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ON UNIQUENESS THEOREMS FOR HOLOMORPHIC CURVES SHARING HYPERSURFACES WITHOUT COUNTING MULTIPLICITY*

ПРО ТЕОРЕМИ ЄДИНОСТІ ДЛЯ ГОЛОМОРФНИХ КРИВИХ, ЩО РОЗДІЛЯЮТЬ ГІПЕРПЛОЩИНИ БЕЗ ВРАХУВАННЯ КРАТНОСТІ

We prove some uniqueness theorems for algebraically nondegenerate holomorphic curves sharing hypersurfaces ignoring multiplicity.

Доведено деякі теореми єдиності для алгебраїчно невироджених голоморфних кривих, що розділяють гіперплощини без врахування кратності.

1. Introduction. In 1926, R. Nevanlinna proved that two non-constant meromorphic maps $f, g\colon \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ satisfying $f^{-1}(a_j) = g^{-1}(a_j)$, for $a_1, \ldots, a_5 \in \mathbb{P}^1(\mathbb{C})$ distinct, we must have $f \equiv g$. In 1975, H. Fujimoto [5] generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Since that time, this problem has been studied intensively. In this paper we give some uniqueness results for algebraically nondegenerate holomorphic curves sharing sufficiently many nonlinear hypersurfaces in a projective space. To state our results, we first introduce some notations.

Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and Q be the homogeneous polynomial of degree d in $n+1$ variables with coefficients in $\mathbb{C}$ defining D, we define

$$\bar{E}_f(D) := \left\{ z \in \mathbb{C} \,\middle|\, Q \circ f(z) = 0 \quad \text{ignoring multiplicity} \right\},$$

$$E_f(D) := \left\{ (z, m) \in \mathbb{C} \times \mathbb{N} \,\middle|\, Q \circ f(z) = 0 \quad \text{and} \quad \operatorname{ord}_{Q \circ f}(z) = m \right\},$$

where $\operatorname{ord}_\gamma(z)$ is the order of holomorphic functions γ at z. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of hypersurfaces, we define

$$\bar{E}_f(\mathcal{D}) := \bigcup_{D \in \mathcal{D}} \bar{E}_f(D) \qquad \text{and} \qquad E_f(\mathcal{D}) := \bigcup_{D \in \mathcal{D}} E_f(D).$$

In 1975, H. Fujimoto [5] proved the following theorem.

Theorem A. *Let $\mathcal{H} = \{H_1, \ldots, H_{3n+2}\}$ be a collection of $3n+2$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$, and $f, g\colon \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be meromorphic maps such that $f(\mathbb{C}^m) \not\subset H$ and $g(\mathbb{C}^m) \not\subset H$ for any $H \in \mathcal{H}$. If*

$$E_f(H_j) = E_g(H_j) \qquad \textit{for any} \qquad H_j \in \mathcal{H}$$

then $f \equiv g$.

By Theorem A, the linearly nondegenerate meromorphic maps are uniquely determined by $3n+2$ hyperplanes in general position. In the last years, many uniqueness theorems for holomorphic maps with hyperplanes have been established. For the case hypersurfaces, in 2008, by using the second main theorem with ramification of An-Phuong [1], H. T. Phuong [7], M. Dulock and M. Ru [4] proved some uniqueness

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theorems for algebraically nondegenerate holomorphic curves. Recently, G. Dethloff and T. V. Tan [3] proved one uniqueness theorem for meromorphic maps in the case moving hypersurfaces. Our contribution is to give some unicity results for algedraically nondegenerate holomorphic curves sharing sufficiently many hypersurfaces in general position for Veronese embedding.

Now let D_j be a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$, which is defined by a homogeneous polynomial Q_j of degree d. Then

$$Q_j(z_0, \ldots, z_n) = \sum_{k=0}^{n_d} a_k z_0^{i_{k0}} \ldots z_n^{i_{kn}},$$

where $i_{k0} + \ldots + i_{kn} = d$ for $k = 0, 1, \ldots, n_d$ and $n_d = \binom{n+d}{n} - 1$. We denote by $\mathbf{a} = (a_0, \ldots, a_{n_d})$ the vector associated with D_j (or with Q_j).

Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of arbitrary hypersurfaces and Q_j be the homogeneous polynomial in $\mathbb{C}|z_0, \ldots, z_n|$ of degree d_j defining D_j for $j = 1, \ldots, q$. We define the *minimal index of degrees of* $\mathcal{D}$ by

$$\delta_{\mathcal{D}} := \min\{d_1, \ldots, d_q\}.$$

Let $m_{\mathcal{D}}$ be the least common multiple of the d_j for $j = 1, \ldots, q$ and denote

$$n_{\mathcal{D}} = \binom{n + m_{\mathcal{D}}}{n} - 1.$$

For $j = 1, \ldots, q$, we set $Q_j^* = Q_j^{m_{\mathcal{D}}/d_j}$ and let $\mathbf{a}_j^*$ be the vector associated with Q_j^*. The collection $\mathcal{D}$ is said to be in *general position for Veronese embedding* if $q > n_{\mathcal{D}}$ and for any distinct $i_1, \ldots, i_{n_{\mathcal{D}}+1} \in \{1, \ldots, q\}$, the vectors $\mathbf{a}_{i_1}^*, \ldots, \mathbf{a}_{i_{n_{\mathcal{D}}+1}}^*$ are linearly independent.

Recall that the collection $\mathcal{D} = \{D_1, \ldots, D_q\}$ is said to be in N-subgeneral position if $q > N$ and for any distinct $i_1, \ldots, i_{N+1} \in \{1, \ldots, q\}$

$$\bigcap_{k=1}^{N+1} D_{i_k} = \varnothing,$$

where N be a positive integer such that $N \geq n$. It is seen that, for hyperplanes, general position for Veronese embedding is equivalent to the usual concept of hyperplanes in general position (namely n-subgeneral position). For hypersurfaces, general position for Veronese embedding implies $n_{\mathcal{D}}$-subgeneral position.

The following results we obtained in this paper.

Theorem 1.1. *Let f and g be algebraically nondegenerate holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q \geq n_{\mathcal{D}} + 2 + 2n_{\mathcal{D}}^2/\delta_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.*

Theorem 1.2. *Let f and g be algebraically nondegenerate holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q \geq n_{\mathcal{D}} + 2 +$*

$+\ 2n_{\mathcal{D}}/\delta_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that

(a) $f(z) = g(z)$ *for all* $z \in \bar{E}_f(\mathcal{D}) \bigcup \bar{E}_g(\mathcal{D})$,

(b) $\bar{E}_f(D_i) \bigcap \bar{E}_f(D_j) = \varnothing$ *and* $\bar{E}_g(D_i) \bigcap \bar{E}_g(D_j) = \varnothing$ *for all* $i \neq j \in \in \{1, \ldots, q\}$.

Then $f \equiv g$.

Theorems 1.1 and 1.2 are uniqueness theorems for algebraically nondegenerate meromorphic curves in the case hypersurface, they have shown the sufficient conditions for two algebraically nondegenerate meromorphic curves being equivalent. Note that, if $n = 1$, hypersurfaces are distinct points, Theorem 1.2 becomes Nevanlinna's five theorem. And in Theorem 1.2 too, if $m_{\mathcal{D}} = 1$, $\delta_{\mathcal{D}} = 1$, $n_{\mathcal{D}} = n$ and the number of hyperplanes is $3n + 2$ as in Fujimoto's result. Furthermore, the number of hypersurfaces in our results is smaller that in Dulock – Ru's result.

2. Preliminaries from Nevanlinna – Cartan theory. In this section, we introduce some notations in Nevanlinna – Cartan theory and recall some results, which are necessary for proofs of our main results.

Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map and $f = (f_0, \ldots, f_n)$ be a reduced representative of f. The Nevanlinna – Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \left\| f\left(re^{i\theta}\right) \right\| d\theta,$$

where $\| f(z) \| = \max\{| f_0(z)|, \ldots, | f_n(z)|\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. Let Q be the homogeneous polynomial of degree d defining D. The proximity function of f is defined by

$$m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\left\| f\left(re^{i\theta}\right) \right\|^d}{\left| Q \circ f\left(re^{i\theta}\right) \right|} d\theta.$$

Let $n_f(r, D)$ be the number of zeros of $Q \circ f$ in the disk $|z| \leq r$, counting multiplicity. For any positive integer k, let $n_f(r, D, \leq k)$ be the number of zeros having multiplicity $\leq k$ of $Q \circ f$ in the disk $|z| \leq r$, counting multiplicity and let $n_f(r, D, > k)$ be the number of zeros having multiplicity $> k$ of $Q \circ f$ in the disk $|z| \leq r$, counting multiplicity. The integrated counting functions are defined by

$$N_f(r, D) = \int_0^r \frac{n_f(t, D) - n_f(0, D)}{t} dt + n_f(0, D) \log r,$$

$$N_{f,\leq k}(r, D) = \int_0^r \frac{n_f(t, D, \leq k) - n_f(0, D, \leq k)}{t} dt + n_f(0, D, \leq k) \log r,$$

$$N_{f,>k}(r, D) = \int_0^r \frac{n_f(t, D, > k) - n_f(0, D, > k)}{t} dt + n_f(0, D, > k) \log r.$$

For any positive integers Δ, k, let $n_f^{\Delta}(r, D)$ be the number of zeros of $Q \circ f$ in the disk $|z| \le r$, where any zero is counted with multiplicity if its multiplicity is less than or equal to Δ, and Δ times otherwise. Let $n_f^{\Delta}(r, D, \le k)$ (resp. $n_f^{\Delta}(r, D, > k)$) be the number of zeros having multiplicity $\le k$ (resp. $> k$) of $Q \circ f$ in the disk $|z| \le r$, where any zero is counted if its multiplicity is less than or equal to Δ, and Δ times otherwise, too. The integrates truncated counting functions are defined by

$$N_f^{\Delta}(r, D) = \int_0^r \frac{n_f^{\Delta}(t, D) - n_f^{\Delta}(0, D)}{t} dt + n_f^{\Delta}(0, D) \log r ,$$

$$N_{f,\le k}^{\Delta}(r, D) = \int_0^r \frac{n_f^{\Delta}(t, D, \le k) - n_f^{\Delta}(0, D, \le k)}{t} dt + n_f^{\Delta}(0, D, \le k) \log r ,$$

$$N_{f,>k}^{\Delta}(r, D) = \int_0^r \frac{n_f^{\Delta}(t, D, > k) - n_f^{\Delta}(0, D, > k)}{t} dt + n_f^{\Delta}(0, D, > k) \log r .$$

Next, we will recall the following lemma which show properties of integrated counting functions. The proof can be found in [7].

Lemma 2.1. *With the above notations we have*

1) $N_f(r, D) = N_{f,\le k}(r, D) + N_{f,>k}(r, D)$,

2) $N_f^{\Delta}(r, D) = N_{f,\le k}^{\Delta}(r, D) + N_{f,>k}^{\Delta}(r, D)$,

3) $N_f^{\Delta}(r, D) \le N_f(r, D)$,

4) $N_f^1(r, D) \le N_f^{\Delta}(r, D) \le \Delta N_f^1(r, D)$,

5) $N_{f,\le k}^1(r, D) \le N_{f,\le k}^{\Delta}(r, D) \le \Delta N_{f,\le k}^1(r, D)$,

6) $N_{f,>k}^1(r, D) \le N_{f,>k}^{\Delta}(r, D) \le \Delta N_{f,>k}^1(r, D)$,

7) $\dfrac{1}{k+1} N_{f,\le k}^{\Delta}(r, D) + N_{f,>k}^{\Delta}(r, D) \le \dfrac{\Delta}{k+1} N_f(r, D)$.

First main theorem. *Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. If $f(\mathbb{C}) \not\subset D$, then for every real number r with* $0 < r < \infty$,

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

In 1933, H. Cartan [2] proved the following theorem.

Theorem 2.1. *Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic map, and let H_j, $1 \le j \le q$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then*

$$\big(q - (n+1) - \varepsilon\big) T_f(r) \le \sum_{j=1}^{q} N_f^n(r, H_j) + S_f(r),$$

where $S_f(r) = o\big(T_f(r)\big)$ and inequality holds for all large r outside a set of finite Lebesgue measure.

3. Proofs of Theorems 1.1 and 1.2. To prove our theorems we need the following lemma.

Lemma 3.1. *Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically nondegenerate holomorphic* *map and* $D_1, \ldots, D_q$ *be hypersurfaces in* $\mathbb{P}^n(\mathbb{C})$ *of degree* $d_j, j = 1, \ldots, q$, *such that the collection* $\mathcal{D} = \{D_1, \ldots, D_q\}$ *is in general position for Veronese embedding in* $\mathbb{P}^n(\mathbb{C})$. *Then for every* $\varepsilon > 0$,

$$(q - n_{\mathcal{D}} - 1 - \varepsilon) T_f(r) \le \sum_{j=1}^{q} \frac{1}{d_j} N_f^{n_{\mathcal{D}}}(r, D_j) + S_f(r),$$

where inequality holds for all large r *outside a set of finite Lebesgue measure.*

Proof. Let $f = (f_0, \ldots, f_n)$ be a reduced representative of f, where $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ without common zeros, and $Q_j, j \le 1 \le q$, be the homogeneous polynomials in $\mathbb{C}[z_0, \ldots, z_n]$ of degree d_j defining D_j. Of course we may assume that $q \ge n_{\mathcal{D}} + 1$.

We first claim that it suffices to prove the lemma in the case that all of the d_j are equal to $m_{\mathcal{D}}$. Indeed, if we have the lemma in that case, then we know that for $\varepsilon > 0$ as in statement of the lemma that

$$(q - n_{\mathcal{D}} - 1 - \varepsilon) T_f(r) \le \sum_{j=1}^{q} \frac{1}{m_{\mathcal{D}}} N_f^{n_{\mathcal{D}}}(r, Q_j^{m_{\mathcal{D}}/d_j}) + S_f(r).$$

Note that if $z \in \mathbb{C}$ is the zero of $Q_j \circ f$ with multiplicity a then z is the zero of $Q_j^{m_{\mathcal{D}}/d_j} \circ f$ with multiplicity $a \frac{m_{\mathcal{D}}}{d_j}$. This implies that

$$N_f^{n_{\mathcal{D}}}\left(r, Q_j^{m_{\mathcal{D}}/d_j}\right) \le \frac{m_{\mathcal{D}}}{d_j} N_f^{n_{\mathcal{D}}}(r, Q_j).$$

Hence

$$(q - n_{\mathcal{D}} - 1 - \varepsilon) T_f(r) \le \sum_{j=1}^{q} \frac{1}{m_{\mathcal{D}}} N_f^{n_{\mathcal{D}}}(r, Q_j^{m_{\mathcal{D}}/d_j}) + S_f(r) \le$$

$$\le \sum_{j=1}^{q} \frac{1}{d_j} N_f^{n_{\mathcal{D}}}(r, Q_j) + S_f(r).$$

Therefore, without loss of generality, we can assume that $D_1, \ldots, D_q$ have a same degree of $m_{\mathcal{D}}$.

We recall the *lexicographic ordering* on the n-tuples of natural numbers: let $J = \{j_1, \ldots, j_n\}$, $I = \{i_1, \ldots, i_n\} \in \mathbb{N}^n$, $J < I$ if and only if for some $b \in \{1, \ldots, n\}$ we have $j_l = i_l$ for $l < b$ and $j_b < i_b$. With the n-tuples $I = \{i_1, \ldots, i_n\}$ of non-negative integers, we denote $\sigma(I) := \sum_j i_j$.

Let $(z_0 : \ldots : z_n)$ be a homogeneous coordinates in $\mathbb{P}^n(\mathbb{C})$ and let $\{I_0, \ldots, I_{n_{\mathcal{D}}}\}$ be a set of $(n+1)$-tuples such that $\sigma(I_j) = m_{\mathcal{D}}, j = 0, \ldots, n_{\mathcal{D}}$, and $I_i < I_j$ for $i < j \in \in \{0, \ldots, n_{\mathcal{D}}\}$. For $\mathbf{z} = (z_0 : \ldots : z_n) \in \mathbb{P}^n(\mathbb{C})$, we write $\mathbf{z}^I$ as $z_0^{i_0} \ldots z_n^{i_n}$ where $I = \{i_0, \ldots, i_n\} \in \{I_0, \ldots, I_{n_{\mathcal{D}}}\}$. Then we may write the set of monomials of degree $m_{\mathcal{D}}$ as $\{\mathbf{z}^{I_0}, \ldots, \mathbf{z}^{I_{n_{\mathcal{D}}}}\}$.

Denote

$$\varrho_{m_{\mathcal{D}}} : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$$

be the *Veronese embedding of degree* $m_{\mathcal{D}}$. Let $(w_0 : \ldots : w_{n_{\mathcal{D}}})$ be a homogeneous coordinate in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$. Then ϱ_d is given by

$$\varrho_{m_{\mathcal{D}}}(\mathbf{z}) = \big(w_0(\mathbf{z}) : \ldots : w_{m_{\mathcal{D}}}(\mathbf{z})\big), \quad \text{where} \quad w_j(\mathbf{z}) = \mathbf{z}^{I_j}, j = 0, \ldots, n_{\mathcal{D}}.$$

Now we set

$$F = (F_0 : \ldots : F_{n_{\mathcal{D}}}) = \varrho_{m_{\mathcal{D}}} \circ f,$$

then $F_j = \mathbf{f}^{I_j}, j = 0, \ldots, n_{\mathcal{D}}$. Then F is a holomorphic map from $\mathbb{C}$ to $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$ and $\mathbf{F} = (F_0, \ldots, F_{n_{\mathcal{D}}})$ is a reduced representative of F. We know from the assumption that f is algebraically nondegenerate hence F is linearly nondegenerate.

For any hypersurface $D_j \in (D_1, \ldots, D_q)$, let $\mathbf{a}_j = (a_{j0}, \ldots, a_{jn_{\mathcal{D}}})$ be the vector associated with D_j, we set

$$L_j = a_{j0}w_0 + \ldots + a_{in_{\mathcal{D}}}w_{n_{\mathcal{D}}}.$$

Then L_j is a linear form in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$. Let H_j be a hyperplane in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$, which is made by L_j, we say that the hyperplane H_j associated with D_j. Hence for the collection of hypersurfaces $(D_1, \ldots, D_q)$ in $\mathbb{P}^n(\mathbb{C})$, we have the collection of hyperplanes $(H_1, \ldots, H_q)$ of associate hyperplanes in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$. By the assumption that $(D_1, \ldots, D_q)$ is in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$, we have that $(H_1, \ldots, H_q)$ is in general position in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$.

By the definition of $\mathbf{F}$ we have for any $j = 1, \ldots, q$,

$$\mathbf{a}_j.\mathbf{F} := \sum_{k=0}^{n_{\mathcal{D}}} a_j.F_k = D_j(\mathbf{f}).$$

So

$$N_f(r, D_j) = N_F(r, H_j) \quad \text{and} \quad N_f^{n_{\mathcal{D}}}(r, D_j) = N_F^{n_{\mathcal{D}}}(r, H_j). \tag{3.1}$$

Furthermore by the first main theorem,

$$T_F(r) = m_{\mathcal{D}}T_f(r) + O(1), \quad S_F(r) = S_f(r). \tag{3.2}$$

With the assumption of Lemma 3.1, applying Theorem 2.1 to holomorphic map $F: \mathbb{C} \to \mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$ and the hyperplanes $H_j, j = 1, \ldots, q$, we have

$$(q - n_{\mathcal{D}} - 1 - \varepsilon)T_F(r) \le \sum_{j=1}^{q} N_F^{n_{\mathcal{D}}}(r, H_j) + S_F(r) \tag{3.3}$$

holds for all large r outside a set of finite Lebesgue measure.

Combining with (3.1), (3.2) and (3.3) together, we have

$$(q - n_{\mathcal{D}} - 1 - \varepsilon)m_{\mathcal{D}}T_f(r) \le \sum_{j=1}^{q} N_f^{n_{\mathcal{D}}}(r, D_j) + S_f(r).$$

This concludes the proof of the lemma.

***Proof of Theorem* 1.1.** Assume for the sake contradiction that $f \not\equiv g$. Then there are two numbers, $\alpha, \beta \in \{0, \ldots, n\}$, $\alpha \neq \beta$ such that $f_\alpha g_\beta \not\equiv f_\beta g_\alpha$. Let k be a sufficiently large positive integer, which will be chosen later, and ε be a real number such that $0 < \varepsilon < 1$. For any $D_j \in \mathcal{D}$, for all large r outside a set of finite Lebesgue measure, by the first main theorem we have

$$N_f^{n_\mathcal{D}}(r, D_j) = N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + N_{f,>k}^{n_\mathcal{D}}(r, D_j) =$$

$$= \frac{k}{k+1} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{1}{k+1} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + N_{f,>k}^{n_\mathcal{D}}(r, D_j) \le$$

$$\le \frac{k}{k+1} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{n_\mathcal{D}}{k+1} N_{f,\le k}^{1}(r, D_j) + n_\mathcal{D} N_{f,>k}^{1}(r, D_j) \le$$

$$\le \frac{k}{k+1} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{n_\mathcal{D}}{k+1} N_{f,\le k}(r, D_j) + \frac{n_\mathcal{D}}{k+1} N_{f,>k}(r, D_j) \le$$

$$\le \frac{k}{k+1} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{n_\mathcal{D}}{k+1} N_f(r, D_j) \le$$

$$\le \frac{k}{k+1} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{d_j n_\mathcal{D}}{k+1} T_f(r) + O(1),$$

where d_j is degree of D_j. So

$$\frac{1}{d_j} N_f^{n_\mathcal{D}}(r, D_j) \le \frac{k}{d_j(k+1)} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{n_\mathcal{D}}{k+1} T_f(r) + O(1).$$

It implies that

$$\sum_{j=1}^{q} \frac{1}{d_j} N_f^{n_\mathcal{D}}(r, D_j) \le \frac{k}{k+1} \sum_{j=1}^{q} \frac{1}{d_j} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{q n_\mathcal{D}}{k+1} T_f(r) + O(1). \quad (3.4)$$

From Lemma 3.1, the inequality (3.4) becomes

$$(q - n_\mathcal{D} - 1 - \varepsilon) T_f(r) \le \frac{k}{k+1} \sum_{j=1}^{q} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{q n_\mathcal{D}}{k+1} T_f(r) + S_f(r) \le$$

$$\le \frac{k}{\delta_\mathcal{D}(k+1)} \sum_{j=1}^{q} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + \frac{q n_\mathcal{D}}{k+1} T_f(r) + S_f(r).$$

This is equivalent to

$$\left(q - \frac{q n_\mathcal{D}}{k+1} - n_\mathcal{D} - 1 - \varepsilon\right) T_f(r) \le \frac{k}{\delta_\mathcal{D}(k+1)} \sum_{j=1}^{q} N_{f,\le k}^{n_\mathcal{D}}(r, D_j) + S_f(r).$$

So

$$\big(q(k + 1 - n_\mathcal{D}) - (n_\mathcal{D} + 1 + \varepsilon)(k+1)\big) T_f(r) \le$$

$$\leq \frac{k}{\delta_{\mathcal{D}}} \sum_{j=1}^{q} N_{f,\leq k}^{n_{\mathcal{D}}}(r, D_j) + S_f(r) \leq$$

$$\leq \frac{n_{\mathcal{D}} k}{\delta_{\mathcal{D}}} \sum_{j=1}^{q} N_{f,\leq k}^{1}(r, D_j) + S_f(r). \tag{3.5}$$

Assume that $z_0 \in \mathbb{C}$ is a zero of $D_j \circ f$ with multiplicity less than or equal to k, then $z_0 \in \bar{E}_f(\mathcal{D}) \bigcup \bar{E}_g(\mathcal{D})$. This implies that $g(z_0) = f(z_0)$, so $\dfrac{f_\alpha(z_0)}{f_\beta(z_0)} = \dfrac{g_\alpha(z_0)}{g_\beta(z_0)}$, namely z_0 is a zero of the function $\dfrac{f_\alpha}{f_\beta} - \dfrac{g_\alpha}{g_\beta}$. Note that by the hypothesis $\mathcal{D}$ is in general position for Veronese embedding then there exist at most $n_{\mathcal{D}}$ hypersurfaces D_j in $\mathcal{D}$ such that $D_j \circ f(z_0) = 0$. This implies that

$$\sum_{j=1}^{q} N_{f,\leq k}^{1}(r, D_j) \leq n_{\mathcal{D}} N_{f_\alpha/f_\beta - g_\alpha/g_\beta}(r).$$

Therefore, by properties of counting function, (3.5) becomes

$$\big(q(k+1-n_{\mathcal{D}}) - (n_{\mathcal{D}}+1+\varepsilon)(k+1)\big) T_f(r) \leq$$

$$\leq \frac{n_{\mathcal{D}}^2 k}{\delta_{\mathcal{D}}} N_{f_\alpha/f_\beta - g_\alpha/g_\beta}(r) + S_f(r) \leq \frac{n_{\mathcal{D}}^2 k}{\delta_{\mathcal{D}}} \big(T_f(r) + T_g(r)\big) + S_f(r). \tag{3.6}$$

Similarly for the holomorphic map g, we have

$$\big(q(k+1-n_{\mathcal{D}}) - (n_{\mathcal{D}}+1+\varepsilon)(k+1)\big) T_g(r) \leq$$

$$\leq \frac{n_{\mathcal{D}}^2 k}{\delta_{\mathcal{D}}} \big(T_f(r) + T_g(r)\big) + S_g(r). \tag{3.7}$$

Adding the inequalities (3.6) and (3.7), we have

$$\big(q(k+1-n_{\mathcal{D}}) - (n_{\mathcal{D}}+1+\varepsilon)(k+1)\big)\big(T_f(r) + T_g(r)\big) \leq$$

$$\leq \frac{2 n_{\mathcal{D}}^2 k}{\delta_{\mathcal{D}}} \big(T_f(r) + T_g(r)\big) + S_f(r) + S_r(r).$$

This concludes that

$$q(k+1-n_{\mathcal{D}}) - (n_{\mathcal{D}}+1+\varepsilon)(k+1) - \frac{2 n_{\mathcal{D}}^2 k}{\delta_{\mathcal{D}}} \leq \frac{S_f(r) + S_g(r)}{T_f(r) + T_g(r)}$$

holds for all large r. Let $r \to \infty$, we have

$$q(k+1-n_{\mathcal{D}}) - (n_{\mathcal{D}}+1+\varepsilon)(k+1) - \frac{2 n_{\mathcal{D}}^2 k}{\delta_{\mathcal{D}}} \leq 0.$$

This is equivalent to

$$k\big(q\delta_{\mathcal{D}} - (n_{\mathcal{D}}+1+\varepsilon)\delta_{\mathcal{D}} - 2 n_{\mathcal{D}}^2\big) + \big(q - qn_{\mathcal{D}} - (n_{\mathcal{D}}+1+\varepsilon)\big)\delta_{\mathcal{D}} \leq 0. \tag{3.8}$$

If we take

$$k > \frac{(qn_{\mathcal{D}} - q + n_{\mathcal{D}} + 1 + \varepsilon)\,\delta_{\mathcal{D}}}{\delta_{\mathcal{D}} - (n_{\mathcal{D}} + 1 + \varepsilon)\,\delta_{\mathcal{D}} - 2\,n_{\mathcal{D}}^2},$$

then since the hypothesis that $q \geq n_{\mathcal{D}} + 2 + \dfrac{2\,n_{\mathcal{D}}^2}{\delta_{\mathcal{D}}}$, we have a contradiction. Hence $f_i g_j \equiv f_j g_i$ for any $i \neq j \in \{0, \ldots, n\}$, namely $f \equiv g$. This is the conclusion of the proof of Theorem 1.1.

***Proof of Theorem* 1.2.** Assume for the sake contradiction that $f \not\equiv g$. Then there are two numbers $\alpha, \beta \in \{0, \ldots, n\}$, $\alpha \neq \beta$ such that $f_\alpha g_\beta \not\equiv f_\beta g_\alpha$. Let k be a sufficiently large positive integer, which will be chosen later, and ε be a real number such that $0 < \varepsilon < 1$. With the hypothesis in Theorem 1.2 and the proof of Theorem 1.1, we have

$$\big(q(k + 1 - n_{\mathcal{D}}) - (n_{\mathcal{D}} + 1 + \varepsilon)(k + 1)\big)\, T_f(r) \leq$$

$$\leq \frac{n_{\mathcal{D}} k}{\delta_{\mathcal{D}}} \sum_{j=1}^{q} N^1_{f,\leq k}(r, D_j) + S_f(r). \tag{3.9}$$

We know that, if $z_0 \in \mathbb{C}$ is a zero of $D_j \circ f$ with multiplicity less than or equal to k, then z_0 will be a zero of the function $\dfrac{f_\alpha}{f_\beta} - \dfrac{g_\alpha}{g_\beta}$. By the hypothesis we have

$$\overline{E}_f(D_i) \bigcap \overline{E}_f(D_j) = \varnothing$$

for any pair $i \neq j \in \{1, \ldots, q\}$. So if z_0 is a zero of $D_j \circ f$ then z_0 will be not a zero of $D_j \circ f$ for all $i \neq j \in \{1, \ldots, q\}$. Hence

$$\sum_{j=1}^{q} N^1_{f,\leq k}(r, D_j) \leq N_{(f_\alpha / f_\beta) - (g_\alpha / g_\beta)}(r) \leq T_f(r) + T_g(r) + O(1).$$

Therefore, (3.9) becomes

$$\big(q(k + 1 - n_{\mathcal{D}}) - (n_{\mathcal{D}} + 1 + \varepsilon)(k + 1)\big)\, T_f(r) \leq$$

$$\leq \frac{n_{\mathcal{D}} k}{\delta_{\mathcal{D}}} \big(T_f(r) + T_g(r)\big) + S_f(r). \tag{3.10}$$

Similarly for the holomorphic map g, we have

$$\big(q(k + 1 - n_{\mathcal{D}}) - (n_{\mathcal{D}} + 1 + \varepsilon)(k + 1)\big)\, T_g(r) \leq$$

$$\leq \frac{n_{\mathcal{D}} k}{\delta_{\mathcal{D}}} \big(T_f(r) + T_g(r)\big) + S_g(r). \tag{3.11}$$

Since the inequalities (3.10) and (3.11), we have

$$\big(q(k + 1 - n_{\mathcal{D}}) - (n_{\mathcal{D}} + 1 + \varepsilon)(k + 1)\big)\big(T_f(r) + T_g(r)\big) \leq$$

$$\leq \frac{2\,n_{\mathcal{D}} k}{\delta_{\mathcal{D}}} \big(T_f(r) + T_g(r)\big) + S_f(r) + S_g(r).$$

Hence

$$q\delta_{\mathcal{D}}(k+1-n_{\mathcal{D}})-\delta_{\mathcal{D}}(n_{\mathcal{D}}+1+\varepsilon)(k+1) - 2n_{\mathcal{D}}k \le \frac{S_f(r)+S_g(r)}{T_f(r)+T_g(r)}$$

holds for all large r. Let $r \to \infty$, we have

$$k\big(q\delta_{\mathcal{D}}-(n_{\mathcal{D}}+1+\varepsilon)\delta_{\mathcal{D}}-2n_{\mathcal{D}}\big)+\big(q-n_{\mathcal{D}}-(n_{\mathcal{D}}+1+\varepsilon)\big)\delta_{\mathcal{D}} \le 0.$$

If we take

$$k > \frac{(qn_{\mathcal{D}}-q+n_{\mathcal{D}}+1+\varepsilon)\delta_{\mathcal{D}}}{q\delta_{\mathcal{D}}-(n_{\mathcal{D}}+1+\varepsilon)\delta_{\mathcal{D}}-2n_{\mathcal{D}}},$$

then since the hypothesis that $q \ge n_{\mathcal{D}}+2+\dfrac{2n_{\mathcal{D}}}{\delta_{\mathcal{D}}}$, we have a contradiction. Hence $f_ig_j \equiv f_jg_i$ for any $i \ne j \in \{0, \ldots, n\}$, namely $f \equiv g$. This is the conclusion of the proof of Theorem 1.2.

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