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ESTIMATES FOR WEIGHTED EIGENVALUES OF FOURTH-ORDER ELLIPTIC OPERATOR WITH VARIABLE COEFFICIENTS*

ОЦІНКИ ЗВАЖЕНИХ ВЛАСНИХ ЗНАЧЕНЬ ЕЛІПТИЧНОГО ОПЕРАТОРА ЧЕТВЕРТОГО ПОРЯДКУ ІЗ ЗМІННИМИ КОЕФІЦІЄНТАМИ

We investigate the Dirichlet weighted eigenvalue problem for a fourth-order elliptic operator with variable coefficients in a bounded domain in $\mathbb{R}^n$. We establish a sharp inequality for its eigenvalues. It yields an estimate for the upper bound of the $(k+1)$-th eigenvalue in terms of the first k eigenvalues. Moreover, we also obtain estimates for some special cases of this problem. In particular, our results generalize the Wang – Xia inequality (J. Funct. Anal. – 2007. – **245**) for the clamped plate problem to a fourth-order elliptic operator with variable coefficients.

Досліджено задачу Діріхле про зважені власні значення для еліптичного оператора четвертого порядку із змінними коефіцієнтами в обмеженій області із $\mathbb{R}^n$. Встановлено точну нерівність для її власних значень, з якої випливає оцінка для верхньої межі $(k+1)$-го власного значення через перші k власних значень. Також отримано оцінки для цієї задачі у деяких окремих випадках. Зокрема, наші результати узагальнюють нерівність Ванга – Ксі (J. Funct. Anal. – 2007. – **245**) для затиснутої пластини на випадок еліптичного оператора четвертого порядку із змінними коефіцієнтами.

1. Introduction. As we know, there have been some remarkable results about estimates for eigenvalues of elliptic operators with constant coefficients such as the Laplacian Δ, the biharmonic operator Δ^2 and so on. For the Dirichlet Laplacian problem, we refer to [2, 3, 6, 8, 10, 18, 20] and the excellent survey [1] of Ashbaugh. Let us give a brief survey on some results about the Dirichlet eigenvalue problem of the biharmonic operator (also called the clamped plate problem):

$$\Delta^2 u = \lambda u, \qquad \text{in } \Omega,$$
$$u|_{\partial\Omega} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$ and ν denotes the outward unit normal vector to $\partial\Omega$. Let λ_r be the r-th eigenvalue of problem (1.1). In their pioneering work [16], Payne, Pólya and Weinberger proved

$$\lambda_{k+1} - \lambda_k \leq \frac{8(n+2)}{n^2}\frac{1}{k}\sum_{r=1}^{k}\lambda_r. \tag{1.2}$$

In 1984, Hile and Yeh [11] generalized (1.2) to

$$\frac{n^2 k^{3/2}}{8(n+2)}\left(\sum_{r=1}^{k}\lambda_r\right)^{-1/2} \leq \sum_{r=1}^{k}\frac{\lambda_r^{1/2}}{\lambda_{k+1}-\lambda_r}$$

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by using an improved method of Hile and Protter [8]. In 1990, Hook [9], Chen and Qian [4] independently obtained

$$\frac{n^2k^2}{8(n+2)} \le \sum_{r=1}^{k} \frac{\lambda_r^{1/2}}{\lambda_{k+1}-\lambda_r} \sum_{r=1}^{k} \lambda_r^{1/2}.$$

In 2006, Cheng and Yang [5] established a universal inequality

$$\lambda_{k+1} \le \frac{1}{k}\sum_{r=1}^{k}\lambda_r + \left[\frac{8(n+2)}{n^2}\right]^{1/2} \frac{1}{k}\sum_{r=1}^{k}\left[\lambda_r(\lambda_{k+1}-\lambda_r)\right]^{1/2}.$$

This also gave an affirmative answer for a question introduced by Ashbaugh in [1]. In 2007, Wang and Xia [19] further derived a sharper inequality

$$\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \le \frac{8(n+2)}{n^2}\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\lambda_r \tag{1.3}$$

for an n-dimensional complete minimal submanifold in a Euclidean space.

Elliptic operators with variable coefficients are also very important in analysis and applications (see [7, 12]). However, there have been fewer references on estimates for eigenvalues of elliptic operators with variable coefficients. To the author's knowledge, Hook [9], Qian and Chen [15], Sun [17] considered second order elliptic operators with variable coefficients and obtained some inequalities of eigenvalues.

For simplicity, we use the following notations:

$$D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}.$$

In this paper, we are concerned about the Dirichlet weighted eigenvalue problem of fourth-order elliptic operator $\sum_{i,j=1}^{n} D_{ij}\big(a_{ij}(x)D_{ij}\big)$ with variable coefficients $a_{ij}(x)$, which is described by

$$\begin{aligned} &\sum_{i,j=1}^{n} D_{ij}\big(a_{ij}(x)D_{ij}u\big) = \Lambda\rho u, \qquad \text{in } \Omega, \\ &u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}\bigg|_{\partial\Omega} = 0, \end{aligned} \tag{1.4}$$

where ρ is a positive continuous function (also called the density) on $\overline{\Omega}$ and the functions $a_{ij}(x) = a_{ji}(x) \in C^2(\overline{\Omega})$ for $i,j = 1,\ldots,n$.

A prototype of fourth-order elliptic operator $\sum_{i,j=1}^{n} D_{ij}\big(a_{ij}(x)D_{ij}\big)$ is the biharmonic operator Δ^2. Namely, if $\rho(x) \equiv 1$ and $a_{ij}(x) \equiv 1$ for $i,j = 1,\ldots,n$ simultaneously, problem (1.4) becomes problem (1.1). Moreover, since the weight function ρ denotes the density in applications, problem (1.4) have more applications. For example, weighted estimates are intelligent in some filtering and identification problems (cf. [13, 14]).

The goal of this paper is to obtain some estimates for eigenvalues of problem (1.4). In Theorem 2.1, we establish a sharp inequality for its eigenvalues. Noticing that (2.1) is a quadratic inequality of Λ_{k+1}, we give an estimate for the upper bound of the $(k+1)$-th eigenvalue Λ_{k+1} in terms of the first k eigenvalues in Theorem 2.2. From these results, we can find the influences of variable coefficients $a_{ij}(x)$ and the weight function $\rho(x)$ on estimates of eigenvalues of problem (1.4). Furthermore, we derive some brief estimates for eigenvalues of some special cases in Corollaries 2.1 – 2.3. Our results generalize and extend some previous results for the clamped plate problem. In particular, inequality (1.3) of Wang and Xia [19] is a corollary of Theorem 2.1.

2. Results and their proofs.

Theorem 2.1. *Let Λ_r be the r-th weighted eigenvalue of problem* (1.4). *Denote by*

$$\sigma = \Big(\min_{x\in\overline{\Omega}} \rho(x)\Big)^{-1}, \qquad \tau = \Big(\max_{x\in\overline{\Omega}} \rho(x)\Big)^{-1}$$

and

$$\zeta = \max_{x\in\overline{\Omega}} \Big[\sum_{l=1}^{n} |D_l(a_{ll}(x))|^2\Big]^{1/2}.$$

Suppose that the functions $a_{ij}(x)$ satisfy:

$$0 < \xi \le a_{ij}(x) \le \eta, \quad i,j = 1,\ldots,n,$$

where ξ and η are two positive constants. Then we have

$$\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2 \le$$

$$\le \frac{8(n+2)\sigma^2\eta}{n^2\tau^2\xi}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\Lambda_r + \frac{8\sigma^{\frac{9}{4}}\zeta}{n^2\tau^2\xi^{3/4}}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\Lambda_r^{3/4}. \tag{2.1}$$

Proof. Denote by u_r the r-th weighted orthonormal eigenfunction of problem (1.4) corresponding to eigenvalues Λ_r, $r = 1,2,\ldots,k$. Namely u_r satisfies

$$\sum_{i,j=1}^{n} D_{ij}\big(a_{ij}(x)D_{ij}u_r\big) = \Lambda_r\rho u_r, \qquad \text{in } \Omega,$$

$$u_r|_{\partial\Omega} = \frac{\partial u_r}{\partial\nu}\Big|_{\partial\Omega} = 0,$$

$$\int_{\Omega}\rho u_r u_s = \delta_{rs}.$$

Let $x = (x_1,\ldots,x_n)$ be the Cartesian coordinate functions of $\mathbb{R}^n$. We define the trial functions φ_{rl} by

$$\varphi_{rl} = x_l u_r - \sum_{s=1}^{k} b_{rs}^{l} u_s, \quad \text{for } r = 1,\ldots,k, \text{ and } l = 1,\ldots,n,$$

where

$$b^l_{rs} = \int\limits_\Omega \rho x_l u_r u_s = b^l_{sr}.$$

Then, for $r, s = 1, \ldots, k$, and $l = 1, \ldots, n$, it is easy to check

$$\int\limits_\Omega \rho \varphi_{rl} u_s = 0 \tag{2.2}$$

and

$$\int\limits_\Omega \rho \varphi_{rl} x_l u_r = \int\limits_\Omega \rho \varphi^2_{rl}.$$

Hence the Rayleigh – Ritz inequality in variation method reads as

$$\Lambda_{k+1} \leq \frac{\int_\Omega \varphi_{rl} \sum_{i,j=1}^n D_{ij}\big(a_{ij}(x) D_{ij}\varphi_{rl}\big)}{\int_\Omega \rho\varphi^2_{rl}}. \tag{2.3}$$

Since

$$\sum_{i,j=1}^n D_{ij}\left[a_{ij}(x) D_{ij}(x_l u_r)\right] =$$

$$= \sum_{i,j=1}^n \delta_{il} D_{ij}\big(a_{ij}(x) D_j u_r\big) + \sum_{i,j=1}^n \delta_{jl} D_{ij}\big(a_{ij}(x) D_i u_r\big) + \sum_{i,j=1}^n \delta_{il} D_j\big(a_{ij}(x) D_{ij} u_r\big) +$$

$$+ \sum_{i,j=1}^n \delta_{jl} D_i\big(a_{ij}(x) D_{ij} u_r\big) + \sum_{i,j=1}^n x_l D_{ij}\big(a_{ij}(x) D_{ij} u_r\big) =$$

$$= 2\sum_{i=1}^n D_{il}\big(a_{il}(x) D_i u_r\big) + 2\sum_{i=1}^n D_i\big(a_{il}(x) D_{il} u_r\big) + x_l \Lambda_r \rho u_r,$$

we have

$$\int\limits_\Omega \varphi_{rl} \sum_{i,j=1}^n D_{ij}\big(a_{ij}(x) D_{ij}\varphi_{rl}\big) =$$

$$= \int\limits_\Omega \varphi_{rl} \Bigg[2\sum_{i=1}^n D_{il}\big(a_{il}(x) D_i u_r\big) + 2\sum_{i=1}^n D_i\big(a_{il}(x) D_{il} u_r\big) +$$

$$+ x_l \Lambda_r \rho u_r - \sum_{s=1}^k b^l_{rs} \Lambda_s \rho u_s \Bigg] =$$

$$= 2\int\limits_\Omega \varphi_{rl} \sum_{i=1}^n \left[D_{il}\big(a_{il}(x) D_i u_r\big) + D_i\big(a_{il}(x) D_{il} u_r\big)\right] + \Lambda_r \int\limits_\Omega \rho \varphi^2_{rl}. \tag{2.4}$$

Substituting (2.4) into (2.3), we obtain

$$(\Lambda_{k+1}-\Lambda_r)\int\limits_\Omega \rho\varphi_{rl}^2 \leq$$

$$\leq 2\int\limits_\Omega x_l u_r \sum_{i=1}^n \Big[D_{il}\big(a_{il}(x)D_i u_r\big)+D_i\big(a_{il}(x)D_{il}u_r\big)\Big]+\sum_{s=1}^k b_{rs}^l c_{rs}^l, \tag{2.5}$$

where

$$c_{rs}^l=-2\int\limits_\Omega u_s\sum_{i=1}^n\Big[D_{il}\big(a_{il}(x)D_i u_r\big)+D_i\big(a_{il}(x)D_{il}u_r\big)\Big]=-c_{sr}^l.$$

Using integration by parts, we deduce

$$\Lambda_r b_{rs}^l=\int\limits_\Omega x_l u_s\sum_{i,j=1}^n D_{ij}\big(a_{ij}(x)D_{ij}u_r\big)=$$

$$=\int\limits_\Omega u_r\left\{2\sum_{i=1}^n\Big[D_{il}\big(a_{il}(x)D_i u_s\big)+D_i\big(a_{il}(x)D_{il}u_s\big)\Big]+\lambda_s\rho x_l u_s\right\}=$$

$$=-c_{sr}^l+\Lambda_s b_{rs}^l.$$

It yields

$$c_{rs}^l=(\Lambda_r-\Lambda_s)b_{rs}^l. \tag{2.6}$$

At the same time, we have

$$2\int\limits_\Omega x_l u_r\sum_{i=1}^n\Big[D_{il}\big(a_{il}(x)D_i u_r\big)+D_i\big(a_{il}(x)D_{il}u_r\big)\Big]=$$

$$=-2\int\limits_\Omega u_r\sum_{i=1}^n D_i\Big[a_{il}(x)D_{il}\big(x_l u_r\big)\Big]+2\int\limits_\Omega x_l u_r\sum_{i=1}^n D_i\big(a_{il}(x)D_{il}u_r\big)=$$

$$=-2\int\limits_\Omega\left[u_r\sum_{i=1}^n D_i\big(a_{il}(x)D_i u_r\big)+u_r D_l\big(a_{ll}(x)D_l u_r\big)+u_r a_{ll}(x)D_{ll}u_r\right]=w_{rl}, \tag{2.7}$$

where

$$w_{rl}=2\int\limits_\Omega\left[\sum_{i=1}^n a_{il}(x)|D_i u_r|^2+2a_{ll}(x)|D_l u_r|^2+D_l\big(a_{ll}(x)\big)u_r D_l u_r\right].$$

Substituting (2.6) and (2.7) into (2.5), we arrive at

$$(\Lambda_{k+1}-\Lambda_r)\int\limits_\Omega\rho\varphi_{rl}^2\leq w_{rl}+\sum_{s=1}^k(\Lambda_r-\Lambda_s)(b_{rs}^l)^2.$$

Using (2.2), we have

$$-2(\Lambda_{k+1}-\Lambda_r)^2\int\limits_\Omega \varphi_{rl} D_l u_r =$$

$$= -2(\Lambda_{k+1}-\Lambda_r)^2\int\limits_\Omega \sqrt{\rho}\varphi_{rl}\left(\frac{1}{\sqrt{\rho}}D_l u_r - \sqrt{\rho}\sum_{s=1}^k d_{rs}^l u_s\right) \le$$

$$\le \delta_r(\Lambda_{k+1}-\Lambda_r)^3\int\limits_\Omega \rho\varphi_{rl}^2 + \frac{\Lambda_{k+1}-\Lambda_r}{\delta_r}\int\limits_\Omega\left(\frac{1}{\sqrt{\rho}}D_l u_r - \sqrt{\rho}\sum_{s=1}^k d_{rs}^l u_s\right)^2 =$$

$$= \delta_r(\Lambda_{k+1}-\Lambda_r)^3\int\limits_\Omega \rho\varphi_{rl}^2 + \frac{\Lambda_{k+1}-\Lambda_r}{\delta_r}\left[\int\limits_\Omega \frac{1}{\rho}|D_l u_r|^2 - \sum_{s=1}^k (d_{rs}^l)^2\right], \qquad (2.8)$$

where the constants δ_r form a decreasing sequence of positive numbers. At the same time, using integration by parts again, we get

$$-2\int\limits_\Omega \varphi_{rl} D_l u_r = -2\int\limits_\Omega x_l u_r D_l u_r + 2\sum_{s=1}^k b_{rs}^l d_{rs}^l = \int\limits_\Omega u_r^2 + 2\sum_{s=1}^k b_{rs}^l d_{rs}^l, \qquad (2.9)$$

where

$$d_{rs}^l = \int\limits_\Omega u_s D_l u_r = -d_{sr}^l.$$

Substituting (2.9) into (2.8) and taking sum on r from 1 to k, we obtain

$$\sum_{r=1}^k (\Lambda_{k+1}-\Lambda_r)^2\int\limits_\Omega u_r^2 + 2\sum_{r,s=1}^k (\Lambda_{k+1}-\Lambda_r)^2 b_{rs}^l d_{rs}^l \le$$

$$\le \sum_{r,s=1}^k \delta_r(\Lambda_{k+1}-\Lambda_r)^2(\Lambda_r-\Lambda_s)(b_{rs}^l)^2 - \sum_{r,s=1}^k \frac{1}{\delta_r}(\Lambda_{k+1}-\Lambda_r)(d_{rs}^l)^2 +$$

$$+2\sum_{r=1}^k \delta_r(\Lambda_{k+1}-\Lambda_r)^2\left[\int\limits_\Omega \sum_{i=1}^n a_{il}(x)|D_i u_r|^2 + 2\int\limits_\Omega a_{ll}(x)|D_l u_r|^2 +\right.$$

$$\left. + \int\limits_\Omega D_l(a_{ll}(x)) u_r D_l u_r\right] + \sum_{r=1}^k \frac{1}{\delta_r}(\Lambda_{k+1}-\Lambda_r)\int\limits_\Omega \frac{1}{\rho}|D_l u_r|^2. \qquad (2.10)$$

Because $\{\delta_r\}_{r=1}^\infty$ is is a decreasing sequence, one can get

$$\sum_{r,s=1}^k \delta_r(\Lambda_{k+1}-\Lambda_r)^2(\Lambda_r-\Lambda_s)(b_{rs}^l)^2 \le -\sum_{r,s=1}^k \delta_r(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\Lambda_s)^2(b_{rs}^l)^2. \qquad (2.11)$$

Using (2.11) and

$$2\sum_{r,s=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2 b_{rs}^l d_{rs}^l = -2\sum_{r,s=1}^{k}(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\Lambda_s) b_{rs}^l d_{rs}^l,$$

we can eliminate the unwanted terms in both sides of (2.10) and obtain the following general inequality:

$$\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2 \le$$

$$\le 2\sum_{r=1}^{k}\delta_r(\Lambda_{k+1}-\Lambda_r)^2\left[\int_\Omega\sum_{i=1}^{n}a_{il}(x)|D_iu_r|^2+2\int_\Omega a_{ll}(x)|D_lu_r|^2+\right.$$

$$\left.+\int_\Omega D_l(a_{ll}(x))u_rD_lu_r\right]+\sum_{r=1}^{k}\frac{1}{\delta_r}(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}|D_lu_r|^2. \qquad (2.12)$$

Taking sum on l from 1 to n in (2.12), we have

$$n\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2 \le$$

$$\le 2\sum_{r=1}^{k}\delta_r(\Lambda_{k+1}-\Lambda_r)^2\left[\int_\Omega\sum_{i,l=1}^{n}a_{il}(x)|D_iu_r|^2+2\int_\Omega\sum_{l=1}^{n}a_{ll}(x)|D_lu_r|^2+\right.$$

$$\left.+\int_\Omega\sum_{l=1}^{n}D_l(a_{ll}(x))u_rD_lu_r\right]+\sum_{r=1}^{k}\frac{1}{\delta_r}(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}\sum_{l=1}^{n}|D_lu_r|^2. \qquad (2.13)$$

Now we need to calculate and estimate some terms in (2.13). It is easy to find that

$$0<\tau=\tau\int_\Omega\rho u_r^2\le\int_\Omega u_r^2\le\sigma\int_\Omega\rho u_r^2=\sigma. \qquad (2.14)$$

Combining

$$\int_\Omega u_r\sum_{i,j=1}^{n}D_{ij}(a_{ij}D_{ij}u_r)=\Lambda_r\int_\Omega\rho u_r^2=\Lambda_r$$

and

$$\int_\Omega u_r\sum_{i,j=1}^{n}D_{ij}(a_{ij}D_{ij}u_r)=$$

$$=\int_\Omega\sum_{i,j=1}^{n}a_{ij}|D_{ij}u_r|^2\ge\xi\int_\Omega\sum_{i,j=1}^{n}|D_{ij}u_r|^2=\xi\int_\Omega|\Delta u_r|^2,$$

we have

$$\int\limits_{\Omega} |\Delta u_r|^2 \leq \xi^{-1}\Lambda_r.$$

Hence it yields

$$\int\limits_{\Omega} \sum_{l=1}^{n} |D_l u_r|^2 = \int\limits_{\Omega} |\nabla u_r|^2 \leq \left[\int\limits_{\Omega} u_r^2 \int\limits_{\Omega} |\Delta u_r|^2\right]^{1/2} \leq \sigma^{1/2}\xi^{-1/2}\Lambda_r^{1/2}, \qquad (2.15)$$

where ∇ denotes the gradient operator. At the same time, since

$$\sum_{l=1}^{n} D_l(a_{ll}(x))u_r D_l u_r \leq$$

$$\leq |u_r| \left[\sum_{l=1}^{n} |D_l(a_{ll}(x))|^2\right]^{1/2} \left[\sum_{l=1}^{n} |D_l u_r|^2\right]^{1/2} \leq \zeta |u_r||\nabla u_r|,$$

we have

$$\int\limits_{\Omega} \sum_{l=1}^{n} D_l(a_{ll}(x))u_r D_l u_r \leq \zeta \int\limits_{\Omega} |u_r||\nabla u_r| \leq \zeta \left[\int\limits_{\Omega} u_r^2 \int\limits_{\Omega} |\nabla u_r|^2\right]^{1/2} \leq$$

$$\leq \zeta\sigma^{3/4}\xi^{-1/4}\Lambda_r^{1/4}. \qquad (2.16)$$

Substituting (2.14), (2.15) and (2.16) into (2.13), we obtain

$$n\tau \sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)^2 \leq \sigma^{1/2}\xi^{-1/2} \left\{ \sum_{r=1}^{k} \frac{1}{\delta_r} (\Lambda_{k+1} - \Lambda_r)\sigma\Lambda_r^{1/2} + \right.$$

$$\left. +2\sum_{r=1}^{k} \delta_r (\Lambda_{k+1} - \Lambda_r)^2 \left[(n+2)\eta\Lambda_r^{1/2} + \zeta(\sigma\xi)^{1/4}\Lambda_r^{1/4}\right] \right\}. \qquad (2.17)$$

Then, putting

$$\delta_r = \frac{\left\{ \dfrac{2\sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)\sigma\Lambda_r^{1/2}\left[(n+2)\eta\Lambda_r^{1/2} + \zeta(\sigma\xi)^{1/4}\Lambda_r^{1/4}\right]}{\sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)^2} \right\}^{1/2}}{2\left[(n+2)\eta\Lambda_r^{1/2} + \zeta(\sigma\xi)^{1/4}\Lambda_r^{1/4}\right]}$$

in (2.17), we have

$$n\tau \left[\sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)^2\right]^{1/2} \leq$$

$$\leq 2\sigma\xi^{-1/2} \left\{ 2\sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)\Lambda_r^{1/2} \left[(n+2)\eta\Lambda_r^{1/2} + \zeta(\sigma\xi)^{1/4}\Lambda_r^{1/4}\right] \right\}^{1/2}.$$

Therefore, (2.1) holds.

Theorem 2.1 is proved.

Inequality (2.1) is sharp. It implies an estimate for the upper bound of Λ_{k+1}. In fact, (2.1) is a quadratic inequality of Λ_{k+1}. Solving it, we can obtain an upper bound of Λ_{k+1} in terms of the first k eigenvalues.

Theorem 2.2. *Under the assumptions of Theorem* 2.1, *we have*

$$\Lambda_{k+1} \le \frac{1}{k}\left[(1+E)\sum_{r=1}^{k}\Lambda_r + F\sum_{r=1}^{k}\Lambda_r^{3/4}\right] +$$

$$+\left\{\left[(1+E)\frac{1}{k}\sum_{r=1}^{k}\Lambda_r + F\frac{1}{k}\sum_{r=1}^{k}\Lambda_r^{3/4}\right]^2 - \frac{1}{k}\left[(1+2E)\sum_{r=1}^{k}\Lambda_r^2 + 2F\sum_{r=1}^{k}\Lambda_r^{7/4}\right]\right\}^{1/2},$$

where

$$E = \frac{4(n+2)\sigma^2\eta}{n^2\tau^2\xi} \quad \text{and} \quad F = \frac{4\sigma^{\frac{9}{4}}\zeta}{n^2\tau^2\xi^{3/4}}.$$

From (2.1), we can find the influences of variable coefficients $a_{ij}(x)$ and the weight function $\rho(x)$ on estimates of eigenvalues of problem (1.4). Besides the lower and upper bounds of $a_{ij}(x)$, estimate (2.1) depends on $\zeta = \max_{x\in\overline{\Omega}}\left[\sum_{l=1}^{n}|D_l(a_{ll}(x))|^2\right]^{1/2}$. Namely if $A(x) = (a_{ij}(x))_{n\times n}$ denotes the $(n\times n)$-matrix with components $a_{ij}(x)$, it depends on the diagonal elements of the matrix $A(x)$. In particular, when $a_{ll}(x)$ are all constants for $l = 1,\ldots,n$, it holds $\zeta = 0$. Therefore, for this special case, we have the following corollaries.

***Corollary* 2.1.** *Under the assumptions of Theorem* 2.1, *if* $a_{ll}(x)$ *are all constants for* $l = 1,\ldots,n$, *we have*

$$\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2 \le \frac{8(n+2)\sigma^2\eta}{n^2\tau^2\xi}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\Lambda_r. \tag{2.18}$$

***Remark* 2.1.** It is not difficult to find that inequality (1.3) in [19] is a corollary of Theorem 2.2.

Since (2.18) is also a quadratic inequality of Λ_{k+1}, we can get an estimate for the upper bound of Λ_{k+1} in terms of the first k eigenvalues.

***Corollary* 2.2.** *Under the assumptions of Corollary* 2.1, *we have*

$$\Lambda_{k+1} \le \left[1+\frac{4(n+2)\sigma^2\eta}{n^2\tau^2\xi}\right]\frac{1}{k}\sum_{r=1}^{k}\Lambda_r + \left\{\left[\frac{4(n+2)\sigma^2\eta}{n^2\tau^2\xi}\frac{1}{k}\sum_{r=1}^{k}\Lambda_r\right]^2 -\right.$$

$$\left. - \left[1+\frac{8(n+2)\sigma^2\eta}{n^2\tau^2\xi}\right]\frac{1}{k}\sum_{s=1}^{k}\left(\Lambda_s - \frac{1}{k}\sum_{r=1}^{k}\Lambda_r\right)^2\right\}^{1/2}. \tag{2.19}$$

Then, using (2.19) and the Cauchy – Schwarz inequality

$$\sum_{r=1}^{k}\Lambda_r^2 \ge \frac{1}{k}\left(\sum_{r=1}^{k}\Lambda_r\right)^2,$$

we can get a weaker but more explicit inequality.

***Corollary* 2.3.** *Under the assumptions of Corollary* 2.1, *we have*

$$\Lambda_{k+1} \leq \left[1 + \frac{8(n+2)\sigma^2\eta}{n^2\tau^2\xi}\right] \frac{1}{k} \sum_{r=1}^{k} \Lambda_r.$$

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