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Q-PERMUTABLE SUBGROUPS OF FINITE GROUPS*

Q-ПЕРЕСТАВНІ ПІДГРУПИ СКіНЧЕННИХ ГРУП

A subgroup H of a group G is called Q-permutable in G if there exists a subgroup B of G such that (1) $G = HB$ and (2) if H_1 is a maximal subgroup of H containing H_{QG}, then $H_1B = BH_1 < G$, where H_{QG} is the largest permutable subgroup of G contained in H. In this paper we prove that: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F^*(H)$ having no supersolvable supplement in G is Q-permutable in G, then $G \in \mathcal{F}$.

Підгрупу H групи G називають Q-переставною в G, якщо існує підгрупа B групи G така, що: 1) $G = HB$ та 2) якщо H_1 — максимальна підгрупа H, що містить H_{QG}, то $H_1B = BH_1 < G$, де H_{QG} є найбільшою переставною підгрупою G, що міститься в H. У цій роботі доведено наступне твердження. Нехай $\mathcal{F}$ — насичена формація, що містить $\mathcal{U}$, а G — група з нормальною підгрупою H такою, що $G/H \in \mathcal{F}$. Якщо кожна максимальна підгрупа кожної нециклічної силовської підгрупи $F^*(H)$, що не має надрозв'язного доповнення в G, є Q-переставною в G, то $G \in \mathcal{F}$.

1. Introduction. All groups considered in this paper are finite. Our terminology and notation are standard (see [2, 6, 12]). In what follows, $\mathcal{U}$ denotes the formation of all supersolvable groups.

It has been of interest to use the supplementation of subgroups to characterize the structure of a group. In this context, Hall and Kegel proved some interesting results for solvable groups (see [5, 8, 9]). Recently, by considering some special supplemented subgroups, Wang introduced the concept of c-normal [14] and Ballester–Bolinches, Guo and Wang introduced the notion of c-supplemented subgroups [1]. More recently, A. N. Skiba introduced the concept of weakly s-permutable subgroups [13] and Miao and Lempken introduced the definition of $\mathcal{M}$-supplemented subgroups [10]. They used certain types of supplement to study conditions for solvability and supersolvability of finite groups.

In these paper, we continue this work and introduce the concept of Q-permutable subgroups.

Definition 1.1. *A subgroup H of a group G is called Q-permutable in G if there exists a subgroup B of G such that* (1) $G = HB$ *and* (2) *if H_1 is a maximal subgroup of H containing H_{QG}, then $H_1B = BH_1 < G$, where H_{QG} is the largest permutable subgroup of G contained in H.*

Recall that a subgroup H is called $\mathcal{M}$-supplemented in G, if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for every maximal subgroup H_1 of H. Moreover, a subgroup H is called weakly s-permutable in G if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{sG}$ where H_{sG} is the largest s-permutable subgroup of G contained in H.

The following examples indicate that the Q-permutability of subgroups cannot be deduced from Skiba's result nor from other related results.

***Example* 1.1.** Let $G = S_4$ be the symmetric group of degree 4 and $H = \langle(1234)\rangle$ be a cyclic subgroup of order 4. Then $G = HA_4$ where A_4 is the alternating group of

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degree 4. Clearly, since $A_4 \trianglelefteq G$, we have A_4 permutes all maximal subgroups of H and hence H is Q-permutable in G. On the other hand, we have $H_{sG} = 1$. Otherwise, if H is s-permutable in G, then H is normal in G, a contradiction. If $H_{sG} = \langle(13)(24)\rangle$ is s-permutable in G, then we also have $\langle(13)(24)\rangle$ is normal in G, a contradiction. Therefore we know that H is not weakly s-permutable in G.

Example **1.2.** Let $G = S_4$ be the symmetrical group of degree 4 and H be a Sylow 2-subgroup of G. Clearly, H is Q-permutable in G and $G = HA_4$. Furthermore, H is not $\mathcal{M}$-supplemented in G.

2. Preliminaries. For the sake of convenience, we first list here some known results which will be useful in the sequel.

Lemma 2.1. *Let G be a group. Then*:

(1) *If H is Q-permutable in G, $H \leq M \leq G$, then H is Q-permutable in M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is Q-permutable in G if and only if H/N is Q-permutable in G/N.*

(3) *Let π be a set of primes. Let K be a normal π'-subgroup and H be a π-subgroup of G. If H is Q-permutable in G, then HK/K is Q-permutable in G/K.*

(4) *Let R be a solvable minimal normal subgroup of G and R_1 be a maximal subgroup of R. If R_1 is Q-permutable in G, then R is a cyclic group of prime order.*

(5) *Let P be a p-subgroup of G where p is a prime divisor of $|G|$. If P is Q-permutable in G, then there exists a subgroup B of G such that $|G\colon TB| = p$ for any maximal subgroup T of P containing P_{QG}.*

Proof. (1) If H is Q-permutable in G, then there exists a subgroup B of G such that $G = HB$ and $TB < G$ for any maximal subgroup T of H with $H_{QG} \leq T$. Since $H \leq M \leq G$, we have $H_{QG} \leq H_{QM}$. Thus we may set $L = M \cap B$. Clearly, $L = M \cap B \leq M$ and $M = M \cap HB = H(M \cap B) = HL$. Since $TB < G$ for every maximal subgroup T of H with $H_{QM} \leq T$, we easily see that $TL = T(M \cap B) = M \cap TB$ is a proper subgroup of M.

(2) This follows easily from the definition of Q-permutable subgroups.

(3) If H is Q-permutable in G, then there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for any maximal subgroup H_1 of H containing H_{QG}. Clearly, $(HK/K)(BK/K) = G/K$. For any maximal subgroup T/K of HK/K containing $(HK/K)_{Q(G/K)}$, since K is a normal π'-subgroup and H is a π-subgroup of G, we have $T = T_1K$ where T_1 is a maximal subgroup of H containing H_{QG}. Therefore $(T_1K/K)(BK/K) = T_1BK/K = (BK/K)(T_1K/K) < G/K$. Otherwise, if $T_1BK = G$, then $|G\colon T_1B| = |K\colon K \cap T_1B|$ is a π'-number, on the other hand, $|G\colon T_1B| = |HB\colon T_1B|$ is a π-number, which is a contradiction.

Conversely, if HK/K is Q-permutable in G/K by the subgroup B/K, we easily verify that H is Q-permutable in G by B.

(4) If R_1 is permutable in G, then $R = R_1$ since the minimal normal subgroup of G is also a minimal permutable subgroup of G [11]), a contradiction. On the other hand, if $(R_1)_{QG} \leq R_1$, then there exists a subgroup B of G such that $G = R_1B$ and $TB = BT < G$ for any maximal subgroup T of R_1 with $(R_1)_{QG} \leq T$. If $R \cap B = R$, then $B = G$, a contradiction. If $R \cap B = 1$, then R is a cyclic subgroup of prime order.

(5) If P is Q-permutable in G, then there exists a subgroup B of G such that $G = PB$ and $TB = BT < G$ for any maximal subgroup T of P with $P_{QG} \leq T$. Since $|P\colon T| = p$, we get $|G| = |PB| = p|T||B|/|P \cap B| = (p/|(P \cap B)\colon (T \cap B)|) \cdot |TB|$.

As p is a prime and $TB < G$, we conclude that $P \cap B = T \cap B$ and $|G\colon TB| = p$. Now the claim follows.

Lemma 2.2 ([4], Theorem 1.8.17)**.** *Let N be a nontrivial solvable normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.3 ([16], Theorem 4.6)**.** *If H is a subgroup of G with $|G\colon H| = p$, where p is the smallest prime divisor of $|G|$, then $H \trianglelefteq G$.*

Lemma 2.4 ([3], main theorem)**.** *Suppose a finite group G has a Hall π-subgroup where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.5. *Let G be a finite group and P be a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P has a p-nilpotent supplement or is Q-permutable in G, then $G/O_p(G)$ is solvable p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of smallest order. Furthermore we have,

(1) $O_p(G) \neq 1$.

If $1 < O_p(G) \leq P$, then $G/O_p(G)$ satisfies the hypotheses and the minimal choice of G implies that $G/O_p(G) \cong (G/O_p(G))/O_p(G/O_p(G))$ is p-nilpotent, a contradiction.

(2) $O_p(G) = 1$.

Let P_1 be a maximal subgroup of P. If $|P| = p$, then G is p-nilpotent by Burnside p-nilpotent Theorem, a contradiction. So we may assume $|P| \geqslant p^2$. By hypotheses, if P_1 has a p-nilpotent supplement in G, then there exists a subgroup K of G such that $G = P_1K$ and K is p-nilpotent. Therefore we have $K_{p'} \trianglelefteq K$ where $K_{p'}$ is a Hall p'-subgroup of K and of course of G. Hence $G = P_1 N_G(K_{p'})$. If $P \cap N_G(K_{p'}) = P$, then $K_{p'} \trianglelefteq G$, a contradiction. If $P \cap N_G(K_{p'}) = L$ where L is a maximal subgroup of P, then $|G\colon N_G(K_{p'})| = |P\colon P \cap N_G(K_{p'})| = |P\colon L| = p$ and hence $N_G(K_{p'}) \trianglelefteq G$ by Lemma 2.3, a contradiction. So we may assume $P \cap N_G(K_{p'}) \leq L_2 < L_1$ where L_1 is a maximal subgroup of P and L_2 is a maximal subgroup of L_1. If L_1 has a p-nilpotent supplement in G, then there exists a p-nilpotent subgroup H such that $G = L_1H$. With the similar discussion we have $G = L_1 N_G(H_{p'})$ where $H_{p'}$ is the Hall p'-subgroup of H and of course of G. By Lemma 2.4, there exists an element x of P such that $N_G(K_{p'}) = (N_G(H_{p'}))^x$. Therefore $G = L_1 N_G(H_{p'}) = (L_1 N_G(H_{p'}))^x = L_1 N_G(K_{p'})$. Furthermore, $P = P \cap L_1 N_G(K_{p'}) = L_1(P \cap N_G(K_{p'})) = L_1$, a contradiction. Hence L_1 is Q-permutable in G, there exists a subgroup B of G such that $G = L_1B$ and $TB < G$ for any maximal subgroup T of L_1 with $(L_1)_{QG} \leq T$. Moreover, $(L_1)_{QG} \leq O_p(G) = 1$ and hence L_1 is $\mathcal{M}$-supplemented in G in this case. Therefore $L_2B < G$ and $|G\colon L_2B| = p$ by Lemma 2.1(5). Since p is the smallest prime divisor of $|G|$, Lemma 2.3 implies that $L_2B \trianglelefteq G$. We have $G = L_1B = PB = PL_2B$ and $P \cap L_2B = L_2(P \cap B)$ is the Sylow p-subgroup of L_2B. Clearly, $L_2(P \cap B)$ is the maximal subgroup of P. By hypotheses if $L_2(P \cap B)$ is Q-permutable in G, then $L_2(P \cap B)$ is $\mathcal{M}$-supplemented in G and hence $\mathcal{M}$-supplemented in L_2B by [10] (Lemma 2.1(1)). So L_2B is p-nilpotent by [10] (Lemma 2.11). Therefore G is p-nilpotent, a contradiction.

So we may assume $L_2(P \cap B)$ has a p-nilpotent supplement in G. With the similar discussion as above, there exists a p-nilpotent subgroup S such that $G = L_2(P \cap$

$\cap B)S = L_2(P \cap B)N_G(S_{p'})$ where $S_{p'}$ is a Hall p'-subgroup of S and also of G. By Lemma 2.4, there exists an element g of P such that $N_G(K_{p'}) = (N_G(S_{p'}))^g$. Therefore $G = L_2(P \cap B)N_G(S_{p'}) = (L_2(P \cap B)N_G(S_{p'}))^g = L_2(P \cap B)N_G(K_{p'})$. Furthermore, $P = P \cap L_2(P \cap B)N_G(K_{p'}) = L_2(P \cap B)(P \cap N_G(K_{p'})) = L_2(P \cap B)$, a contradiction.

Therefore $G/O_p(G)$ is p-nilpotent.

Lemma 2.6 ([7], Chapter X)**.** *Let G be a group and N a subgroup of G. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Then*:

(1) *if N is normal in G, then $F^*(N) \leq F^*(G)$;*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{Soc}\,(F(G)C_G(F(G))/F(G)$;*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$;*

(4) *$C_G(F^*(G)) \leq F(G)$;*

(5) *let $P \trianglelefteq G$ and $P \leq O_p(G)$; then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$;*

(6) *if K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Lemma 2.7 ([10], Lemma 2.7)**.** *Let G be a finite group with normal subgroups H and L and let $p \in \pi(G)$. Then the following hold*:

1) *If $L \leq \Phi(G)$, then $F(G/L) = F(G)/L$.*

2) *If $L \leq H \cap \Phi(G)$, then $F(H/L) = F(H)/L$.*

3) *If H is a p-group and $L \leq \Phi(H)$, then $F^*(G/L) = F^*(G)/L$.*

4) *If $L \leq \Phi(G)$ with $|L| = p$, then $F^*(G/L) = F^*(G)/L$.*

5) *If $L \leq H \cap \Phi(G)$ with $|L| = p$, then $F^*(H/L) = F^*(H)/L$.*

Lemma 2.8 ([15], Theorem 3.1)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, G a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If for every maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

3. Main results.

Theorem 3.1. *Let G be a group and H a normal subgroup of G with $G/H \in \mathcal{U}$. If every maximal subgroup of every noncyclic Sylow subgroup of H has a supersolvable supplement or is Q-permutable in G, then G is supersolvable.*

Proof. Assume that the theorem is false and let G be a counterexample with minimal order. Then we have following claims:

(1) G is solvable.

By hypotheses and Lemma 2.5, $H/O_r(H)$ is solvable r-nilpotent where r is the smallest prime divisor of $|H|$ and hence G is solvable. Let L be a minimal normal subgroup of G contained in H. Clearly, L is an elementary abelian p-group for some prime divisor p of $|G|$.

(2) $G/L \in \mathcal{U}$ and L is the unique minimal normal subgroup of G contained in H such that $H \cap \Phi(G) = 1$. Furthermore, $L = F(H) = C_H(L)$.

Firstly, we check that $(G/L, H/L)$ satisfies the hypotheses for (G, H). We know that $H/L \trianglelefteq G/L$ and $(G/L)/(H/L) \cong G/H$ is supersolvable. Let $\overline{Q} = QL/L$ be a Sylow q-subgroup of H/L. We may assume that Q is a Sylow q-subgroup of H. If $p = q$, we may assume that $L \leq P$, where P is a Sylow p-subgroup of H. If $L \leq P = Q$, then every maximal subgroup of P/L is of the form P_1/L where P_1 is a maximal subgroup of P. If P_1/L has no supersolvable supplement in G/L, then P_1 has no supersolvable supplement in G, by hypotheses, P_1 is Q-permutable in G and hence P_1/L is Q-permutable in G/L by Lemma 2.1(3). Now we assume that $p \neq q$. Let $\overline{Q_1}$ be a maximal subgroup of a Sylow q-subgroup of $\overline{H}$. Without loss of generality, we may assume that $\overline{Q_1} = Q_1L/L$ where Q_1 is a maximal subgroup of a Sylow q-subgroup of H. Clearly, if Q_1L/L has no supersolvable supplement in G/L, then Q_1L/L is Q-permutable in G/L by Lemma 2.1(3). Hence G/L satisfies the hypotheses of the theorem. The minimal choice of G implies that $G/L \in \mathcal{U}$. Since $\mathcal{U}$ is a saturated formation, we know that L is the unique minimal normal subgroup of G which is contained in H and $L \nleq \Phi(G)$. By Lemma 2.2 we have $F(H) = L$. The solvability of H implies that $L \leq C_H(L) = C_H(F(H)) \leq F(H)$ and so $C_H(L) = L = F(H)$.

(3) L is a Sylow subgroup of H.

Let q be the largest prime divisor of $|H|$ and Q be a Sylow q-subgroup of H. Since G/L is supersolvable, we have H/L is supersolvable. Consequently, LQ/L char $H/L \trianglelefteq G/L$ and hence $LQ \trianglelefteq G$. If $p = q$, then $L \leq Q \trianglelefteq G$. Therefore $Q \leq F(H) = L$ and L is a Sylow q-subgroup of H as desired.

Now we assume $p < q$. Let P be a Sylow p-subgroup of H. Clearly, P is not cyclic. Otherwise, $G/L \in \mathcal{U}$ implies that $G \in \mathcal{U}$. Then $L \leq P$ and $PQ = PLQ$ is a subgroup of H. Note that every maximal subgroup of noncyclic Sylow subgroup of PQ having no supersolvable supplement in PQ is Q-permutable in PQ by Lemma 2.1(1). Therefore PQ satisfies the hypotheses for G. If $PQ < G$, the minimal choice of G implies that PQ is supersolvable; in particular, $Q \trianglelefteq PQ$. Hence $LQ = L \times Q$ and $Q \leq C_H(L) \leq L$, a contradiction.

Now we may assume that $G = PQ = H$ and $L < P$. Since $G/L \in \mathcal{U}$, $LQ \trianglelefteq G$. By the Frattini argument, $G = LN_G(Q)$. Note that $L \cap N_G(Q)$ is normalized by $N_G(Q)$ and L. Since L is the unique minimal normal subgroup of G, we have $L \cap N_G(Q) = 1$. Let P_2 be a Sylow p-subgroup of $N_G(Q)$. Then LP_2 is a Sylow p-subgroup of G. Choose a maximal subgroup P_1 of LP_2 such that $P_2 \leq P_1$. Clearly, $L \nleq P_1$ and hence $(P_1)_G = 1$. If P_1 is Q-permutable in G, then there exists a subgroup B of G such that $G = P_1B$ and $TB < G$ for any maximal subgroup T of P_1 containing $(P_1)_{QG}$. Furthermore, $(P_1)_{QG} = 1$. Otherwise, if $(P_1)_{QG} \neq 1$, then we have $(P_1)_{QG} = L$ since $1 < (P_1)_{QG} \leq O_p(G) = L$ and a minimal normal subgroup of G and also is the minimal permutable subgroup of G by [11], contrary to $L \nleq P_1$. We may choose a maximal subgroup T of P_1 with $P_2 \leq T$. Otherwise, $P_2 = P_1$, then we have $|L| = p$ and hence $G/L \in \mathcal{U}$ implies that G is supersolvable, a contradiction. By Lemma 2.1(5), $|G \colon TB| = p$. Therefore $L \leq TB$ or $L \cap TB = 1$. If $L \cap TB = 1$, then $|G \colon TB| = |L| = p$, a contradiction. Therefore $L \leq TB$ and hence $LP_2 \leq TB$, contrary to $|G \colon TB| = p$.

So P_1 has a supersolvable supplement in G, that is, there exists a supersolvable subgroup K of G such that $G = P_1K$. In fact, K has a normal p-complement Q_1 which is also a Sylow q-subgroup of G. By Sylow's theorem, there exists an element $g \in L$ such that $Q_1^g = Q$. Since $P_1 \trianglelefteq LP_2$, we have that $G = P_1K = (P_1K)^g = P_1K^g$. Since $K^g \cong K$ has a normal Sylow q-subgroup and $Q = Q_1^g \leq K^g$, it follows that $K^g \leq N_G(Q)$. Since $LP_2 = LP_2 \cap G = LP_2 \cap P_1K^g = P_1(LP_2 \cap K^g)$, we have that $LP_2 \cap K^g \nleq P_2$. Otherwise, $LP_2 \leq P_1P_2 = P_1$, a contradiction. Therefore P_2 is a proper subgroup of $P_3 = < P_2, LP_2 \cap K^g >$. On the other hand, since both P_2 and K^g are contained in $N_G(Q)$, P_3 is a p-subgroup of $N_G(Q)$ which contains a Sylow p-subgroup P_2 of $N_G(Q)$ as a proper subgroup, which is a contradiction.

(4) $G \in \mathcal{U}$.

Let L_1 be a maximal subgroup of L. If L_1 has a supersolvable supplement in G, then there exists a supersolvable subgroup K of G such that $G = L_1K$. Since L is a minimal normal subgroup of G, we have $L \cap K \in \{1, L\}$. If $L \cap K = L$, then $G = L_1K = K$, a contradiction. If $L \cap K = 1$, then $|L| = p$, also a contradiction. Thus we have that L_1 is Q-permutable in G. In this case we know that L is a cyclic subgroup by Lemma 2.1(4), a contradiction.

The final contradiction completes our proof.

***Corollary* 3.1.** *Let G be a group. If every maximal subgroup of every noncyclic Sylow subgroup of G having no supersolvable supplement in G is Q-permutable in G, then G is supersolvable .*

Theorem 3.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every noncyclic Sylow subgroup of H having no supersolvable supplement in G is Q-permutable in G, then $G \in \mathcal{F}$.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Since the pair (H, H) satisfies the hypotheses for the pair (G, H) with $H/H \in \mathcal{U}$, H is supersolvable by Theorem 3.1.

Now let p be the largest prime in $\pi(H)$ and $P \in \mathrm{Syl}_p(H)$; so $P = O_p(H) \trianglelefteq G$. Let L be a minimal normal subgroup of G contained in P. Using similar arguments as for the proof of Claim (2) of Theorem 3.1 we easily establish that $G/L \in \mathcal{F}$ and that L is the unique minimal normal subgroup of G contained in H; moreover, $L = F(H) = C_H(L)$ is noncyclic and $H \cap \Phi(G) = 1$.

Clearly, $\Omega_1(Z(P)) \trianglelefteq G$ and so $L \leq \Omega_1(Z(P))$; hence $P \leq C_H(L) = L$ and thus $L = P \in \mathrm{Syl}_p(H)$. The same arguments as the last step of the proof of Theorem 3.1 now yield a contradiction.

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F(H)$ having no supersolvable supplement in G is Q-permutable in G, then $G \in \mathcal{F}$.*

Proof. Assume that the assertion is false and choose G to be a counterexample of minimal order.

By Lemma 2.8, we may pick a maximal subgroup M of G not containing $F(H)$.

Actually, since $F(H) \nleq M$, there at least exists a prime p of $\pi(|H|)$ with $O_p(H) \nleq M$. Then $G = O_p(H)M$ and $O_p(H) \cap M \trianglelefteq G$. If $|O_p(H)| = p$, then $|G\colon M| = p$ and hence $G \in \mathcal{F}$ by Lemma 2.8, a contradiction. Let M_p be a Sylow p-subgroup of M. Then we know that $G_p = O_p(H)M_p$ is a Sylow p-subgroup of G. Now, let P_1 be a maximal subgroup of G_p containing M_p and set $P_2 = P_1 \cap O_p(H)$. Then $P_1 = P_2M_p$. Moreover, $P_2 \cap M_p = O_p(H) \cap M_p$, so $|O_p(H)\colon P_2| = |O_p(H)M_p\colon P_2M_p| = |G_p\colon P_1| = p$, that is, P_2 is a maximal subgroup of $O_p(H)$. Hence $P_2(O_p(H) \cap M)$ is a subgroup of $O_p(H)$. By the maximality of P_2 in $O_p(H)$, we have $P_2(O_p(H) \cap M) = P_2$ or $O_p(H)$.

1) If $P_2(O_p(H) \cap M) = O_p(H)$, then $G = O_p(H)M = P_2M$. Notice that $O_p(H) \cap M = P_2 \cap M$. So $O_p(H) = P_2$, a contradiction.

2) $P_2(O_p(H) \cap M) = P_2$, that is, $O_p(H) \cap M \leq P_2$. Clearly, $O_p(H) \cap M \trianglelefteq G$, so $O_p(H) \cap M \leq (P_2)_G$. On the other hand, if P_2 has a supersolvable supple-

ment in G, then there exists a supersolvable subgroup N of G such that $G = P_2N$. Set $K = (P_2)_GN$, then $G = P_2N = P_2K$ and $K/K \cap (P_2)_G = K/(P_2)_G = (P_2)_GN/(P_2)_G \cong N/N \cap (P_2)_G \in \mathcal{U} \subseteq \mathcal{F}$.

Now, we consider the following cases.

a) $K < G$. Suppose that K_1 is a maximal subgroup of G containing K. Then $O_p(H)\cap K_1 \trianglelefteq G$, which implies that $(O_p(H)\cap K_1)M$ is a subgroup of G. If $(O_p(H)\cap K_1)M = G = O_p(H)M$, then $O_p(H) \cap K_1 = O_p(H)$ since $(O_p(H) \cap K_1) \cap M = O_p(H) \cap M$. This implies that $O_p(H) \leq K_1$, and hence $G = O_p(H)K_1 = K_1$, which is contrary to the above hypotheses on K_1. Thus $(O_p(H) \cap K_1)M = M$ and hence $O_p(H) \cap K_1 \leq M$. Furthermore, $P_2 \cap K \leq O_p(H) \cap K \leq O_p(H) \cap M \leq (P_2)_G \leq P_2 \cap K$, that is, $O_p(H) \cap K = O_p(H) \cap M = P_2 \cap K$. This is contrary to $G = P_2K = O_p(H)K$.

b) $K = (P_2)_GN = G$. In this case, if $(P_2)_G = 1$, then $N = G \in \mathcal{F}$, a contradiction. So we may assume that $(P_2)_G \neq 1$. Thus $(P_2)_GM = M$ or G. If $(P_2)_GM = G$, then $G = (P_2)_GM = O_p(H)M = P_2M$. Note that $O_p(H) \cap M = P_2 \cap M$, so $O_p(H) = P_2$, a contradiction. Therefore $(P_2)_GM = M$. It follows from $(P_2)_G \leq O_p(H) \cap M \leq (P_2)_G$ that $O_p(H) \cap M = (P_2)_G$. By hypotheses, $G/(P_2)_G \in \mathcal{U}$ implies that $|G/(P_2)_G \colon M/(P_2)_G| = |G \colon M| = p$. This is contrary to the choice of M.

So we may assume that P_2 is Q-permutable in G. There exists a subgroup B of G such that $P_2B = G$ and $TB < G$ for any maximal subgroup T containing $(P_2)_{QG}$. Assume P_2 is permutable in G. The maximality of M in G implies $P_2M = M$ or G. If $P_2M = G$, then $G = O_p(H)M = P_2M$ and hence $O_p(H) = P_2$ since $O_p(H) \cap M = P_2 \cap M$, a contradiction. Thus $O_p(H)\cap M = P_2 \cap M = P_2$ and hence $|F(H)\colon F(H)\cap M| = |G\colon M| = |O_p(H)\colon O_p(H) \cap M| = p$, a contradiction.

Finally we may assume $(P_2)_{QG} < P_2$. For any maximal subgroup T of P_2 containing $(P_2)_{QG}$, we have $|G\colon TB| = p$ by Lemma 2.1(5). Clearly, TB is a maximal subgroup of G. Then $O_p(H) \cap TB \trianglelefteq G$, which implies that $(O_p(H) \cap TB)M$ is a subgroup of G. If $(O_p(H) \cap TB)M = G = O_p(H)M$, then $O_p(H) \cap TB = O_p(H)$ since $(O_p(H) \cap TB) \cap M = O_p(H) \cap M$. This leads to $O_p(H) \leq TB$, and hence $G = O_p(H)TB = TB$, which is contrary to the above hypotheses on TB. Thus $O_p(H) \cap TB \leq M$. Furthermore, $P_2 \cap TB \leq O_p(H) \cap TB \leq O_p(H) \cap M \leq (P_2)_{QG} \leq P_2 \cap TB$, from this, $O_p(H) \cap TB = O_p(H) \cap M = P_2 \cap TB$. This is contrary to $G = P_2B = O_p(H)B$.

The final contradiction completes our proof.

***Corollary* 3.2.** *Let G be a group with a solvable normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F(H)$ having no supersolvable supplement in G is Q-permutable in G, then $G \in \mathcal{U}$.*

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F^*(H)$ having no supersolvable supplement in G is Q-permutable in G, then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is false and choose G to be a counterexample of minimal order; so in particular, $H \neq 1$. We consider the following two cases.

Case 1. $\mathcal{F} = \mathcal{U}$.

By Corollary 3.1 we easily verify that $F^*(H)$ is supersolvable and hence $F(H) = F^*(H) \neq 1$ by Lemma 2.6(3). Since H satisfies the hypotheses of the theorem, the minimal choice of G implies that H is supersolvable if $H < G$. Then $G \in \mathcal{U}$ by Corollary 3.2, a contradiction. Thus we have

(1) $H = G$, $F^*(G) = F(G) \neq 1$.

Let S be a proper normal subgroup of G containing $F^*(G)$. By Lemma 2.6(1), $F^*(G) = F^*(F^*(G)) \leq F^*(S) \leq F^*(G)$, so $F^*(S) = F^*(G)$. By hypotheses and Lemma 2.1(1), every maximal subgroup of every noncyclic Sylow subgroup of $F^*(S)$ having no supersolvable supplement in S is Q-permutable in S. Hence S is supersolvable by the minimal choice of G and we get

(2) Every proper normal subgroup of G containing $F^*(G)$ is supersolvable.

Suppose now that $\Phi(O_p(G)) \neq 1$ for some $p \in \pi(F(G))$. By Lemma 2.6(5) we have $F^*(G/\Phi(O_p(G))) = F^*(G)/\Phi(O_p(G))$. Using Lemma 2.1(2) we observe that the pair $(G/\Phi(O_p(G)), F^*(G)/\Phi(O_p(G)))$ satisfies the hypotheses of the theorem. The minimal choice of G then implies $G/\Phi(O_p(G)) \in \mathcal{U}$. Since $\mathcal{U}$ is a saturated formation, we get $G \in \mathcal{U}$, a contradiction. Thus we have

(3) If $p \in \pi(F(G))$, then $\Phi(O_p(G)) = 1$ and $O_p(G)$ is elementary abelian; in particular, $F^*(G) = F(G)$ is abelian and $C_G(F(G)) = F(G)$.

If L is a minimal normal subgroup of G contained in $F(G)$ and $|L| = p$ where $p \in \pi(F(G))$, then set $C = C_G(L)$. Clearly, $F(G) \leq C \trianglelefteq G$. If $C < G$, then C is solvable by (2). On the other hand, since G/C is cyclic, then we have G is solvable, a contradiction. So we may assume $C = G$. Now we have $L \leq Z(G)$. Then we consider subgroup G/L. By Lemma 2.6(6), we have $F^*(G/L) = F^*(G)/L = F(G)/L$. In fact, G/L satisfies the condition of the theorem by Lemma 2.1. Therefore the minimal choice of G implies that $G/L \in \mathcal{U}$ and hence G is supersolvable, a contradiction. This proves

(4) There is no normal subgroup of prime order in G contained in $F(G)$.

If every Sylow subgroup of $F(G)$ is cyclic,then $F(G) = H_1 \times \ldots \times H_r$ where H_i, $i = 1, \ldots, r$, is the cyclic Sylow subgroup of $F(G)$ and hence $G/C_G(H_i)$ is abelian for any $i \in \{1, \ldots, r\}$. Moreover, we have $G/\bigcap_{i=1}^{r} C_G(H_i) = G/C_G(F(G))$ is abelian and hence $G/F(G)$ is abelian since $C_G(F(G)) = C_G(F^*(G)) \leq F(G)$. Therefore G is solvable, a contradiction. This proves that

(5) There exists noncyclic Sylow subgroup $O_p(G)$ of $F(G)$ for some prime $p \in$ $\in \pi(F(G))$.

Let P_1 be a maximal subgroup of $O_p(G)$. If P_1 has a supersolvable supplement in G, then there exists a supersolvable subgroup K of G such that $G = P_1K =$ $= O_p(G)K$. Clearly, $G/O_p(G) \cong K/K \cap O_p(G)$ is supersolvable and hence G is solvable, a contradiction. So we obtain that

(6) Every maximal subgroup of every noncyclic Sylow subgroup of $F(G)$ has no supersolvable supplement in G.

Set $R = O_p(G) \cap \Phi(G)$. If $R = 1$, then by Lemma 2.2, $O_p(G)$ is the direct product of some minimal normal subgroup of G. So we may assume that $O_p(G) = R_1 \times \ldots \times R_t$, where R_i is a minimal normal subgroup of G, $i = 1, 2, \ldots, t$. Consider the maximal subgroup P_1 of P, where P_1 has the following form:

$$P_1 = R_1 \times \ldots \times R_{i-1} \times {R_i}^* \times R_{i+1} \times \ldots \times R_t.$$

Where ${R_i}^*$ is a maximal subgroup of R_i for some i. By hypotheses and (6), P_1 is Q-permutable in G. Let T denote the normal subgroup $R_1 \times \ldots \times R_{i-1} \times R_{i+1} \times \ldots \times R_t$ of G, then $P_1 = {R_i}^*T$. Let T denote the normal subgroup $R_1 \times \ldots \times R_{i-1} \times R_{i+1} \times \ldots \times R_t$

of G, then $P_1 = R_i{}^*T$. Clearly, $T \leq (P_1)_{QG}$ and P/T is the minimal normal subgroup of G/T. Since $(P_1)_{QG}/T$ is a permutable subgroup of G/T, by [11] a minimal normal subgroup of G is also the minimal permutable subgroup of G, we have $(P_1)_{QG} = T$. By hypotheses P_1 is Q-permutable in G, there exists a subgroup B of G such that $G = P_1B$ and $SB = BS < G$ for any maximal subgroup S of P_1 containing $(P_1)_{QG} = T$. By Lemma 2.1(5) we have $|G\colon SB| = p$. Since R_i is the minimal normal subgroup of G, we have $R_i \cap SB \in \{1, R_i\}$. Clearly, if $R_i \leq SB$, then we have $SB = R_iSB = G$, a contradiction. Hence we have $R_i \nleq SB$, we know that $|R_i| = p$, contrary to (4). This contradiction leads to

(7) $R = O_p(G) \cap \Phi(G) \neq 1$.

Let Q be a Sylow q-subgroup of $F(G)$, and let L be a minimal normal subgroup of G contained in R, where $q \neq p$. Then Q is elementary abelian by (3). By the definition of a generalized Fitting subgroup, $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G/L), E(G/L)] = 1$, where $E(G/L)$ is the layer of G/L. Since $L \leq \Phi(G)$, $F(G/L) = F(G)/L$. Now set $E/L = E(G/L)$. Since Q is normal in G and $[F(G)/L, E/L] = 1$, $[Q, E] \leq Q \cap L = 1$, i.e., $[Q, E] = 1$. Therefore $F(G)E \leq C_G(Q)$. If $C_G(Q) < G$, then $C_G(Q)$ is supersolvable by (2). Thus $E(G/L) = E/L$ is supersolvable. The semisimplicity of $E(G/L)/Z(E(G/L))$ implies that $E(G/L) = Z(E(G/L))$. So $E(G/L) \leq F(G/L)$ and $F^*(G/L) = F(G)/L$, with the same argument in (3), we have that G/L satisfies the hypotheses of the theorem. By the minimal choice of G, G/L is supersolvable and so is G, a contradiction. If $C_G(Q) = G$, then $Q \leq Z(G)$. By Lemma 2.6(6), $F^*(G/Q) = F^*(G)/Q = F(G)/Q$. Similarly, G/Q is supersolvable and so is G by Corollary 3.2, a contradiction. This verifies

(8) $F(G) = O_p(G)$.

If $R = O_p(G)$, then by hypotheses every maximal subgroup P_1 of $O_p(G)$ is Q-permutable in G. That is, there exists a subgroup B of G such that $G = P_1B$ and $TB < G$ for any maximal subgroup T of P_1 containing $(P_1)_{QG}$. Then $G = P_1B = B$ since $P_1 \leq \Phi(G)$, a contradiction. Hence $R \neq O_p(G)$. Now $\Phi(G/R) = 1$. Then by Lemma 2.2, $O_p(G)/R = (H_1/R) \times \ldots \times (H_m/R)$, where H_i/R, $i = 1, \ldots, m$, are minimal normal in G/R. With the same argument as in (7), we know that H_i/R, $i = 1, \ldots, m$, are all of order p because all maximal subgroups of $O_p(G)/R$ are Q-permutable in G/R by Lemma 2.1(2). Again, since $O_p(G)$ is an elementary abelian p-group, H_i is of the form $\langle x_i\rangle R$, $i = 1, \ldots, m$. This proves

(9) $O_p(G) = \langle x_1\rangle \times \ldots \times \langle x_m\rangle \times R$ where $\langle x_i\rangle \neq 1$ and $\langle x_i\rangle R \trianglelefteq G$, $i = 1, \ldots, m$.

Now let L be a minimal normal subgroup of G contained in R and set $\overline{G} := G/L$. Clearly, $F(\overline{G}) = F(G)/L = O_p(G)/L$, because $L \leq \Phi(G)$. If $F^*(\overline{G}) = F(\overline{G})$, then we easily verify that G/L satisfies the hypotheses of the theorem, and thus $G/L \in \mathcal{U}$ by the minimal choice of G. Since $L \leq \Phi(G)$ and $\mathcal{U}$ is saturated, we get $G \in \mathcal{U}$, a contradiction. Therefore $F^*(\overline{G}) = F(\overline{G})E(\overline{G}) > F(\overline{G})$ and so there exists a perfect normal subgroup E in G such that $EL/L = E(\overline{G})$. Clearly, $O_p(G)E$ is a nonsolvable normal subgroup of G; hence, by (2) and (1), $G = O_p(G)E$. In particular, $\overline{G} = \overline{O_p(G)}E(\overline{G}) = F^*(\overline{G})$ with $[\overline{O_p(G)}, E(\overline{G})] = [F(\overline{G}), E(\overline{G})] = 1$ and hence $[O_p(G), E] \leq L$. Since L is minimal normal in G and since $[O_p(G), E] \trianglelefteq G$ as well as $C_G(O_p(G)) = O_p(G)$, we get $[O_p(G), E] = L$. Therefore we have the following:

(10) $G = O_p(G)E$ with $L = [O_p(G), E] \leq O_p(G) \cap E$, $\overline{G} = \overline{O_p(G)}\,\overline{E} = F^*(\overline{G})$ and $\overline{O_p(G)} \leq Z(\overline{G})$.

Now assume that M is a minimal normal subgroup of G contained in $O_p(G)$ with $M \neq L$. Then $\overline{M} = ML/L$ is a minimal normal subgroup of $\overline{G}$ contained in $Z(\overline{G})$. As $M \cap L = 1$, we get $|M| = |\overline{M}| = p$, contrary to (4). This proves

(11) L is the unique minimal normal subgroup of G contained in $O_p(G)$.

Now let T be a complement of L in $O_p(G)$ and set $P_1 := TL_1$ where L_1 is a maximal subgroup of L. Then P_1 is a maximal subgroup of $O_p(G)$ and so, by hypotheses, is Q-permutable in G with $G = P_1B = PB$ and $SB = BS < G$ for any maximal subgroup S containing $(P_1)_{QG}$. Obviously, $(P_1)_{QG}$ is normalized by $O_p(G)O^p(G) = O_p(G)O^p(E) = O_p(G)E = G$. Since $(P_1)_{QG}$ does not contain L, we conclude $(P_1)_{QG} = 1$ by (11).

By Lemma 2.1(5), $|G\colon SB| = p$. Clearly, SB is a maximal subgroup of G, and $L \cap SB \in \{1, L\}$. If $L \cap SB = 1$, then $|L| = p$, this is contrary to (4). So we have $L \leq SB$ for any maximal subgroup S of P_1. Furthermore, if $L \cap P_1 = 1$, then we also have $|L| = p$, a contradiction. So we get $L \cap P_1 \neq 1$. We claim that $L \cap P_1 \leq S$ for any maximal subgroup S of P_1. Otherwise, there exists a maximal subgroup S of P_1 such that $L \cap P_1 \nleq S$. So we consider $SB = (L \cap P_1)SB = P_1B = G$, a contradiction. Based on the discussion as above, we have $1 < L \cap P_1 \leq \Phi(P_1) \leq \Phi(O_p(G))$, contrary to (3), thereby completing the proof for Case I .

Case 2. $\mathcal{F} \neq \mathcal{U}$.

By case 1, H is supersolvable. Particularly, H is solvable and hence $F^*(H) = F(H)$. Therefore $G \in \mathcal{F}$ by Corollary 3.2.

The final contradiction completes our proof.

***Corollary* 3.3.** *Let G be a group with a normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F^*(H)$ having no supersolvable supplement in G is Q-permutable in G, then $G \in \mathcal{U}$.*

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