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MULTIDIMENSIONAL RANDOM MOTION WITH UNIFORMLY DISTRIBUTED CHANGES OF DIRECTION AND ERLANG STEPS

БАГАТОВИМІРНИЙ ВИПАДКОВИЙ РУХ ІЗ РІВНОМІРНО РОЗПОДІЛЕНИМИ ЗМІНАМИ НАПРЯМКУ ТА КРОКАМИ ЕРЛАНГА

In this paper we study transport processes in $\mathbb{R}^n$, $n \geq 1$, having non-exponential distributed sojourn times or non-Markovian step durations. We use the idea that the probabilistic properties of a random vector are completely determined by those of its projection on a fixed line, and using this idea we avoid many of the difficulties appearing in the analysis of these problems in higher dimensions. As a particular case, we find the probability density function in three dimensions for 2-Erlang distributed sojourn times.

Досліджено процеси переносу в $\mathbb{R}^n$, $n \geq 1$, що мають неекспоненціально розподілений час перебування або немарковську тривалість кроків. Використано ідею про те, що ймовірнісні властивості випадкового вектора цілком визначаються такими самими властивостями його проекцій на фіксовану пряму. Цей підхід дозволив уникнути багатьох складностей, що з'являються при дослідженні цих проблем у вимірностях вищого порядку. Як окремий випадок, знайдено функцію щільності ймовірності у тривимірному випадку для часу перебування з 2-розподілом Ерланга.

1. Introduction. One-dimensional non-Markovian generalizations of the telegrapher's random process were obtained in [1, 2] with velocities alternating at Erlang-distributed sojourn times. Uniformly distributed direction of motion or isotropic motion has been studied by Pinsky [3] for transport processes on Riemannian manifold and by Orsingher and De Gregorio in higher dimensions [4]. However, most of the papers on multidimensional random motion are devoted to analysis of models in which motions are driven by a homogeneous Poisson process (see [3 – 6] and references therein). The recent work of Le Caer [7] departs from this trend since he is studying uniformly distributed orientation random motion with Pearson – Dirichlet distributed steps in a multidimensional random walk setting. In this work, we consider random motions with uniformly distributed directions on the multidimensional space $\mathbb{R}^n$, $n \geq 1$, with Erlang distributed step lengths. Our analysis is based on random evolutions on a semi-Markov media.

Let us consider the renewal process $\xi(t) = \max\{m \geq 0\colon \tau_m \leq t\}$, $t \geq 0$, where $\tau_m = \sum_{k=1}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, ...$, are i.i.d. random variables with a distribution function $G(t)$ such that there exists the probability density function (pdf) $g(t) = \dfrac{d}{dt} G(t)$.

We assume that a particle starting from the coordinate origin $(0, 0, \ldots, 0)$ of the space $\mathbb{R}^n$, at time $t = 0$, continues its motion with a constant absolute velocity v along the direction $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_0^{(n)} = (x_1, x_2, \ldots, x_n)$ is a random n-dimensional vector uniformly distributed on the unit sphere $\Omega_1^{n-1} = \{(x_1, x_2, \ldots, x_n)\colon x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}$.

At instant τ_1 the particle changes its direction to $\boldsymbol{\eta}_1^{(n)}$, where $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$ are i.i.d. random vectors on Ω_1^{n-1}. Then, at instant τ_2 the particle changes its direction to $\boldsymbol{\eta}_2^{(n)}$, where $\boldsymbol{\eta}_2^{(n)}$ is also uniformly distributed on Ω_1^{n-1} and independent of $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$, and so on.

Denote by $\mathbf{x}^{(n)}(t)$, $t \geq 0$, the particle position at time t. We have that

$$\mathbf{x}^{(n)}(t) = v \sum_{i=0}^{\xi(t)} \boldsymbol{\eta}_i^{(n)} \theta_i + v\, \boldsymbol{\eta}_{\xi(t)}^{(n)} \left(t - \tau_{\xi(t)}\right). \tag{1}$$

Basically, Eq. (1) determines the random evolution in the semi-Markov medium $\xi(t)$.

Lemma 1. *The probability distribution of the random vector $\mathbf{x}^{(n)}(t)$ is determined by the probability distribution of its projection $x^{(n)}(t) = v \sum_{i=0}^{\xi(t)} \eta_i^{(n)} \theta_i + v\, \eta_{\xi(t)}^{(n)}(t - \tau_{\xi(t)})$ on a fixed line, where $\eta_i^{(n)}$ is the projection of $\boldsymbol{\eta}_i^{(n)}$ on the line.*

Proof. Let us consider the cumulative distribution function (cdf) $F_{x^{(n)}(t)}(y) = P(x^{(n)}(t) \leq y)$. Then, the characteristic function $\varphi_{\mathbf{x}^{(n)}(t)}(\boldsymbol{\alpha})$ of $\mathbf{x}^{(n)}(t)$ is given by

$$\varphi_{\mathbf{x}^{(n)}(t)} = \mathbf{E} \exp\left\{ i \left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t) \right) \right\} = \mathbf{E} \exp\left\{ i \,\|\boldsymbol{\alpha}\| \left(\mathbf{e}, \mathbf{x}^{(n)}(t) \right) \right\} =$$

$$= \mathbf{E} \exp\left\{ i \,\|\boldsymbol{\alpha}\| x^{'(n)}(t) \right\} = \int_0^\infty \exp\left\{ i \,\|\boldsymbol{\alpha}\| y \right\} dF_{x^{(n)}(t)}(y),$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2}$, $x^{(n)}(t)$ is the projection of $\mathbf{x}^{(n)}(t)$ onto the unit vector $\mathbf{e}$ and it has a cdf $F_{x^{(n)}(t)}(y)$.

Lemma 1 is proved.

It is well known that if $f(x_1, x_2, \ldots, x_n) \in L_1(\mathbb{R}^n)$ depends only on $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$, i.e., $f(x_1, x_2, \ldots, x_n) = g(r)$, then the function

$$\varphi(s_1, s_2, \ldots, s_n) = \int_{\mathbb{R}^n} f(\mathbf{x}) \exp\left\{ -i \sum_{k=1}^n s_k x_k \right\} d\mathbf{x}$$

depends only on $s = \|\mathbf{s}\|$. Such functions are called radial functions and for these functions the Fourier transform in several variables goes over into the "Bessel transform" in one variable as follows:

$$\varphi(\mathbf{s}) = \frac{(2\pi)^{n/2}}{s^{(n-2)/2}} \int_0^\infty g(r)\, r^{n/2} J_{(n-2)/2}(sr) dr,$$

where $J_p(x)$ denotes the Bessel function, of the first kind, of order p [10].

Since $\varphi_{\mathbf{x}}(\boldsymbol{\alpha})$ depends only on $\alpha = \|\boldsymbol{\alpha}\|$, meaning that $\varphi_{\mathbf{x}}(\boldsymbol{\alpha}) = \varphi(\alpha)$ then the pdf $f_{\mathbf{x}(t)}(\mathbf{y})$ corresponding to the distribution

$$F_{\mathbf{x}(t)}(\mathbf{y}) = \mathrm{P}\left(v \sum_{i=0}^{\xi(t)+1} \boldsymbol{\eta}_i^{(n)} \theta_i + v\, \boldsymbol{\eta}_{\xi(t)}^{(n)} \left(t - \tau_{\xi(t)}\right) \leq \mathbf{y} \right)$$

depends only on $r = \|\mathbf{y}\|$, that is, $f_{\mathbf{x}(t)}(\mathbf{y}) = h(r)$ and we have

$$\varphi_{\mathbf{x}(t)}(\boldsymbol{\alpha}) = \frac{(2\pi)^{n/2}}{\alpha^{(n-2)/2}} \int\limits_0^\infty h(r)\, r^{n/2} J_{(n-2)/2}(\alpha r) dr.$$

It also follows that if $h(r)$ is continuous on $[0, +\infty)$ and $\int_0^\infty r^{n-1} h(r) dr < \infty$, and if $\int_0^\infty \alpha^{n-1} \varphi(\alpha) d\alpha < \infty$, then [10]

$$f_{\mathbf{x}(t)}(\mathbf{y}) = h(r) = \frac{1}{(2\pi)^{n/2}\, r^{(n-2)/2}} \int\limits_0^\infty \varphi(\alpha)\, \alpha^{n/2} J_{(n-2)/2}(\alpha r) d\alpha.$$

Now, let us define $\hat{x}^{(n)}(t) = v \sum_{i=0}^{\xi(t)} \eta_i^{(n)} \theta_i$ and $\Delta(t) = v\, \eta_{\xi(t)}^{(n)} (t - \tau_{\xi(t)})$, and we will denote as $F_{\hat{x}^{(n)}(t)}(y)$ (resp. $F_{\Delta(t)}(y)$) the cdf of $\hat{x}^{(n)}(t)$ (resp. $\Delta(t)$). It is easy to verify that $\hat{x}^{(n)}(t)$ and $\Delta(t)$ are independent. Hence, we have $F_{x^{(n)}(t)}(y) = F_{\hat{x}^{(n)}(t)}(y) * F_{\Delta(t)}(y)$.

Therefore, by using Lemma 1 we can study the cdf of $\mathbf{x}^{(n)}(t)$ but we need to know the cdf of $\hat{x}^{(n)}(t)$ and $\Delta(t)$.

Lemma 2. *Let $F_n(t)$ be the cdf of $\eta_i^{(n)} \theta_i$ and it is of the following form*:

$$F_n(t) = \begin{cases} \dfrac{1}{2} + \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int\limits_0^1 G\left(\frac{t}{x}\right) (1 - x^2)^{(n-3)/2} dx, & if \quad t \geq 0, \\ \dfrac{1}{2} - \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int\limits_0^1 G\left(-\frac{t}{x}\right) (1 - x^2)^{(n-3)/2} dx, & if \quad t < 0. \end{cases} \tag{2}$$

Proof. Let us denote by $f_{\eta_i}(x)$ the pdf of the projection $\eta_i^{(n)}$ of the vector $\boldsymbol{\eta}_i^{(n)}$ onto a fixed line. It is showed in [8] that $f_{\eta_i}(x)$ is of the following form:

$$f_{\eta_i^{(n)}}(x) = \begin{cases} \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} (1 - x^2)^{(n-3)/2}, & if \quad x \in [-1, 1], \\ 0, & if \quad x \notin [-1, 1]. \end{cases} \tag{3}$$

Since $\eta_i^{(n)}$ and θ_i are independent it is easy to verify that the cdf of $\eta_i^{(n)} \theta_i$ is of the form (2).

Lemma 2 is proved.

The process $\gamma(t) = t - \tau_{\xi(t)}$ is a Markov process and it has the following generator operator A [9]:

$$A\varphi(s) = \varphi'(s) + \frac{g(s)}{1 - G(s)} \left(\varphi(0) - \varphi(s)\right), \quad s \geq 0,$$

where $\varphi \in \mathcal{C}^1(\mathbb{R})$.

Lemma 3. *The cdf $F_{\Delta(t)}(s) = \mathrm{P}\left(v\, \eta_{\xi(t)}^{(n)} (t - \tau_{\xi(t)}) \leq s\right)$ is given by*

$$F_{\Delta(t)}(s)=\begin{cases}\dfrac{1}{2}+\dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\displaystyle\int_0^1 F_{\gamma(t)}\left(\frac{s}{vx}\right)(1-x^2)^{(n-3)/2}dx, & if\quad s\geq 0,\\[2ex] \dfrac{1}{2}-\dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\displaystyle\int_0^1 F_{\gamma(t)}\left(-\frac{s}{vx}\right)(1-x^2)^{(n-3)/2}dx, & if\quad s<0.\end{cases}$$

Proof. The cdf $F_{\gamma(t)}(u) = \mathrm{P}(\gamma(t) \leq u)$ satisfies the following Markov renewal equation [9]

$$F_{\gamma(t)}(u)=V(t,u)+\int_0^t g(s)F_{\gamma(t-s)}(u)ds, \tag{4}$$

where $V(t,u) = \mathrm{P}(\gamma(t)\leq u,\tau_1>t) = (1-G(t))\,I_{\{t\leq u\}}$.

Let us define the function $R(t)=\sum_{k=0}^{\infty} g^{*(k)}(t)$, where the symbol $*(n)$ denotes the k-fold convolution of $g(t)$ with itself. Then, Eq. (4) can be rewritten as

$$F_{\gamma(t)}(u)=(V*R)(t,u)=\int_0^t V(t-s,u)dR(s).$$

Since $v\,\eta_{\xi(t)}^{(n)}$ and $\gamma(t)$ are independent that concludes the proof.

2. Evolution in odd-dimensional spaces. Now, let us assume that $n=2l+3$, $l=0,1,2,\ldots$, and θ_k has a $(n-1)$-Erlang distribution, that is $g(t)=\dfrac{\lambda^{n-1}}{\Gamma(n-1)}\,t^{n-2}e^{-\lambda t}$. It follows from Lemma 2 that the pdf $f_n(t)$ of the random variable $\eta_i^{(n)}\theta_i$ has the form

$$f_n(t)=\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\Gamma(n-1)}\lambda\int_0^1\frac{(\lambda t)^{2l+1}}{x^{2l+2}}\,e^{-\lambda t/x}(1-x^2)^l dx$$

or equivalently,

$$f_n(t)=\frac{\lambda\Gamma\left(l+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(l+1)\Gamma(2l+2)}\sum_{k=0}^{l}\binom{l}{k}(-1)^k(\lambda t)^{2k}\int_{\lambda t}^{\infty}s^{2(l-k)}e^{-s}ds,$$

for $t\geq 0$. Furthermore, the following equivalent expression can be found after some algebraic simplifications

$$f_n(t)=\frac{\lambda e^{-\lambda t}}{l!\,2^{2l+1}}\sum_{k=0}^{l}(-1)^k\frac{(2(l-k)!)}{k!(l-k)!}\sum_{m=0}^{2(l-k)}\frac{(\lambda t)^{2k+m}}{m!}. \tag{5}$$

We have $f_n(t)=f_n(-t)$ for the case when $t<0$.

3. Evolution in three dimensions. Let us consider the particular case when $n=3$. Thus, by taking into account Lemma 2, we have that $\eta_i^{(3)}$ is uniformly distributed on $[-1,1]$.

Let random variables θ_k, $k=0,1,2,\ldots$, be 2-Erlang distributed, i.e., $g(t)=\lambda^2te^{-\lambda t}$, $\lambda>0$, $t\geq 0$.

For this particular case, the Laplace transform of $R(t)$, say $\widehat{R}(s)$, is of the form

$$\widehat{R}(s) = \int_0^\infty R(t)e^{-st}dt = \sum_{k=0}^\infty \int_0^\infty g^{*(k)}(t)e^{-st}dt = \sum_{k=0}^\infty \left(\frac{\lambda}{\lambda+s}\right)^{2k} = \frac{(\lambda+s)^2}{s^2+2\lambda s},$$

and the Laplace transform $\widehat{V}(s,u)$ of $V(t,u)$ can be written as

$$\widehat{V}(s,u) = \frac{2\lambda + s - (\lambda us + \lambda^2 us + s + 2)e^{-(\lambda+s)u}}{(\lambda+s)^2}.$$

Therefore, the Laplace transform $\widehat{F}_\gamma(s,u)$ of $F_{\gamma(t)}(u)$ is given by

$$\widehat{F}_\gamma(s,u) = \widehat{R}(s)\widehat{V}(s,u) = \frac{2\lambda + s - (\lambda us + \lambda^2 us + s + 2)e^{-(\lambda+s)u}}{s(s+2\lambda)}. \tag{6}$$

After applying the inverse Laplace transform to $\widehat{F}_\gamma(s,u)$, we obtain for $t > u > 0$

$$F_{\gamma(t)}(u) = e^{-2\lambda t} - (2+\lambda u)e^{-\lambda t}\sinh(\lambda(t-u)) + 2e^{-\lambda t}\sinh(\lambda t) - (\lambda u + 1)e^{-\lambda(2t-u)}.$$

Thus, we have the limit result

$$\lim_{t\to+\infty} F_{\gamma(t)}(u) = 1 - e^{-\lambda u} - \frac{\lambda u}{2}\, e^{-\lambda u}.$$

Taking into account Lemma 3 we can obtain the corresponding expression for $F_{\Delta(t)}(s)$.

It follows from Eq. (5) that $\eta_i\theta_i$ has the Laplace distribution with pdf $f_3(t) = \frac{1}{2}\lambda e^{-\lambda|t|}$.

Therefore, the Fourier transform of $\mathrm{P}\left(v\sum_{i=0}^k \eta_i\theta_i \le y\right)$ is given by

$$\int_{-\infty}^\infty e^{-i\lambda y} d\mathrm{P}\left(v\sum_{i=0}^k \eta_i\theta_i \le y\right) = \left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2}\right)^k.$$

On the other hand, since

$$F_{\hat{x}^{(3)}(t)}(y) = \mathrm{P}\left(v\sum_{i=0}^{\xi(t)} \eta_i^{(3)}\theta_i \le y\right) = \sum_{k=0}^\infty \mathrm{P}\left(v\sum_{i=0}^{k} \eta_i^{(3)}\theta_i \le y\right)\mathrm{P}(\xi(t)=k)$$

then the characteristic function of $\hat{\mathbf{x}}^{(3)}(t)$, say,

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = \mathbf{E}[e^{-i\alpha\hat{x}^{(3)}(t)}] = \int_{-\infty}^\infty e^{-i\alpha y} dF_{\hat{x}^{(3)}(t)}(y)$$

can be calculated as follows

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = \sum_{k=0}^\infty \left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2}\right)^k \mathrm{P}(\xi(t)=k) =$$

$$= e^{-\lambda t}\sum_{k=0}^\infty \left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2}\right)^k \left(\frac{(\lambda t)^{2k}}{2k!} + \frac{(\lambda t)^{2k+1}}{(2k+1)!}\right).$$

Let us define $\Phi = \dfrac{\lambda^2}{\sqrt{\lambda^2 + v^2\alpha^2}}$, then

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = e^{-\lambda t}\left[\cosh \Phi t + \frac{\lambda^2 + v^2\alpha^2}{\lambda^2}\sinh \Phi t\right].$$

Therefore, by using the inverse Fourier transform, we can obtain $F_{\hat{\mathbf{x}}^{(3)}(t)}(\mathbf{y})$.

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