha_L(\tilde{g}(w))\} d(\beta_L(g(a)), \beta_L(\tilde{g}(w))) \geq$$

$$\geq |\alpha_L(g(a)) - \alpha_L(\tilde{g}(w))| + c\min\{\alpha_L(g(a)), \alpha_L(\tilde{g}(w))\} \geq c'\alpha_L(g(a)),$$

where $c' = \min\{c, 1\}$. Then $\alpha_L(g(a)) \leq \lambda d(a, X\setminus W) + s$, where $\lambda = \lambda'/c'$, $s = s'/c'$.

Proposition 2.2 is proved.

3. Auxiliary results. In this section we collect some results needed in the proof of the main result. They are proved in [3] but it turned out that we have also cover the case of functions with infinite values.

A map $f\colon X \to \mathbb{R}_+ \cup \{\infty\}$ is said to be *coarsely proper* if the preimage $f^{-1}([0,c])$ is bounded for every $c \in \mathbb{R}_+$.

Lemma 3.1. *For any function $\varphi\colon X \to \mathbb{R}_+$ with $\varphi(x) \to 0$ as $x \to \infty$ the function $1/\varphi\colon X \to \mathbb{R}_+ \cup \{\infty\}$ is coarsely proper.*

Proposition 3.1. *Let $f\colon X \to \mathbb{R}_+ \cup \{\infty\}$ be a coarsely proper function. There exists an asymptotically Lipschitz proper function $q\colon X \to \mathbb{R}_+$ with $q \leq f$.*

Proof. This was proved in [3] for the case of $f\colon X \to \mathbb{R}_+$ (see Proposition 3.5). This proof also works for our case.

Proposition 3.2. *Let $f_n\colon X \to \mathbb{R}_+ \cup \{\infty\}$ be a sequence of coarsely proper functions. Then there exists a filtration $X = \cup_{n=1}^{\infty} A_n$ and a coarsely proper function $f\colon X \to \mathbb{R}_+$ with $f|A_n \leq n$ and $f|(X\setminus A_n) \leq f_n$ for every n.*

Proof. Let $B_n = \bigcup_{i=1}^{n} f_i^{-1}([0,n])$. The sets B_i are bounded and $B_1 \subset B_2 \subset \ldots$. Therefore, there exist bounded subsets $A_1 \subset A_2 \subset \ldots$ such that $A_n \cap \left(\bigcup_{i=1}^{\infty} B_i\right) = B_n$

and $\bigcup_{i=1}^{\infty} A_i = X$. For $x \in A_n \setminus A_{n-1}$, put $f(x) = n$. Obviously, f is coarsely proper and $f|A_n \leq n$. Now suppose that $x \notin A_n$, then $x \notin B_n$ and therefore $x \notin f_n^{-1}([0,n])$, i.e., $f_n(x) > n \geq f|(X \setminus A_n)$.

Proposition 3.2 is proved.

The following is an easy modification of Lemma 3.6 from [3] and the proof of it works in our case as well.

Lemma 3.2. *Suppose that $f\colon A \to \mathbb{R}_+ \cup \{\infty\}$ is a coarsely proper map defined on a closed subset A of a proper metric space X and $g\colon W \to \mathbb{R}_+$ is a proper asymptotically Lipschitz map such that $g \leq f|W$ and there exist λ, s such that $\lambda d(a, X \setminus W) + s \geq g(a)$ for every $a \in A$. Then there exists a proper asymptotically Lipschitz map $\bar{g}\colon X \to \mathbb{R}_+$ for which $\bar{g} \leq f$ and $\bar{g}|A = g$.*

3.1. Almost geodesic spaces. A metric space X is said to be *almost geodesic* if there exists $C > 0$ such that for every two points $x, y \in X$ there is a short map $f\colon [0, Cd(x,y)] \to X$ with $f(0) = x$, $f(Cd(x,y)) = y$. If in this definition $C = 1$, then we come to the well-known notion of *geodesic* space.

We are going to describe a construction of embedding of a discrete metric space X into an almost geodesic space of the asymptotic dimension $\min\{\mathrm{asdim} X, 1\}$.

For an unbounded discrete metric space X with base point x_0 define a function $f\colon X \to [0,\infty)$ by the formula $f(x) = d(x, x_0)$. Choose a sequence $0 = t_0 < t_1 < t_2 < \ldots$ in $f(X)$ so that $t_{i+1} > 2t_i$ for every i. To every pair of points $x, y \in f^{-1}([t_i, t_{i+1}])$, for some i, attach the line segment $[0, d(x,y)]$ along its endpoints. Let $\hat{X}$ be the union of X and all attached segments. We endow $\hat{X}$ with the maximal metric that agrees with the initial metric on X and the standard metric on every attached segment.

Note that since X is discrete and proper, every set $f^{-1}([t_i, t_{i+1}])$ is finite and therefore $\hat{X}$ is a proper metric space.

Proposition 3.3. *The space $\hat{X}$ is almost geodesic.*

Proof. Suppose that $x, y \in \hat{X}$, then $x \in [x_1, x_2]$, $y \in [y_1, y_2]$, where $x_1, x_2, y_1, y_2 \in X$ and $[x_1, x_2]$, $[y_1, y_2]$ are attached segments. We may suppose that $d(x,y) = d(x, x_1) + d(x_1, y_1) + d(y_1, y)$.

Case 1: There exists i such that $x_1, y_1 \in f^{-1}([t_i, t_{i+1}])$. Then $[x, x_1] \cup [x_1, y_1] \cup [y_1, y]$ is a segment of diameter $d(x,y)$ that connects x and y in $\hat{X}$.

Case 2: $f(x_1) \in [t_i, t_{i+1}]$, $f(y_1) \in [t_j, t_{j+1}]$, where $i \neq j$. Without loss of generality, we may assume that $i < j$.

Obviously, $d(x_1, y_1) \leq d(x,y)$. Since $|t_j - t_{j-1}| \leq d(x_1, y_1)$, we see that $|t_j - t_{j-1}| \leq d(x,y)$. This implies that $t_j/2 \leq d(x,y)$, or equivalently, $t_j \leq d(x,y)$.

Besides, $d(y_1, f^{-1}([0, t_{j-1}]))) \leq d(x_1, y_1) \leq d(a,b)$.

For every $k = i, i+1, \ldots, j_1$ choose $z_k \in f^{-1}(t_k)$. Then

$$d(y_1, z_{j-1}) \leq d(y_1, f^{-1}([0, t_{j-1}]) + \mathrm{diam}\ (f^{-1}([0, t_{j-1}])) \leq$$
$$\leq d(a,b) + 2t_{j-1} \leq d(a,b) + t_j \leq 3d(a,b).$$

We connect x and y by the segment

$$J = [x, x_1] \cup [x_1, z_1] \cup \bigcup_{k=i}^{j-1} [z_k, z_{k+1}]) \cup [z_{j-1}, y_1] \cup [y_1, y].$$

Then

$$\operatorname{diam} J \le d(x, x_1) + d(x_1, z_{i+1}) + \left(\sum_{k=i+1}^{j-1} d(z_k, z_{k+1})\right) + d(z_{j-1}, y_1) + d(y_1, y) =$$

$$= d(x, y) + 2t_{i+1} + \sum_{k=i+1}^{j-1} 2t_{k+1} + 5d(x, y) + d(x, y) \le$$

$$\le 7d(x, y) + 2(t_{i+1} + \ldots + t_j) \le 7d(x, y) + 4t_j \le 15d(x, y).$$

Proposition 3.3 is proved.

We need a version of the fact proved in [3] for geodesic spaces.

Proposition 3.4. *Let $f\colon X \to Y$ be a coarse uniform map of an almost geodesic space X. Then f is asymptotically Lipschitz.*

Proof. Let C be a constant from the definition of almost geodesic space. Suppose $x, y \in X$, then there exists a short map $\alpha\colon [0, Cd(x, y)] \to X$ such that $\alpha(0) = x$, $\alpha(Cd(x, y)) = y$. There exist points $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = Cd(x, y)$, where $k \le [d(x, y)] + 1$, such that $|t_i - t_{i-1}| \le C$ for every $i = 1, \ldots, k$.

Since f is coarse uniform, there exists $R > 0$ such that $d(f(x'), f(y')) < R$ whenever $d(x', y') \le C$. Then

$$d(f(x), f(y)) \le \sum_{i=1}^{k} d(f(\alpha(t_i)), f(\alpha(t_{i-1}))) \le$$

$$\le kR \le ([d(x, y)] + 1)R \le Rd(x, y) + 2R.$$

Proposition 3.4 is proved.

4. Asymptotic extension dimension. Let P be an object of the category $\mathcal{A}$. For any object X of $\mathcal{A}$ the *Kuratowski notation* $X\tau P$ means the following: for every proper asymptotically Lipschitz map $f\colon A \to P$ defined on a closed subset A of X there is a proper asymptotically Lipschitz extension of f onto X.

Denote by $\mathcal{L}$ the class of compact absolute Lipschitz neighborhood Euclidean extensors (ALNER). Following [4], we define a preorder relation $\le$ on $\mathcal{L}$. For $L_1, L_2 \in \mathcal{L}$, we have $L_1 \le L_2$ if and only if $X\tau\mathcal{O}L_1$ implies $X\tau\mathcal{O}L_2$ for all proper metric spaces X. This preorder relation leads to the following equivalence relation $\sim$ on $\mathcal{L}$: $L_1 \sim L_2$ if and only if $L_1 \le L_2$ and $L_2 \le L_1$. We denote by $[L]$ the equivalence class containing $L \in \mathcal{L}$. The class $[L]$ is called the *asymptotic extension dimension type* of L. The mentioned preorder relation induces the partial order relation on all the asymptotic extension dimension types.

For a proper metric space X, we say that its *asymptotic extension dimension does not exceed* $[\mathcal{O}L]$ (briefly as-ext-$\dim X \le [\mathcal{O}L]$ whenever $X\tau\mathcal{O}L$.

If as-ext-$\dim X \le [\mathcal{O}L]$, then the equality as-ext-$\dim X = [\mathcal{O}L]$ means the following. If we also have as-ext-$\dim X \le [\mathcal{O}L']$, then $[\mathcal{O}L] \le [\mathcal{O}L']$.

From the result on extension of asymptotically Lipschitz functions ([3]; see also [6]), the element $[*]$ is maximal.

Theorem 4.1. *Let L be a compact metric ALNER. The following conditions are equivalent*:

1) as-ext-$\dim X \le [\mathcal{O}L]$;
2) ext-$\dim \nu X \le [L]$.

Proof. 1) $\Rightarrow$ 2). Assume that as-ext-dim $X \leq [\mathcal{O}L]$. Let $\varphi \colon C \to L$ be a map defined on a closed subset C of νX. Since $L \in \mathrm{ANE}$, there exists an extension $\varphi' \colon V \to L$ of φ over a closed neighborhood V of C in $\bar{X} = X \cup \nu X$. Then $\mathrm{Var}_R \varphi'(x) \to 0$ as $x \to \infty$, for any fixed $R > 0$. By Lemma 3.1, the function

$$f_n \colon V \cap X \to \mathbb{R}_+ \cup \{\infty\}, \quad f_n(x) = \frac{1}{\mathrm{Var}_R \varphi'(x)},$$

is coarsely proper, for every $n \in \mathbb{N}$. By Proposition 3.2, there is a coarsely proper function $f \colon V \cap X \to \mathbb{R}_+$ and a filtration $V \cap X = \bigcup_{n=1}^{\infty} A_n$ such that $f|A_n \leq n$ and $f|(X \setminus A_n) \leq f_n$. By Proposition 3.5 from [3], there is an asymptotically Lipschitz function $q \colon V \cap X \to \mathbb{R}_+$ with $q \leq f$. We suppose that q is (λ, s)-Lipschitz for some $\lambda, s > 0$. Define the map $g \colon V \cap X \to \mathcal{O}L$ by the formula $g(x) = [\varphi'(x), q(x)]$.

We are going to check that the map $g(x)$ is asymptotically Lipschitz. Let $x, y \in \in V \cap X$ and $n - 1 \leq d(x, y) \leq n$.

Suppose that $x, y \in (V \cap X) \setminus A_n$, then $q(x) \leq f_n(x)$, $q(y) \leq f_n(y)$. We have

$$d(g(x), g(y)) = |q(x) - q(y)| + \min\{q(x), q(y)\} d(\varphi'(x), \varphi'(y)) \leq$$

$$\leq \lambda d(x, y) + s + \min\{q(x), q(y)\} \mathrm{Var}_n \varphi'(x) \leq \lambda d(x, y) + s + 1.$$

If $x \in A_n$, then $q(x) \leq n$ and we obtain

$$d(g(x), g(y)) \leq \lambda d(x, y) + s + n d(\varphi'(x), \varphi'(y)) \leq$$

$$\leq \lambda d(x, y) + s + n \mathrm{diam}\, L \leq \lambda d(x, y) + s + (d(x, y) + 1) \mathrm{diam}\, L \leq$$

$$\leq (\lambda + \mathrm{diam}\, L) d(x, y) + (s + \mathrm{diam}\, L).$$

We argue similarly if $y \in A_n$.

Now, by the assumption, there is an asymptotically Lipschitz extension $\bar{g} \colon X \to \to \mathcal{O}L$ of g. Consider the composition $\eta_L \nu \bar{g} \colon \nu X \to \mathcal{O}L$. Obviously, $\eta_L \nu \bar{g}|C = \varphi$. We conclude that ext-dim$\nu X \leq [L]$.

2) $\Rightarrow$ 1). Let $f \colon A \to \mathcal{O}L$ be an asymptotically Lipschitz map defined on a proper closed subset A of a proper metric space X. By Proposition 2.2, there is a proper asymptotically Lipschitz map $\tilde{f} \colon W \to \mathcal{O}L$ defined on a neighborhood W of A and constants λ, s such that $\alpha_L f(a) \leq \lambda d(a, X \setminus W) + s$ for all $a \in A$. Denote by $\varphi \colon \nu X \to \to L$ an extension of the composition $\eta_L \nu \tilde{f}$. Since L is an absolute neighborhood extensor, there exists an extension $\psi \colon V \to L$ of φ onto a closed neighborhood of νX in the Higson compactification $\bar{X}$. Extend ψ to a map $\hat{\psi} \colon (V \cap X)\hat{} \to L$ as follows. Let J be a segment attached to V with endpoints a and b. We require that $\hat{\psi}$ linearly maps J onto a geodesic segment in L with endpoints $\psi(a)$ and $\psi(b)$.

Show that $\hat{\psi}$ is a slowly oscillating map. Since ψ is slowly oscillating, for every $\varepsilon > 0$ and $R > 0$ there exists $K > 0$ such that $\mathrm{Var}_R \psi(x) < \varepsilon$ whenever $d(x, x_0) > > K$. Suppose that $\hat{\psi}$ is not slowly oscillating, then there exist $R > 0$, $C > 0$, and sequences (x_1^i), (x_2^i) in $(V \cap X)\hat{}$ such that $d(x_1^i, x_2^i) < R$, $x_1^i \to \infty$, $x_2^i \to \infty$ and $d(\hat{\psi}(x_1^i), \hat{\psi}(x_2^i)) > C$ for every i. We assume that $x_1^i \in [a_1^i, b_1^i]$, $x_2^i \in [a_2^i, b_2^i]$, for every i, where $a_1^i, b_1^i, a_2^i, b_2^i \in X \cap V$. Without loss of generality we may assume that $a_1^i \to \infty$ and there exists $C_1 > 0$ such that $d(\hat{\psi}(x_1^i), \hat{\psi}(a_1^i)) > C_1$ for every i. If $d(a_1^i, b_1^i) < K$ for all i and some $K > 0$, then $d(\hat{\psi}(x_1^i), \hat{\psi}(a_1^i)) < d(\hat{\psi}(a_1^i), \hat{\psi}(b_1^i)) \to 0$, and we obtain a contradiction. Therefore, we may assume that $d(a_1^i, b_1^i) \to \infty$. Then $d(a_1^i, x_1^i)/d(a_1^i, b_1^i) < R/d(a_1^i, b_1^i) \to 0$ and therefore, by the definition of the map $\hat{\psi}$, $d(\hat{\psi}(x_1^i), \hat{\psi}(a_1^i))/d(\hat{\psi}(a_1^i), \hat{\psi}(b_1^i)) \to 0$. Then obviously $d(\hat{\psi}(x_1^i), \hat{\psi}(a_1^i)) \to 0$ and we obtain a contradiction.

Since the map $\tilde{f}$ is asymptotically Lipschitz, there exists $K > 0$ such that for any $a \in W$ we have

$$\operatorname{diam}\,(\alpha_L \tilde{f}(N_1(a)) + \alpha_L \tilde{f}(a)\operatorname{diam}\,(\psi(N_1(a)) \leq K.$$

Define the function $r\colon (X \cap V)\check{} \to \mathbb{R}_+ \cup \{\infty\}$ by the formula $r(x) = K/(\psi(N_1(x)))$. We have $f(a) \leq r(a)$ for every $a \in A$. The function r is asymptotically proper and by Proposition 3.1, there exists a (λ', s')-Lipschitz function $\bar{f}\colon X \to \mathbb{R}_+$, for some λ', s', with $\bar{f} \leq r$ and $\bar{f}|A = \alpha_L f$.

Define a map $g\colon (X \cap V)\check{} \to \mathcal{O}L$ by the formula $g(x) = (\psi(x), \bar{f}(x))$. Obviously, $g|A = f$. We are going to show that g is a coarse uniform map.

Suppose $x, y \in X$, $d(x, y) < 1$, then

$$d(g(x), g(y)) \leq |\bar{f}(x) - \bar{f}(y)| + \min\{\bar{f}(x), \bar{f}(y)\}d(\psi(x), \psi(y)) \leq \lambda' + s' + K.$$

Note that, since $\bar{f}$ is proper, g is also proper. Since g is coarse uniform, by Proposition 3.4, g is asymptotically Lipschitz. Therefore, as-ext-$\dim X \leq [\mathcal{O}L]$.

Theorem 4.1 is proved.

***Corollary* 4.1** (Finite Sum Theorem)**.** *Suppose X is a proper metric space, $X = X_1 \cup X_2$, where X_1, X_2 are closed subsets of X with* as-ext-$\dim X_i \leq [\mathcal{O}L]$, $i = 1, 2$, *for some $L \in \mathcal{L}$. Then* as-ext-$\dim X \leq [\mathcal{O}L]$.

Proof. Since $\nu X = \nu X_1 \cup \nu X_2$, the result follows from Theorem 4.1 and the finite sum theorem for extension dimension (see [7]).

5. Remarks and open questions.

***Question* 5.1.** Does the equality as-ext-$\dim \mathbb{R}^n = S^n$ hold?

***Question* 5.2.** Let L_1, L_2 be finite polyhedra in euclidean spaces endowed with the induced metric. Is the inequality $[L_1] \leq [L_2]$ introduced in [4] equivalent to the inequality $[L_1] \leq [L_2]$ as in Section 4?

One can define a counterpart of the asymptotic extension dimension by using warped cones instead of open cones. Following [8] we review this construction briefly. Let $\mathcal{F}$ be a foliation on a compact smooth manifold V. Let N be any complementary subbundle to $T\mathcal{F}$ in TM. Choose Euclidean metrics g_N in N and $g_{\mathcal{F}}$ in $T\mathcal{F}$. The *foliated warped cone* $\mathcal{O}_{\mathcal{F}}$ is the manifold $V \times [0, \infty)/V \times \{0\}$ equipped with the metric induced for $t \geq 1$ by the Riemannian metric $g_R + g_{\mathcal{F}} + t^2 g_N$. Since we are interested by the asymptotic properties of the warped cones, the metric structure on any bounded neighborhood of the vertex of the cone is irrelevant.

***Question* 5.3.** Is the obtained warped cone an absolute neighborhood extensor in the asymptotic category?

An affirmative answer to this question would allow us to define asymptotic extension dimension theory with the values in warped cones.

***Question* 5.4.** Can one characterize the dimension of the sublinear corona (see [9]) in terms of the asymptotic extension dimension?

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