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ON THE GENERALIZED CONVOLUTION FOR F_c, F_s, AND $K-L$ INTEGRAL TRANSFORMS*

ПРО УЗАГАЛЬНЕНУ ЗГОРТКУ ДЛЯ F_c, F_s ТА $K-L$ ІНТЕГРАЛЬНИХ ПЕРЕТВОРЕНЬ

We study new generalized convolutions $f \overset{\gamma}{*} g$ with weight function $\gamma(y) = y$ for the Fourier cosine, Fourier sine, and Kontorovich – Lebedev integral transforms in weighted function spaces with two parameters $L(\mathbb{R}_+, x^\alpha e^{-\beta x} dx)$. These generalized convolutions satisfy the factorization equalities

$$F_{\left\{ s \atop c \right\}}(f \overset{\gamma}{*} g)_{\left\{ 1 \atop 2 \right\}}(y) = y(F_{\left\{ c \atop s \right\}} f)(y)(K_{iy} g) \quad \forall y > 0.$$

We establish a relationship between these generalized convolutions and known convolutions, and also relations that associate them with other convolution operators. As an example, we use these new generalized convolutions for the solution of a class of integral equations with Toeplitz-plus-Hankel kernels and a class of systems of two integral equations with Toeplitz-plus-Hankel kernels.

Вивчаються нові узагальнені згортки $f \overset{\gamma}{*} g$ з ваговою функцією $\gamma(y) = y$ для косинус-Фур'є, синус-Фур'є та Конторовича – Лебедєва інтегральних перетворень у вагових функціональних просторах з двома параметрами $L(\mathbb{R}_+, x^\alpha e^{-\beta x} dx)$. Для цих узагальнених згорток справджуються функціональні рівності

$$F_{\left\{ s \atop c \right\}}(f \overset{\gamma}{*} g)_{\left\{ 1 \atop 2 \right\}}(y) = y(F_{\left\{ c \atop s \right\}} f)(y)(K_{iy} g) \quad \forall y > 0.$$

Одержано співвідношення між цими узагальненими згортками та відомими згортками, а також відповідні співвідношення з іншими операторами згорток. Як приклад, ці нові узагальнені згортки застосовано до класу інтегральних рівнянь з сумою ядер Тепліца і Ганкеля, а також до класу системи двох інтегральних рівнянь з сумою ядер Тепліца і Ганкеля.

Introduction. The commutative convolution of two functions f and g for the Fourier cosine transform is well known [16]:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(x+y) + g(|x-y|)]dy, \quad x > 0. \tag{0.1}$$

For $f, g \in L_1(\mathbb{R}_+)$, this convolution belongs to $L_1(\mathbb{R}_+)$, and the following identity holds:

$$F_c(f * g)(y) = (F_c f)(y)(F_c g)(y) \quad \forall y \in \mathbb{R}, \tag{0.2}$$

where F_c denotes the Fourier cosine transform [16]. In 1967, Kakichev gave a constructive method for defining a convolution with weight function for an arbitrary integral transform (see [11]). On the basis of this method, a convolution of two functions f and g with weight function $\gamma(y) = \sin y$ for the Fourier sine transform was introduced in [11]:

$$(f \underset{F_s}{\overset{\gamma}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[\operatorname{sign}(x+y-1)g(|x+y-1|) + \operatorname{sign}(x-y+1)g(|x-y+1|) -$$

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$$-g(x+y+1)-\operatorname{sign}(x-y-1)g(|x-y-1|)]dy, \quad x>0. \tag{0.3}$$

For $f,g\in L_1(\mathbb{R}_+)$, the convolution $f \overset{\gamma}{\underset{F_c}{*}} g$ belongs to $L_1(\mathbb{R}_+)$ and the following factorization identity holds:

$$F_s(f \overset{\gamma}{\underset{F_s}{*}} g)(y)=\sin y(F_sf)(y)(F_sg)(y) \quad \forall y>0. \tag{0.4}$$

In 1998, Kakichev and Thao introduced a constructive method for defining a generalized convolution with weight function for three arbitrary integral transforms (see [12]), which seems to be very important in convolution theory. The following noncommutative generalized convolution of two functions f and g for the Fourier sine and Fourier cosine transforms was studied in [16]:

$$(f \underset{1}{*} g)(x)=\frac{1}{\sqrt{2\pi}}\int\limits_0^\infty f(u)[g(|x-u|)-g(x+u)]du, \quad x>0. \tag{0.5}$$

If $f,g\in L_1(\mathbb{R}_+)$, then $(f \underset{1}{*} g)$ belongs to $L_1(\mathbb{R}_+)$ and satisfies the following identity:

$$F_s(f \underset{1}{*} g)(y)=(F_sf)(y)(F_cg)(y) \quad \forall y>0. \tag{0.6}$$

Here, F_s is the Fourier sine transform [17]. The following commutative generalized convolution of two functions f and g for the Fourier cosine and sine transforms was defined in [13]:

$$(f \underset{2}{*} g)(x)=\frac{1}{\sqrt{2\pi}}\int\limits_0^\infty f(u)[\operatorname{sign}(u-x)g(|u-x|)+g(u+x)]du, \quad x>0. \tag{0.7}$$

For $f,g\in L_1(\mathbb{R}_+)$, this generalized convolution belongs to $L_1(\mathbb{R}_+)$ and the following factorization equality holds:

$$F_c(f \underset{2}{*} g)(y)=(F_sf)(y)(F_sg)(y) \quad \forall y>0. \tag{0.8}$$

The generalized convolution with weight function $\gamma(y)=\sin y$ for the Fourier cosine and sine transforms of f and g has the form [14]

$$(f \overset{\gamma}{\underset{3}{*}} g)(x)=\frac{1}{2\sqrt{2\pi}}\int\limits_0^\infty f(u)[g(|x+u-1|)+g(|x-u+1|)-g(x+u+1)-$$

$$-g(|x-u-1|)]du, \quad x>0. \tag{0.9}$$

For $f,g\in L_1(\mathbb{R}_+)$, $(f \overset{\gamma}{\underset{3}{*}} g)(x)$ also belongs to $L_1(\mathbb{R}_+)$ and

$$F_c(f \overset{\gamma}{\underset{1}{*}} g)(y)=\sin y\,(F_sf)(y)(F_cg)(y) \quad \forall y>0. \tag{0.10}$$

The Kontorovich – Lebedev transform is of the form [17]

$$K_{iy}[f]=\int\limits_0^\infty K_{iy}(x)f(x)dx,$$

where $K_{ix}(t)$ is the modified Bessel function [2].

Throughout this paper, we are interested in the following function spaces of two parameters:

$$L_p^{\alpha,\beta}(\mathbb{R}_+) \equiv L_p(\mathbb{R}_+; x^\alpha e^{-\beta x}dx), \qquad \alpha \in \mathbb{R}, \quad 0 < \beta \leqslant 1.$$

The norm of a function f in this space is defined as follows:

$$\|f\|_{L_p^{\alpha,\beta}(\mathbb{R}_+)} = \left(\int\limits_0^\infty |f(x)|^p x^\alpha e^{-\beta x} dx \right)^{1/p}.$$

In recent years, there has been much interest in convolution theory for integral transforms, and several interesting applications have been considered (see [4, 6, 7, 18, 20]), in particular, the integral equations with Toeplitz-plus-Hankel kernel [10, 15, 19]

$$f(x) + \int\limits_0^\infty [k_1(x+y) + k_2(x-y)] f(y)dy = g(x), \quad x > 0, \tag{0.11}$$

where k_1, k_2, g are known functions and f is an unknown function. The problem of solving this equation in a closed form in the general case of a Toeplitz kernel k_1 and Hankel kernel k_2 remains open. Many partial cases of this equation can be solved in a closed form with the help of convolutions and generalized convolutions and have interesting applications in biology and medicine (see [8, 9]).

In this paper, we construct and investigate two generalized convolutions for the Fourier cosine, Fourier sine, and Kontorovich – Lebedev transforms in the function spaces $L_p^{\alpha,\beta}(\mathbb{R}_+)$. Applications to the solution, in a closed form, of a class of integral equations with Toeplitz-plus-Hankel kernels and systems of two integral equations with Toeplitz-plus-Hankel kernels are considered.

1. Noncommutative generalized convolutions.

Definition 1.1. *The generalized convolution of two functions f and g with weight function $\gamma(y) = y$ for the Fourier cosine, Fourier sine, and Kontorovich – Lebedev integral transforms is defined as follows:*

$$(f \overset{\gamma}{*} g)_{\left\{ {1 \atop 2} \right\}}(x) = \frac{1}{2} \int\limits_{\mathbb{R}_+^2} v \Big[\sinh(x+u) e^{-v\cosh(x+u)} \pm \sinh(x-u) e^{-v\cosh(x-u)} \Big] f(u) g(v) du dv. \tag{1.1}$$

For convenience, throughout this paper we use the following notation:

$$\theta_{\left\{ {1 \atop 2} \right\}}(x,u,v) = \sinh(x+u) e^{-v\cosh(x+u)} \pm \sinh(x-u) e^{-v\cosh(x-u)}, \quad x > 0.$$

Theorem 1.1. *Let $f \in L_1(\mathbb{R}_+)$, $g \in L_1^{0,\beta}(\mathbb{R}_+)$, and $0 < \beta \leqslant 1$. Then the generalized convolution* (1.1) *belongs to $L_1(\mathbb{R}_+)$ and satisfies the factorization equalities*

$$F_{\left\{ {s \atop c} \right\}} (f \overset{\gamma}{*} g)_{\left\{ {1 \atop 2} \right\}}(y) = y (F_{\left\{ {c \atop s} \right\}} f)(y) (K_{iy} g) \quad \forall y > 0. \tag{1.2}$$

Moreover, the following estimates are true:

$$\left\| (f \overset{\gamma}{*} g)_{\left\{ {1 \atop 2} \right\}} \right\|_{L_1(\mathbb{R}_+)} \leqslant \|f\|_{L_1(\mathbb{R}_+)} \|g\|_{L_1^{0,\beta}(\mathbb{R}_+)}. \tag{1.3}$$

Furthermore, the generalized convolution (1.1) *belongs to* $C_0(\mathbb{R}_+)$, *and the following Parseval-type identities are true*:

$$(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty y(F_{\left\{\substack{c\\s}\right\}} f)(y)(K_{iy}g) \begin{Bmatrix} \sin xy \\ \cos xy \end{Bmatrix} dy. \tag{1.4}$$

Proof. We have

$$\int\limits_0^\infty v|\sinh(x+u)^{-v\cosh(x+u)} \pm \sinh(x-u)^{-v\cosh(x-u)}|dx \leqslant$$

$$\leqslant \int\limits_0^\infty v\big[|\sinh(x+u)|^{-v\cosh(x+u)} + |\sinh(x-u)|^{-v\cosh(x+u)}\big]dx =$$

$$= \int\limits_u^\infty v\sinh t e^{-v\cosh t}dt + \int\limits_{-u}^\infty v|\sinh t|e^{-v\cosh t}dt = 2\int\limits_0^\infty v\sinh t e^{-v\cosh t}dt = 2e^{-v}. \tag{1.5}$$

Using (1.5) and the Fubini theorem, we get

$$\int\limits_0^\infty |(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x)|dx \leqslant \int\limits_{\mathbb{R}_+^2} e^{-v}|f(u)||g(v)|dudv \leqslant$$

$$\leqslant \int\limits_{\mathbb{R}_+^2} e^{-\beta v}|f(u)||g(v)|dudv = \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1^{0,\beta}(\mathbb{R}_+)}. \tag{1.6}$$

This implies the existence of the generalized convolution (1.1) in $L_1(\mathbb{R}_+)$ and the validity of relation (1.3). Note that integral (1.1) is absolutely convergent. Indeed, we have

$$v\sinh(x+u)e^{-v(\cosh(x+u)-1)} = v\sinh(x+u)e^{-v\sinh(x+u)\tanh\frac{x+u}{2}} \leqslant$$

$$\leqslant v\sinh(x+u)\big(e^{\tanh\frac{x+u}{2}}\big)^{-v\sinh(x+u)} \leqslant e^{-\tanh\frac{x+u}{2}} \leqslant 1. \tag{1.7}$$

Similarly,

$$|v\sinh(x-u)e^{-v(\cosh(x-u)-1)}| \leqslant 1.$$

Then relation (1.7) yields

$$\left|\theta_{\left\{\substack{1\\2}\right\}}(x,u,v)\right| \leqslant 2e^{-v}. \tag{1.8}$$

Therefore,

$$|(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x)| \leqslant \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1^{0,\beta}(\mathbb{R}_+)}, \tag{1.9}$$

Further, using relation (12.1.1) from [3, p. 130] (Theorem 2) and the Fubini theorem, we get

$$(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x) = \frac{2}{\pi}\int\limits_{\mathbb{R}_+^3} yf(u)g(v)\begin{Bmatrix}\sin xy\cos yu\\ \cos xy\sin yu\end{Bmatrix}K_{iy}(v)dudvdy =$$

$$= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty y(F_{\left\{\begin{smallmatrix}c\\s\end{smallmatrix}\right\}}f)(y)(K_{iy}g)\begin{Bmatrix}\sin xy\\ \cos xy\end{Bmatrix}dy.$$

Then the Parseval-type identity (1.4) holds for $f \in L_1(\mathbb{R}_+)$ and $g \in L_1^{0,\beta}(\mathbb{R}_+)$, $0<\beta\leqslant 1$.

Using the Parseval-type identity (1.4), relation (1.9), and the reverse formula of the Fourier cosine and Fourier sine transforms, one can easily obtain the factorization property (1.2).

On the other hand, integral (1.1) is absolutely convergent in x, and it follows from the Riemann–Lebesgue lemma that $(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}} \in C_0(\mathbb{R}_+)$.

Theorem 1.1 is proved.

An extension of Theorem 1.1 to the spaces $L_p(\mathbb{R}_+)$ and $L_p^{0,\beta}(\mathbb{R}_+)$ is given as follows:

Theorem 1.2. *Let* $f \in L_p(\mathbb{R}_+)$, $g \in L_q^{0,\beta}(\mathbb{R}_+)$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, $p,q>1$, *and* $0<\beta\leqslant 1$. *Then the generalized convolutions* (1.1) *exist for all* $x>0$, *belong to* $L_r^{\alpha,\gamma}(\mathbb{R}_+)$, $\alpha>-1$, $\gamma>0$, $r\geqslant 1$, *and satisfy the relation*

$$\|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant \gamma^{-(\alpha+1)/r}\Gamma^{1/r}(\alpha+1)\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q^{0,\beta}(\mathbb{R}_+)}. \tag{1.10}$$

If, in addition, $f \in L_1(\mathbb{R}_+)\cap L_p(\mathbb{R}_+)$, *then the factorization equalities* (1.2) *are true. Moreover, the generalized convolutions* (1.1) *belong to* $C_0(\mathbb{R}_+)$, *and the Parseval-type identities* (1.3) *are true.*

Proof. Using the Hölder inequality and relations (1.5) and (1.8), we obtain

$$|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)| \leqslant \frac{1}{2}\left(\int\limits_{\mathbb{R}_+^2}|f(u)|^p 2e^{-v}dudv\right)^{1/p}\left(\int\limits_0^\infty |g(v)|^q 2e^{-v}dv\right)^{1/q} =$$

$$= \left(\int\limits_0^\infty |f(u)|^p du\right)^{1/p}\left(\int\limits_0^\infty |g(v)|^q e^{-v}dv\right)^{1/q} = \|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q^{0,\beta}(\mathbb{R}_+)}. \tag{1.11}$$

On the other hand, using relations (3.225.3) from [3, p. 165] and (1.11), we get

$$\|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant \left(\int\limits_0^\infty x^\alpha e^{-\gamma x}dx\right)^{1/r}\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q^{0,\beta}(\mathbb{R}_+)} =$$

$$= \left(\gamma^{-(\alpha+1)}\Gamma(\alpha+1)\right)^{1/r}\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q^{0,\beta}(\mathbb{R}_+)}, \quad \alpha>-1, \quad \gamma>0.$$

Estimate (1.10) is proved.

Furthermore, since $L_q^{0,\beta}(\mathbb{R}_+) \subset L_1^{0,\beta}(\mathbb{R}_+)$, Theorem 1.1 shows that $(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}} \in L_1(\mathbb{R}_+)$. Therefore, using (1.11), relation (12.1.1) from [3, p. 130] (Theorem 2), and the Fubini theorem, we obtain the Parseval-type identity (1.4), and, hence, the factorization identity (1.2) is true. Finally, it follows from relation (1.11) and the Riemann – Lebesgue lemma that $(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}} \in C_0(\mathbb{R}_+)$.

Theorem 1.2 is proved.

***Corollary* 1.1.** *Under the same assumptions as in Theorem* 1.2, *the generalized convolutions* (1.1) *exist for all positive* x, *are continuous, belong to* $L_p(\mathbb{R}_+)$, *and satisfy the following estimate*:

$$\|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)\|_{L_p(\mathbb{R}_+)} \leqslant \|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q^{0,\beta}(\mathbb{R}_+)}. \tag{1.12}$$

In particular, for $p = 2$, *we obtain a Parseval identity of the Fourier type*:

$$\int_0^\infty |(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)|^2 dx = \int_0^\infty |y(F_{\left\{\begin{smallmatrix}c\\s\end{smallmatrix}\right\}} f)(y)(K_{iy}g)|^2 dy. \tag{1.13}$$

Proof. Using relations (1.5), (1.7), and (1.8) and the Hölder inequality, we get

$$\int_0^\infty |(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)|^p dx \leqslant \frac{1}{2^p} \int_0^\infty \left\{ \int_{\mathbb{R}_+^2} \Big| f(u)|^p v| \sinh(x+u) e^{-v\cosh(x+u)} \pm \right.$$

$$\pm \sinh(x-u) e^{-v\cosh(x-u)} \Big| dudv \left(\int_{\mathbb{R}_+^2} |g(v)|^q v| \sinh(x+u) e^{-v\cosh(x+u)} \pm \right.$$

$$\left. \left. \pm \sinh(x-u) e^{-v\cosh(x-u)} | dudv \right)^{p/q} \right\} dx \leqslant$$

$$\leqslant \frac{1}{2^p} \int_{\mathbb{R}_+^2} |f(u)|^p 2e^{-v} dudv \left(\int_0^\infty |g(v)|^q 2e^{-v} dv \right)^{p/q}.$$

Therefore,

$$\|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)\|_{L_p(\mathbb{R}_+)} \leqslant \|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q^{0,\beta}(\mathbb{R}_+)}.$$

Estimate (1.12) is proved. Moreover, $(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)$ is continuous and belongs to $L_p(\mathbb{R}_+)$.

In the case $p = 2$, we obtain the Parseval identity

$$\|F_{\left\{\begin{smallmatrix}c\\s\end{smallmatrix}\right\}}\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.$$

Therefore, the factorization identity (1.2) yields a Parseval identity of the Fourier type.

Corollary 1.1 is proved.

***Corollary* 1.2.** 1. *Let* $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, $g \in L_2^{0,\beta}(\mathbb{R}_+)$, *and* $0 < \beta \leqslant 1$. *Then the generalized convolutions* (1.1) *exist, are continuous, belong to* $L_r^{\alpha,\gamma}(\mathbb{R}_+)$, $r \geqslant 1, \gamma > 0$, $\alpha > -1$, *and satisfy the following estimate*:

$$\|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant (\gamma^{-(\alpha+1)}\Gamma(\alpha+1))^{1/r}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_2^{0,\beta}(\mathbb{R}_+)}. \tag{1.14}$$

Furthermore, these generalized convolutions satisfy the factorization identities (1.2) *and Parseval-type identities* (1.4).

2. *Let* $f, g \in L_1(\mathbb{R}_+)$. *Then the generalized convolutions* (1.1) *exist, belong to* $L_r^{\alpha,\gamma}(\mathbb{R}_+)$, $r \geqslant 1$, $\gamma > 0$, $\alpha > -1$, *and satisfy the following estimate*:

$$\|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant (\gamma^{-(\alpha+1)}\Gamma(\alpha+1))^{1/r}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}. \tag{1.15}$$

Moreover, the factorization identities (1.2) *and Parseval-type identities* (1.4) *are true.*

Proof. Using the Schwarz inequality and relations (1.7), (1.8), and (1.5), we get

$$|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)| \leqslant \frac{1}{2}\left(\int\limits_{R_+^2} |f(u)|2e^{-v}dv\right)^{1/2}\left(\int\limits_{\mathbb{R}_+^2} |f(u)||g(v)|^2 2e^{-v}dudv\right)^{1/2} \leqslant$$

$$\leqslant \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_2^{0,\beta}(\mathbb{R}_+)}.$$

Therefore, using relation (3.225.3) from [5, p. 165], we obtain

$$\|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant (\gamma^{-(\alpha+1)}\Gamma(\alpha+1))^{1/r}\|f\|_{L_2(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.$$

This yields estimate (1.14). Moreover, by virtue of Theorem 1.2, the factorization identities (1.2) and Parseval-type identities (1.4) are true.

On the other hand, using the Schwarz inequality and relations (1.7) and (1.8), we get

$$|(f \overset{\gamma}{*} g)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x)| \leqslant \frac{1}{2}\left(\int\limits_0^\infty |f(u)||g(v)|2e^{-v}dudv\right)^{1/2}\left(\int\limits_{\mathbb{R}_+^2} |f(u)||g(v)|2e^{-v}dudv\right)^{1/2} \leqslant$$

$$\leqslant \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.$$

Therefore, using relation (3.225.3) from [5, p. 165], we obtain (1.15). Furthermore, using Theorem 1.1, we obtain the factorization identities (1.2) and Parseval-type identities (1.4).

Corollary 1.2 is proved.

Since $L_1^{0,\beta}(\mathbb{R}_+) \subset L_1(\mathbb{R}_+)$, by using relations (1.1), (0.7), and (0.5) one can easily prove the following assertion:

Proposition 1.1. *Let* $f \in L_1(\mathbb{R}_+)$ *and* $g \in L_1^{0,\beta}(\mathbb{R}_+)$. *Then*

$$(f * g)_1(x) = \sqrt{\frac{\pi}{2}}\int\limits_0^\infty vg(v)\big(f(u) \underset{2}{*} \sinh u\, e^{-v\cosh u}\big)(x)dv,$$

$$(f * g)_2(x) = -\sqrt{\frac{\pi}{2}} \int\limits_0^\infty v g(v) \big(f(u) \underset{1}{*} \sinh u \, e^{-v \cosh u}\big)(x) dv.$$

Proof. Using (1.1) and (0.7), we obtain the representation for the convolution $(f * g)_1(x)$. On the other hand, one can easily prove the representation for the convolution $(f * g)_2(x)$ by using (0.5).

Using Theorem 1.1 and relations (0.6), (0.8), (0.4), and (0.10), we obtain the following proposition:

Proposition 1.2. *Let* $f, g \in L_1(\mathbb{R}_+)$, $h \in L_1^{0,\beta}(\mathbb{R}_+)$, *and* $0 < \beta \leqslant 1$. *Then the following equalities are true*:

(a) $f \underset{1}{*} (g * h)_1 = g \underset{1}{*} (f * h)_1$,

(b) $f \underset{2}{*} (g * h)_2 = ((f \underset{1}{*} g) * h)_2$,

(c) $((f \underset{2}{*} g) * h)_2 = f \underset{1}{*} (g * h)_1$,

(d) $f \overset{\gamma}{\underset{3}{*}} (g * h)_1 = ((f \overset{\gamma}{\underset{F_s}{*}} g) * h)_1$.

2. Applications. Integral equations with Toeplitz-plus-Hankel kernels were studied in [15, 19]. In this section, we consider a partial class of these integrals, namely, the integral equations

$$f(x) + \int\limits_0^\infty (\varphi \underset{1}{*} f)(y)[k(x+y) - k(x-y)] dy = h(x), \tag{2.1}$$

where

$$k(t) = \frac{1}{2} \int\limits_0^\infty v \sinh t e^{-v \cosh t} g(v) dv$$

φ, g, and h are given, and f is unknown.

Theorem 2.1. *Let* $\varphi, h \in L_1(\mathbb{R}_+)$ *and* $g \in L_1^{0,\beta}(\mathbb{R}_+)$ *be such that* $1 + y(F_s\varphi)(y) \times \times (K_{iy} g) \neq 0$ *for all positive* y. *Then Eq.* (2.1) *has a unique solution in* $L_1(\mathbb{R}_+)$, *which is determined as follows:*

$$f(x) = h(x) - (h * l)(x),$$

where $l \in L_1(\mathbb{R}_+)$ *is defined by the formula* $(F_c l)(y) = \dfrac{y(F_s\varphi)(y)(K_{iy}g)}{1 + y(F_s\varphi)(y)(K_{iy}g)}$.

Proof. Using Theorem 1.1 and relation (0.6) and applying the Fourier cosine transform to both sides of (2.1), we get

$$(F_c f)(y) + y(F_s\varphi)(y)(F_c f)(y)(K_{iy} g) = (F_c h)(y).$$

By virtue of the factorization equality (1.2), we obtain

$$(F_c f)(y) = (F_c h)(y) \left[1 - \frac{y(F_s\varphi)(y)(K_{iy}g)}{1 + y(F_s\varphi)(y)(K_{iy}g)}\right]. \tag{2.2}$$

According to the Wiener–Levy theorem [1], there is a function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{y(F_s\varphi)(y)(K_{iy}g)}{1 + y(F_s\varphi)(y)(K_{iy}g)}. \tag{2.3}$$

Using (2.2) and (2.3), we obtain the unique solution of (2.1) as follows:

$$f(x) = h(x) - (h * l)(x).$$

Theorem 2.1 is proved.

***Corollary* 2.1.** *The necessary condition for the existence of a solution $f \in L_1(\mathbb{R}_+)$ of Eq.* (2.1) *is as follows*:

$$\|f\|_{L_1(\mathbb{R}_+)} \geqslant \frac{\|h\|_{L_1(\mathbb{R}_+)}}{1 + \|\varphi\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1^{0,\beta}(\mathbb{R}_+)}}.$$

Consider the following system of two integral equations with Toeplitz-plus-Hankel kernels:

$$f(x) + \int_0^\infty g(y)[k_1(x+y) + k_2(x-y)]dy = p(x), \quad x > 0,$$
$$g(x) + \int_0^\infty f(y)[k_3(x+y) + k_4(x-y)] = q(x), \tag{2.4}$$

where p, q, and k_i, $i = \overline{1,4}$, are given functions and f and g are unknown functions. The problem of solving these systems in explicit form remains open. Here, we consider the class

$$f(x) + \int_0^\infty (g * \xi)(y)[k_1(x+y) + k_2(x-y)]dy = p(x), \quad x > 0,$$
$$g(x) + \int_0^\infty (f \underset{1}{*} \eta)(y)[k_3(x+y) + k_4(x-y)]dy = q(x), \tag{2.5}$$

where

$$k_1(t) = \frac{1}{2}\int_0^\infty v \sinh t e^{-v\cosh t}\varphi(v)dv = k_2(t),$$

$$k_3(t) = \frac{1}{2}\int_0^\infty v \sinh t e^{-v\cosh t}\psi(v)dv = -k_4(t),$$

φ, ψ, p, q, ξ, and η are given functions, and f and g are unknown functions.

Theorem 2.2. *Let $\xi, \eta, p, q \in L_1(\mathbb{R}_+)$ and $\varphi, \psi \in L_1^{0,\beta}(\mathbb{R}_+)$ be such that*

$$1 - y^2 (F_c\xi)(y)(F_c\eta)(y)(K_{iy}\varphi)(K_{iy}\psi) \neq 0 \quad \forall y > 0.$$

Then system (2.5) *has the following unique solution $(f, g) \in L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$*:

$$f(x) = p(x) + (p \underset{2}{*} l)(x) - ((\xi \overset{\gamma}{*} \varphi)_1 \underset{2}{*} q)(x) - (((\xi \overset{\gamma}{*} \varphi)_1 \underset{2}{*} q) \underset{2}{*} l)(x),$$

$$g(x) = l(x) + (l * q)(x) - (p \underset{2}{*} (\eta \overset{\gamma}{*} \psi)_1)(x) - (l * (p \underset{2}{*} (\eta \overset{\gamma}{*} \psi)_1))(x),$$

where $l \in L_1(\mathbb{R}_+)$ is defined by the formula

$$(F_c l)(y) = \frac{y^2 (F_c \xi)(y)(F_c \eta)(y)(K_{iy}\varphi)(K_{iy}\psi)}{1 + y^2 (F_c \xi)(y)(F_c \eta)(y)(K_{iy}\varphi)(K_{iy}\psi)}. \tag{2.6}$$

Proof. Using the factorization identity (1.2) and relations (0.2) and (0.6) and applying the Fourier sine transform and the Fourier cosine transform, respectively, to the two equations of system (2.5), we obtain

$$\begin{aligned} &(F_s f)(y) + y(F_c \xi)(y)(K_{iy}\varphi)(F_c g)(y) = (F_s p)(y), \\ &y(F_c \eta)(y)(K_{iy}\psi)(F_s f)(y) + (F_c g)(y) = (F_c q)(y). \end{aligned} \tag{2.7}$$

Solving the above linear system with the use of the Cramer technique and relations (1.2) and (0.8), we get

$$\Delta = 1 - F_s(\xi * \varphi)_1(y) F_s(\eta * \psi)_1(y) = 1 - F_c((\xi * \varphi)_1 \underset{2}{*} (\eta * \psi)_1(y).$$

By virtue of the Wiener – Levy theorem (see [1]), there exists a function $l \in L_1(\mathbb{R}_+)$ defined by (2.6), whence

$$\frac{1}{\Delta} = 1 + (F_c l)(y). \tag{2.8}$$

Using (2.7), (2.8), (1.2), and (0.8), we obtain

$$(F_s f)(y) = [1 + (F_c l)(y)] \begin{vmatrix} (F_s p)(y) & F_s(\xi \overset{\gamma}{*} \varphi)_1(y) \\ (F_c q)(y) & 1 \end{vmatrix} =$$

$$= (F_s p)(y) + F_s(p \underset{2}{*} l)(y) - F_s((\xi \overset{\gamma}{*} \varphi)_1 \underset{2}{*} q)(y) - F_s(((\xi \overset{\gamma}{*} \varphi)_1 \underset{2}{*} q) \underset{2}{*} l)(y).$$

Using inverse Fourier sine formula, we get

$$f(x) = p(x) + (p \underset{2}{*} l)(x) - ((\xi \overset{\gamma}{*} \varphi)_1 \underset{2}{*} q)(x) - (((\xi \overset{\gamma}{*} \varphi)_1 \underset{2}{*} q) \underset{2}{*} l)(x) \in L_1(\mathbb{R}_+). \tag{2.9}$$

By analogy, using (2.8), (1.2), (0.2), and (0.8), we obtain

$$(F_c g)(y) = (F_c l)(y) + F_c(l * q)(y) - F_c(p \underset{2}{*} (\eta \overset{\gamma}{*} \psi)_1)(y) - F_c(l * (p \underset{2}{*} (\eta \overset{\gamma}{*} \psi)_1))(y).$$

Hence,

$$g(x) = l(x) + (l * q)(x) - (p \underset{2}{*} (\eta \overset{\gamma}{*} \psi)_1)(x) - (l * (p \underset{2}{*} (\eta \overset{\gamma}{*} \psi)_1))(x) \in L_1(\mathbb{R}_+). \tag{2.10}$$

Using (2.9) and (2.10), we obtain the unique solution of system (2.5) in $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$.

Theorem 2.2 is proved.

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