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ON AGARWAL – PANG-TYPE INTEGRAL INEQUALITIES*

ПРО НЕРІВНОСТІ ТИПУ АГАРВАЛА – ПАНГА

We establish some new Agarwal – Pang-type inequalities involving second-order partial derivatives. Our results in special cases yield some of interrelated results and provide new estimates for inequalities of this type.

Встановлено деякі нові нерівності типу Агарвала – Панга, що містять частинні похідні другого порядку. В окремих випадках із одержаних результатів випливають деякі пов'язані результати та нові оцінки для нерівностей цього типу.

1. Introduction. In the year 1960, Opial [1] established the following inequality:

Theorem A. *Suppose* $f \in C^1[0,h]$ *satisfies* $f(0) = f(h) = 0$ *and* $f(x) > 0$ *for all* $x \in (0,h)$. *Then the inequality holds*

$$\int_0^h \big|f(x)f'(x)\big|\,dx \le \frac{h}{4}\int_0^h (f'(x))^2 dx, \tag{1.1}$$

where this constant $h/4$ *is best possible.*

Many generalizations, extensions and discretizations of Opial's inequality were established (see e.g. [2 – 15]). For an extensive survey on these inequalities, see [16]. Opial's inequality and its generalizations and extensions play a fundamental role in establishing the existence and uniqueness of initial and boundary-value problems for ordinary and partial differential equations as well as difference equation [8 – 16].

In 1995, Agarwal and Pang [17] proved the following Wirtinger's type inequality and an interesting Opial's type inequality, respectively.

Theorem B. *Let* $\lambda \ge 1$ *be a given real number, and let* $p(t)$ *be a nonnegative and continuous function on* $[0,a]$. *Further, let* $x(t)$ *be an absolutely continuous function on* $[0,a]$, *with* $x(0) = x(a) = 0$. *Then*

$$\int_0^a p(t)|x(t)|^\lambda dt \le \frac{1}{2}\int_0^a [t(a-t)]^{(\lambda-1)/2} p(t)dt \int_0^a \big|x'(t)\big|^\lambda\,dt. \tag{1.2}$$

Theorem C. *Assume that*

(i) l, m, μ *and* v *are nonnegative real numbers such that* $\dfrac{1}{\mu} + \dfrac{1}{v} = 1$, *and* $l\mu \ge 1$,

(ii) $q(t)$ *is a nonnegative and continuous function on* $[0,a]$,

(iii) *let* $x_1(t)$ *and* $x_2(t)$ *are absolutely continuous functions on* $[0,a]$, *with* $x_1(0) = x_1(a) = x_2(0) = x_2(a) = 0$.

Then

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$$\int\limits_0^a q(t)\left[|x_1(t)|^l\left|x_2'(t)\right|^m+|x_2(t)|^l\left|x_1'(t)\right|^m\right]dt\le\left(\frac{1}{2}\int\limits_0^a[t(a-t)]^{(l\mu-1)/2}q^\mu(t)dt\right)^{1/\mu}\times$$

$$\times\int\limits_0^a\left[\frac{1}{\mu}\left(\left|x_1'(t)\right|^{l\mu}+\left|x_2'(t)\right|^{l\mu}\right)+\frac{1}{v}\left(\left|x_1'(t)\right|^{mv}+\left|x_2'(t)\right|^{mv}\right)\right]dt. \tag{1.3}$$

The main purpose of the present paper is to establish Agarwal – Pang-type inequalities involving 2-order partial derivatives. Our results in special cases yield (1.2) and (1.3), respectively.

Theorem 1.1. *Let $\lambda\ge 1$ be a real number, and let $p(s,t)$ be a nonnegative and continuous functions on $[0,a]\times[0,b]$. Further, let $x(s,t)$ be an absolutely continuous function on $[0,a]\times[0,b]$, with $x(s,0)=x(0,t)=x(0,0)=0$ and $x(a,b)=x(a,t)=x(s,b)=0$. Then*

$$\int\limits_0^a\int\limits_0^b p(s,t)|x(s,t)|^\lambda dsdt\le$$

$$\le\frac{1}{2}\left(\int\limits_0^a\int\limits_0^b[st(a-s)(b-t)]^{(\lambda-1)/2}p(s,t)dsdt\right)\int\limits_0^a\int\limits_0^b\left|\frac{\partial^2}{\partial s\partial t}x(s,t)\right|^\lambda dsdt. \tag{1.4}$$

Theorem 1.2. *Assume that*

(i) *l, m, μ and v are nonnegative real numbers such that $\dfrac{1}{\mu}+\dfrac{1}{v}=1$, and $l\mu\ge 1$,*

(ii) *$q(s,t)$ is a nonnegative and continuous function on $[0,a]\times[0,b]$,*

(iii) *let $j=1,2$ and $x_j(s,t)$ are absolutely continuous functions on $[0,a]\times[0,b]$, with $x_j(s,0)=$ $=x_j(0,t)=x_j(0,0)=0$ and $x_j(a,b)=x_j(a,t)=x_j(s,b)=0$.*

Then

$$\int\limits_0^a\int\limits_0^b q(s,t)\left[|x_1(s,t)|^l\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^m+|x_2(s,t)|^l\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^m\right]dsdt\le$$

$$\le\left(\frac{1}{2}\int\limits_0^a\int\limits_0^b[st(a-s)(b-t)]^{(l\mu-1)/2}q^\mu(s,t)dsdt\right)^{1/\mu}\times$$

$$\times\int\limits_0^a\int\limits_0^b\left[\frac{1}{\mu}\left(\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^{l\mu}+\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^{l\mu}\right)+\right.$$

$$\left.+\frac{1}{v}\left(\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^{mv}+\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^{mv}\right)\right]dsdt. \tag{1.5}$$

We also establish the following Opial-type inequality involving 2-order partial derivatives.

Theorem 1.3. *Let $j = 1, 2$ and $\lambda \geq 1$ be a real number, and let $p_j(s,t)$ be a nonnegative and continuous functions on $[0,a] \times [0,b]$. Further, let $x_j(s,t)$ be an absolutely continuous function on $[0,a] \times [0,b]$, with $x_j(s,0) = x_j(0,t) = x_j(0,0) = 0$ and $x_j(a,b) = x_j(a,t) = x_j(s,b) = 0$. Then*

$$\int_0^a \int_0^b \left(p_1(s,t)|x_1(s,t)|^\lambda + p_2(s,t)|x_2(s,t)|^\lambda \right) dsdt \leq$$

$$\leq \frac{1}{2} \left(\frac{ab}{2} \right)^{\lambda - 1} \left[\left(\int_0^a \int_0^b p_1(s,t) dsdt \right) \int_0^a \int_0^b \left| \frac{\partial^2}{\partial s \partial t} x_1(s,t) \right|^\lambda dsdt + \right.$$

$$\left. + \left(\int_0^a \int_0^b p_2(s,t) dsdt \right) \int_0^a \int_0^b \left| \frac{\partial^2}{\partial s \partial t} x_2(s,t) \right|^\lambda dsdt \right]. \tag{1.6}$$

2. Main results and their proofs.

Theorem 2.1. *Let $\lambda \geq 1$ be a real number, and let $p(s,t)$ be a nonnegative and continuous functions on $[0,a] \times [0,b]$. Further, let $x(s,t)$ be an absolutely continuous function on $[0,a] \times [0,b]$, with $x(s,0) = x(0,t) = x(0,0) = 0$ and $x(a,b) = x(a,t) = x(s,b) = 0$. Then*

$$\int_0^a \int_0^b p(s,t)|x(s,t)|^\lambda dsdt \leq$$

$$\leq \frac{1}{2} \left(\int_0^a \int_0^b [st(a-s)(b-t)]^{(\lambda-1)/2} p(s,t) dsdt \right) \int_0^a \int_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^\lambda dsdt. \tag{2.1}$$

Proof. From the hypotheses, we have

$$x(s,t) = \int_0^s \int_0^t \frac{\partial^2}{\partial s \partial t} x(s,t) dsdt.$$

By Hölder's inequality with indices λ and $\lambda/(\lambda - 1)$, it follows that

$$|x(s,t)|^{\lambda/2} \leq \left[\left(\int_0^s \int_0^t \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right| dsdt \right)^\lambda \right]^{1/2} \leq$$

$$\leq (st)^{(\lambda-1)/2} \left(\int_0^s \int_0^t \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^\lambda dsdt \right)^{1/2}. \tag{2.2}$$

Similarly, from

$$x(s,t) = \int\limits_s^a \int\limits_t^b \frac{\partial^2}{\partial s \partial t} x(s,t) dsdt,$$

we obtain

$$|x(s,t)|^{\lambda/2} \le [(a-s)(b-t)]^{(\lambda-1)/2} \left(\int\limits_s^a \int\limits_t^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{\lambda} dsdt \right)^{1/2}. \qquad (2.3)$$

Now a multiplication of (2.2) and (2.3), and by the elementary inequality $2\sqrt{\alpha\beta} \le \alpha + \beta$, $\alpha \ge 0$, $\beta \le 0$ gives

$$|x(s,t)|^{\lambda} \le [st(a-s)(b-t)]^{(\lambda-1)/2} \left(\int\limits_0^s \int\limits_0^t \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{\lambda} dsdt \right)^{1/2} \times$$

$$\times \left(\int\limits_s^a \int\limits_t^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{\lambda} dsdt \right)^{1/2} \le$$

$$\le \frac{1}{2} [st(a-s)(b-t)]^{(\lambda-1)/2} \left(\int\limits_0^s \int\limits_0^t \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{\lambda} dsdt + \int\limits_s^a \int\limits_t^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{\lambda} dsdt \right) =$$

$$= \frac{1}{2} [st(a-s)(b-t)]^{(\lambda-1)/2} \int\limits_0^a \int\limits_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{\lambda} dsdt. \qquad (2.4)$$

Multiplying the both sides of (2.4) by $p(s,t)$ and integrating both sides over t from 0 to b first and then integrating the resulting inequality over s from 0 to a, we obtain

$$\int\limits_0^a \int\limits_0^b p(s,t) |x(s,t)|^{\lambda} dsdt \le$$

$$\le \frac{1}{2} \int\limits_0^a \int\limits_0^b \left([st(a-s)(b-t)]^{(\lambda-1)/2} p(s,t) \int\limits_0^a \int\limits_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{\lambda} dsdt \right) dsdt =$$

$$= \frac{1}{2} \left(\int\limits_0^a \int\limits_0^b [st(a-s)(b-t)]^{(\lambda-1)/2} p(s,t) dsdt \right) \int\limits_0^a \int\limits_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{\lambda} dsdt.$$

***Remark* 2.1.** Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, Theorem 2.1 becomes Theorem B stated in the introduction which was given by Agarwal and Pang [17].

***Remark* 2.2.** Taking for $p(s,t) =$ constant in (2.1), we have

$$\int_0^a\int_0^b |x(s,t)|^\lambda dsdt \leq \frac{1}{2}(ab)^\lambda \left[B\left(\frac{\lambda+1}{2},\frac{\lambda+1}{2}\right)\right]^2 \int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}x(s,t)\right|^\lambda dsdt,$$

where B is the Beta function.

Theorem 2.2. *Let $j = 1,2$ and $\lambda \geq 1$ be a real number, and let $p_j(s,t)$ be a nonnegative and continuous functions on $[0,a]\times[0,b]$. Further, let $x_j(s,t)$ be an absolutely continuous function on $[0,a]\times[0,b]$, with $x_j(s,0) = x_j(0,t) = x_j(0,0) = 0$ and $x_j(a,b) = x_j(a,t) = x_j(s,b) = 0$. Then*

$$\int_0^a\int_0^b \left(p_1(s,t)|x_1(s,t)|^\lambda + p_2(s,t)|x_2(s,t)|^\lambda\right) dsdt \leq$$

$$\leq \frac{1}{2}\left(\frac{ab}{2}\right)^{\lambda-1}\left[\left(\int_0^a\int_0^b p_1(s,t)dsdt\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^\lambda dsdt+\right.$$

$$\left.+\left(\int_0^a\int_0^b p_2(s,t)dsdt\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^\lambda dsdt\right]. \tag{2.5}$$

Proof. From the hypotheses, we have

$$x_1(s,t) = \int_0^s\int_0^t \frac{\partial^2}{\partial s\partial t}x_1(s,t)dsdt \qquad \text{and} \qquad x_1(s,t) = \int_s^a\int_t^b \frac{\partial^2}{\partial s\partial t}x_1(s,t)dsdt.$$

Hence

$$|x_1(s,t)| \leq \frac{1}{2}\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|dsdt.$$

By Hölder's inequality with indices λ and $\lambda/(\lambda-1)$, it follows that

$$p_1(s,t)|x_1(s,t)|^\lambda \leq \frac{1}{2^\lambda}p_1(s,t)\left(\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|dsdt\right)^\lambda \leq$$

$$\leq \frac{1}{2}\left(\frac{ab}{2}\right)^{\lambda-1} p_1(s,t)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^\lambda dsdt. \tag{2.6}$$

Similarly

$$p_2(s,t)|x_2(s,t)|^\lambda \leq \frac{1}{2^\lambda}p_2(s,t)\left(\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|dsdt\right)^\lambda \leq$$

$$\leq \frac{1}{2}\left(\frac{ab}{2}\right)^{\lambda-1} p_2(s,t)\int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^{\lambda} dsdt. \tag{2.7}$$

Taking the sum of (2.6) and (2.7) and integrating the resulting inequalities over t from 0 to b first and then over s from 0 to a, we obtain (2.5).

***Remark* 2.3.** Taking for $x_1(s,t) = x_1(s,t) = x(s,t)$ in (2.5), (2.5) changes to the following inequality:

$$\int_0^a\int_0^b p(s,t)|x(s,t)|^{\lambda} dsdt \leq$$

$$\leq \frac{1}{2}\left(\frac{ab}{2}\right)^{\lambda-1}\left(\int_0^a\int_0^b p(s,t)dsdt\right)\int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}x(s,t)\right|^{\lambda} dsdt. \tag{2.8}$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, (2.8) becomes the following result:

$$\int_0^a p(t)|x(t)|^{\lambda}dt \leq \frac{1}{2}\left(\frac{a}{2}\right)^{\lambda-1}\left(\int_0^a p(t)dt\right)\int_0^a \left|x'(t)\right|^{\lambda} dt. \tag{2.9}$$

This is just a new inequality established by Agarwal and Pang [17]. For $\lambda = 2$ the inequality (2.9) has appear in the work of Traple [18], Pachpatte [19] proved it for $\lambda = 2m$ ($m \geq 1$ an integer).

***Remark* 2.4.** Let $x_j(s,t)$ reduce to $s_j(t)$, $j = 1,2$, and with suitable modifications, (2.5) becomes the following interesting result:

$$\int_0^a \Big(p_1(t)|x_1(t)|^{\lambda} + p_2(t)|x_2(t)|^{\lambda}\Big)\, dt \leq$$

$$\leq \frac{1}{2}\left(\frac{a}{2}\right)^{\lambda-1}\left[\left(\int_0^a p_1(t)dt\right)\int_0^a \left|x_1'(t)\right|^{\lambda} dt + \left(\int_0^a p_2(t)dt\right)\int_0^a \left|x_2'(t)\right|^{\lambda} dt\right].$$

Theorem 2.3. *Assume that*

(i) l, m, μ *and* v *are nonnegative real numbers such that* $\dfrac{1}{\mu} + \dfrac{1}{v} = 1$, *and* $l\mu \geq 1$,

(ii) $q(s,t)$ *is a nonnegative and continuous function on* $[0,a]\times[0,b]$,

(iii) *let* $j = 1,2$ *and* $x_j(s,t)$ *are absolutely continuous functions on* $[0,a]\times[0,b]$, *with* $x_j(s,0) = x_j(0,t) = x_j(0,0) = 0$ *and* $x_j(a,b) = x_j(a,t) = x_j(s,b) = 0$.

Then

$$\int_0^a\int_0^b q(s,t)\left[|x_1(s,t)|^l\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^m + |x_2(s,t)|^l\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^m\right] dsdt \leq$$

$$\leq \left(\frac{1}{2}\int_0^a\int_0^b [st(a-s)(b-t)]^{(l\mu-1)/2} q^{\mu}(s,t)dsdt\right)^{1/\mu} \times$$

$$\times\int_0^a\int_0^b\left[\frac{1}{\mu}\left(\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^{l\mu}+\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^{l\mu}\right)+\right.$$

$$\left.+\frac{1}{v}\left(\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^{mv}+\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^{mv}\right)\right]dsdt. \tag{2.10}$$

Proof. From the Hölder's inequality, the inequality (2.1) and Young's inequality $wz \le \frac{w^\mu}{\mu}+\frac{z^v}{v}$ we have

$$\int_0^a\int_0^b q(s,t)|x_1(s,t)|^l\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^m dsdt \le$$

$$\le\left(\int_0^a\int_0^b q^\mu(s,t)|x_1(s,t)|^{l\mu}dsdt\right)^{1/\mu}\left(\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^{mv}dsdt\right)^{1/v}\le$$

$$\le\left(\frac{1}{2}\int_0^a\int_0^b[st(a-s)(b-t)]^{(l\mu-1)/2}q^\mu(s,t)dsdt\right)^{1/\mu}\left(\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x(s,t)\right|^{l\mu}dsdt\right)^{1/\mu}\times$$

$$\times\left(\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^{mv}dsdt\right)^{1/v}\le$$

$$\le\left(\frac{1}{2}\int_0^a\int_0^b[st(a-s)(b-t)]^{(l\mu-1)/2}q^\mu(s,t)dsdt\right)^{1/\mu}\times$$

$$\times\int_0^a\int_0^b\left(\frac{1}{\mu}\left|\frac{\partial^2}{\partial s\partial t}x_1(s,t)\right|^{l\mu}+\frac{1}{v}\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^{mv}\right)dsdt, \tag{2.11}$$

and similarly

$$\int_0^a\int_0^b q(s,t)|x_1(s,t)|^m\left|\frac{\partial^2}{\partial s\partial t}x_2(s,t)\right|^l dsdt \le$$

$$\le\left(\frac{1}{2}\int_0^a\int_0^b[st(a-s)(b-t)]^{(l\mu-1)/2}q^\mu(s,t)dsdt\right)^{1/\mu}\times$$

$$\times \int_0^a \int_0^b \left(\frac{1}{v} \left| \frac{\partial^2}{\partial s \partial t} x_1(s,t) \right|^{mv} + \frac{1}{\mu} \left| \frac{\partial^2}{\partial s \partial t} x_2(s,t) \right|^{l\mu} \right) dsdt. \tag{2.12}$$

An addition of (2.11) and (2.12) gives the inequality (2.10).

***Remark* 2.5.** Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, (2.10) becomes the following result:

$$\int_0^a q(t) \left[|x_1(t)|^l \left| x_2'(t) \right|^m + |x_2(t)|^l \left| x_1'(t) \right|^m \right] dt \le$$

$$\le \left(\frac{1}{2} \int_0^a [t(a-t)]^{(l\mu-1)/2} q^\mu(t) dt \right)^{1/\mu} \times$$

$$\times \int_0^a \left[\frac{1}{\mu} \left(\left| x_1'(t) \right|^{l\mu} + \left| x_2'(t) \right|^{l\mu} \right) + \frac{1}{v} \left(\left| x_1'(t) \right|^{mv} + \left| x_2'(t) \right|^{mv} \right) \right] dt. \tag{2.13}$$

The inequality (2.13) has appeared in the work of Agarwal and Pang [17].

***Remark* 2.6.** Let $q(s,t) \neq$ constant and taking for $\mu = v = 2$ in (2.10), we obtain

$$\int_0^a \int_0^b q(s,t) \left(|x_1(s,t)|^l \left| \frac{\partial^2}{\partial s \partial t} x_2(s,t) \right|^m + |x_2(s,t)|^l \left| \frac{\partial^1}{\partial s \partial t} x_1(s,t) \right|^m \right) dsdt \le$$

$$\le \left(\frac{1}{8} \int_0^a \int_0^b [st(a-s)(b-t)]^{(2l-1)/2} q^2(s,t) dsdt \right)^{1/2} \times$$

$$\times \int_0^a \int_0^b \left(\left| \frac{\partial^2}{\partial s \partial t} x_1(s,t) \right|^{2l} + \left| \frac{\partial^2}{\partial s \partial t} x_2(s,t) \right|^{2l} + \right.$$

$$\left. + \left| \frac{\partial^2}{\partial s \partial t} x_1(s,t) \right|^{2m} + \left| \frac{\partial^2}{\partial s \partial t} x_2(s,t) \right|^{2m} \right) dsdt. \tag{2.14}$$

Let $x_j(s,t)$, $j = 1,2$, reduce to $s_j(t)$ and with suitable modifications, (2.14) becomes the following result:

$$\int_0^a q(t) \left(|x_1(t)|^l \left| x_2'(t) \right|^m + |x_2(t)|^l \left| x_1'(t) \right|^m \right) dt \le$$

$$\le \left(\frac{1}{8} \int_0^a [t(a-t)]^{(2l-1)/2} q^2(t) dt \right)^{1/2} \times$$

$$\times \int_0^a \left(\left| x_1'(t) \right|^{2l} + \left| x_2'(t) \right|^{2l} + \left| x_1'(t) \right|^{2m} + \left| x_2'(t) \right|^{2m} \right) dt. \qquad (2.15)$$

This is just an inequality established by Agarwal and Pang [17]. The inequality (2.15) is sharper than the following inequality (see [17]):

$$\int_0^a q(t) \left(|x_1(t)|^l \left| x_2'(t) \right|^m + |x_2(t)|^l \left| x_1'(t) \right|^m \right) dt \le$$

$$\le \left(\frac{h^{2l-1}}{4^{l+1}} \int_0^a q^2(t) dt \right)^{1/2} \int_0^a \left(\left| x_1'(t) \right|^{2l} + \left| x_2'(t) \right|^{2l} + \left| x_1'(t) \right|^{2m} + \left| x_2'(t) \right|^{2m} \right) dt.$$

For the integers $l, m \ge 1$, the inequality (2.15) has been obtained by Lin [20].

3. Uniqueness of initial value problem. Here, as application to one of the inequalities obtained in Section 2 we shall prove the uniqueness of the solution of initial value problem involving higher order ordinary equation.

Theorem 3.1. *For the system of differential equations*

$$y_j'' = f_j(t, y_1, y_1', y_2, y_2'), \quad j = 1, 2, \qquad (3.1)$$

together with the initial conditions

$$y_j^{(i)}(0) = y_{j,i}, \quad j = 1, 2, \quad 0 \le i \le 1, \qquad (3.2)$$

we assume that $f_j \colon [0, \tau] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous, and satisfy the Lipschitz condition*

$$|f_j(t, y_{1,0}, y_{1,1}, y_{2,0}, y_{2,1}) - f_j(t, \bar{y}_{1,0}, \bar{y}_{1,1}, \bar{y}_{2,0}, \bar{y}_{2,1})| \le$$

$$\le \sum_{k=0}^{1} \Big[q_{1,j,k}(t) |y_{1,k} - \bar{y}_{1,k}| + q_{2,j,k}(t) |y_{2,k} - \bar{y}_{2,k}| \Big],$$

where the functions $q_{r,j,k} \ge 0$, $1 \le r$, $4j \le 2$, $0 \le k \le 1$, *are continuous on* $[0, \tau]$. *Then the problem* (3.1), (3.2) *has at most one solution on* $[0, \tau]$.

Proof. If the problem (3.1), (3.2) has two solutions $(y_1(t), y_2(t))$, $(\bar{y}_1(t), \bar{y}_2(t))$ then for the functions $x_j(t) = y_j(t) - \bar{y}_j(t)$, $j = 1, 2$, it follows

$$|x_j''(t)|^2 \le \sum_{k=0}^{1} \Big[q_{1,j,k}(t) |x_1^{(k)}(t)| |x_j''(t)| + q_{2,j,k}(t) |x_2^{(k)}(t)| |x_j''(t)| \Big].$$

Summing these two inequalities, and integrating from 0 to t, we obtain

$$\int_0^t \Big[|x_1''(s)|^2 + |x_2''(s)|^2 \Big] ds \le \sum_{k=0}^{1} \int_0^t \bar{q}_k(s) \Big[|x_1^{(k)}(s)| |x_1''(s)| + |x_2^{(k)}(s)| |x_2''(s)| \Big] ds +$$

$$+\sum_{k=0}^{1}\int_{0}^{t}\bar{q}_k^{*}(s)\Big[|x_2^{(k)}(s)||x_2''(s)|+|x_1^{(k)}(s)||x_1''(s)|\Big]ds, \tag{3.3}$$

where $\bar{q}_k(t)=\max(q_{1,1,k}(t),q_{2,2,k}(t))$ and $q_k^{*}(t)=\max(q_{2,1,k}(t),q_{1,2,k}(t))$.

For $0\le k\le 1$, on the right side of (3.3), we apply the inequality (2.10) with $x(s,t)=x(t)$, $l=1$, $\mu=2, m=1$ and with suitable modifications, to obtain the following inequality:

$$\int_{0}^{t}\Big[|x_1''(s)|^2+|x_2''(s)|^2\Big]ds\le K(t)\int_{0}^{t}\Big[|x_1''(s)|^2+|x_2''(s)|^2\Big]ds, \tag{3.4}$$

where $K(t)$ is a continuous function with the property $K(0)=0$. Hence, the inequality (3.4) implies that $y_1(t)=\bar{y}_1(t)$, $y_2(t)=\bar{y}_2(t)$ and $t\in[0,\tau]$.

1. *Opial Z.* Sur une inégalité // Ann. pol. math. – 1960. – **8**. – P. 29 – 32.
2. *Willett D.* The existence-uniqueness theorem for an n-th order linear ordinary differential equation // Amer. Math. Monthly. – 1968. – **75**. – P. 174 – 178.
3. *Das K. M.* An inequality similar to Opial's inequality // Proc. Amer. Math. Soc. – 1969. – **22**. – P. 258 – 261.
4. *Agarwal R. P., Thandapani E.* On some new integrodifferential inequalities // An. sti. Univ. "Al. I. Cuza" Iasi. – 1982. – **28**. – P. 123 – 126.
5. *Yang G. S.* A note on inequality similar to Opial inequality // Tamkang J. Math. – 1987. – **18**. – P. 101 – 104.
6. *Zhao C. J., Cheung W. S.* Sharp integral inequalities involving high-order partial derivatives // J. Inequat. Appl. – 2008. – **2008**. – Article ID 571417. – 10 p.
7. *Agarwal R. P., Pang P. Y. H.* Sharp opial-type inequalities in two variables // Appl. Anal. – 1996. – **56**, № 3. – P. 227 – 242.
8. *Agarwal R. P., Lakshmikantham V.* Uniqueness and nonuniqueness criteria for ordinary differential equations. – Singapore: World Sci., 1993.
9. *Bainov D., Simeonov P.* Integral inequalities and applications. – Dordrecht: Kluwer Acad. Publ., 1992.
10. *Li J. D.* Opial-type integral inequalities involving several higher order derivatives // J. Math. Anal. and Appl. – 1992. – **167**. – P. 98 – 100.
11. *Cheung W. S.* On Opial-type inequalities in two variables // Aequat. Math. – 1989. – **38**. – P. 236 – 244.
12. *Cheung W. S.* Some generalized Opial-type inequalities // J. Math. Anal. and Appl. – 1991. – **162**. – P. 317 – 321.
13. *Godunova E. K., Levin V. I.* On an inequality of Maroni // Mat. Zametki. – 1967. – **2**. – P. 221 – 224.
14. *Mitrinovič D. S.* Analytic inequalities. – Berlin; New York: Springer, 1970.
15. *Yang G. S.* Inequality of Opial-type in two variables // Tamkang J. Math. – 1982. – **13**. – P. 255 – 259.
16. *Agarwal R. P., Pang P. Y. H.* Opial inequalities with applications in differential and difference equations. – Dordrecht: Kluwer Acad. Publ., 1995.
17. *Agarwal R. P.* Sharp Opial-type inequalities involving r-derivatives and their applications // Tohoku Math. J. – 1995. – **47**, № 4. – P. 567 – 593.
18. *Traple J.* On a boundary-value problem for systems of ordinary differential equations of second order // Zesz. nauk. U. J. Pr. Math. – 1971. – **15**. – P. 159 – 168.
19. *Pachpatte B. G.* A note on an inequality ascribed to Wirtinger, // Tamkang J. Math. – 1986. – **17**. – P. 69 – 73.
20. *Lin C. T.* A further generalization of Opial integral inequality // Tamkang J. Math. – 1989. – **27**. – P. 491 – 493.

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