UDC 517.5

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ON SHIBA – WATERMAN SPACE

ПРО ПРОСТІР ШИБИ – УОТЕРМЕНА

We give a necessary and sufficient condition for the inclusion of $\Lambda BV^{(p)}$ in the classes H^q_ω.

Наведено необхідну та достатню умову належності $\Lambda BV^{(p)}$ класам H^q_ω.

In 1980 M. Shiba [9] introduced the class $\Lambda BV^{(p)}$, $1 \le p < \infty$, expanding a fundamental concept of bounded Λ-variation formulated and usefully applied by D. Waterman in 1972 [13].

The main objective of this note is to find a necessary and sufficient condition for the embedding $\Lambda BV^{(p)} \subset H^q_\omega$.

1. Introduction and preliminaries. Let $\Lambda = (\lambda_i)$ be a nondecreasing sequence of positive numbers such that $\sum \dfrac{1}{\lambda_i} = +\infty$ and let p be a number greater than or equal to 1. A function $f\colon [a,b] \to \mathbb{R}$ is said to be of bounded p-Λ-variation on a not necessarily closed subinterval $P \subset \subset [a, b]$ if

$$V(f;P) := \sup \left(\sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i} \right)^{1/p} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping subintervals of P and where $f(I_i) := f(\sup I_i) - f(\inf I_i)$ is the change of the function f over the interval I_i. The symbol $\Lambda BV^{(p)}$ denotes the linear space of all functions of bounded p-Λ-variation with domain $[0,1]$. We will write $V(f)$ instead of $V(f,\, P)$ if $P = [0,1]$. The Shiba – Waterman class $\Lambda BV^{(p)}$ was introduced in 1980 by M. Shiba in [9] and it clearly is a generalization of the well-known Waterman class ΛBV. Some of the basic properties of functions of class $\Lambda BV^{(p)}$ were discussed by R. G. Vyas in [11] recently. More results concerned with the Shiba – Waterman classes and their applications can be found in [1, 2, 4, 6 – 8, 10, 12]. $\Lambda BV^{(p)}$ equipped with the norm $\|f\|_{\Lambda,\,p} := |f(0)| + V(f)$ is a Banach space.

Functions in a Shiba – Waterman class $\Lambda BV^{(p)}$ are regulated [11] (Theorem 2), hence integrable, and thus it makes sense to consider their integral modulus of continuity

$$\omega_q(\delta,\, f) \;=\; \sup_{0 \le h \le \delta} \left(\int\limits_0^{1-h} |f(t+h) - f(t)|^q \right)^{1/q} dt,$$

for $0 \le \delta \le 1$. However, if f is defined on $\mathbb{R}$ instead of on $[0,1]$ and if f is 1-periodic, it is convenient to modify the definition and put

$$\omega_q(\delta,\, f) \;=\; \sup_{0 \le h \le \delta} \left(\int\limits_0^{1} |f(t+h) - f(t)|^q \right)^{1/q} dt,$$

since the difference between the two definitions is then nonessential in all applications of the concept. We will use the second definition in our note, and thus the main Theorem 2.1 will actually deal with 1-periodic functions.

A function $\omega\colon [0,1]\to\mathbb{R}$ is said to be a modulus of continuity if it is nondecreasing, continuous, subadditive and $\omega(0)=0$. If ω is a modulus of continuity, then H^q_ω denotes the class of functions $f\in L_q[0,1]$ for which $\omega_q(\delta,\,f)=O(\omega(\delta))$ as $\delta\to 0+$.

2. On the imbedding of $\Lambda BV^{(p)}$ class in the class H^q_ω. In [3] Goginava gave a necessary and sufficient condition for the inclusion ΛBV in H^q_ω. Also Wang [15] by using an interesting method found a necessary and sufficient condition for the embedding $H^q_\omega\subset\Lambda BV$. Here, we give a necessary condition for the inclusion $\Lambda BV^{(p)}$ in H^q_ω. This work uses [3] and [5] as the bases. If $\omega(\delta)$ is a modulus of continuity, then the following theorem is true.

Theorem 2.1. *For some $p,q\in[1,\infty)$, the inclusion $\Lambda BV^{(p)}\subset H^\omega_q$ holds if and only if*

$$\limsup_{n\to\infty}\frac{1}{\omega(1/n)n^{1/pq}}\max_{1\le m\le n}\frac{m^{1/pq}}{\left(\sum_{i=1}^{m}\frac{1}{\lambda_i}\right)^{1/p}}<+\infty. \tag{1}$$

Proof. *Sufficiency.* We prove an inequality which gives us the sufficiency:

$$\omega\left(\frac{1}{n},f\right)_q\le V(f)\left\{\frac{1}{n^p}\max_{1\le m\le n}\frac{m^{1/p}}{\left(\sum_{i=1}^{m}1/\lambda_i\right)^{q/p}}\right\}^{1/q}.$$

First we recall the following lemma and corollary from [5]:

Lemma 2.1. *Consider the following problem*:

$$F(x)=\sum_{i=1}^{n}x_i^q\to\max\quad\textit{under the condition}\quad\left(\sum_{i=1}^{n}\frac{x_i}{\lambda_i}\right)\le 1\quad\textit{and}$$

$$x_1\ge x_2\ge x_3\ge\ldots\ge x_n\ge 0. \tag{L}$$

Then the solution $x=(x_1,x_2,\ldots,x_n)$ of problem (L) *is among vectors that satisfy conditions*

$$\sum_{i=1}^{n}\frac{x_i}{\lambda_i}=1$$

$x_1=x_2=\ldots=x_k>x_{k+1}=x_{k+2}=\ldots=x_n=0$ *with some k, $1\le k\le n$.*

***Corollary* 2.1.** *The extermal value of problem* (L) *is* $\max_{1\le k\le n}\dfrac{k}{\left(\sum_{i=1}^{k}1/\lambda_i\right)^q}$.

Now, we return to the proof of inequality

$$\left(\omega_q\left(\frac{1}{n},f\right)\right)^q\le\sup_{0<h\le 1/n}\int_0^1|f(x+h)-f(x)|^q\,dx=$$

$$= \sup_{0<h\le 1/n} \sum_{k=1}^{n} \int\limits_{(k-1)/n}^{k/n} |f(x+h)-f(x)|^q\,dx =$$

$$= \sup_{0<h\le 1/n} \sum_{k=1}^{n} \int\limits_{0}^{1/n} \left| f\left(x+\frac{k-1}{n}+h\right) - f\left(x+\frac{k-1}{n}\right)\right|^q dx =$$

$$= \sup_{0<h\le 1/n} \int\limits_{0}^{1/n} \sum_{k=1}^{n} \left| f\left(x+\frac{k-1}{n}+h\right) - f\left(x+\frac{k-1}{n}\right)\right|^q dx \le$$

$$\le \sup_{0<h\le 1/n} \int\limits_{0}^{1/n} n^{1-1/p} \left(\sum_{k=1}^{n} \left| f\left(x+\frac{k-1}{n}+h\right) - f\left(x+\frac{k-1}{n}\right)\right|^{pq}\right)^{1/p} dx.$$

Where the last inequality has been obtained by Hölder inequality. Under the condition $|h| \le \frac{1}{n}$ and fixed x, the segment $I_k(x)$ do not overlap each other and their union does not exceed P. Let enumerate the intervals I_k in decreasing of values $|f(I_k)|$ we get

$$|f(I_1)| \ge |f(I_2)| \ge \ldots \ge |f(I_n)|, \qquad \left(\sum_{k=1}^{n} \frac{|f(I_k)|^p}{\lambda_k}\right)^{1/p} \le V(f).$$

Therefore taking into account the Lemma 2.1 we get

$$\left(\omega_q\left(\frac{1}{n}, f\right)\right)^q \le \sup_{0<h\le 1/n} \int\limits_{0}^{1/n} n^{1-1/p} \left(\sum_{k=1}^{n} \left| f\left(x+\frac{k-1}{n}+h\right) - f\left(x+\frac{k-1}{n}\right)\right|^{pq}\right)^{1/p} dx \le$$

$$\le n^{1-1/p} \int\limits_{0}^{1/n} V^q(f) \max_{1\le k\le n} \frac{k^{1/p}}{\left(\sum_{i=1}^{k} 1/\lambda_i\right)^{q/p}}\,dx =$$

$$= \frac{1}{n^p} V^q(f) \max_{1\le k\le n} \frac{k^{1/p}}{\left(\sum_{i=1}^{k} 1/\lambda_i\right)^{q/p}}.$$

Necessity. Our proof uses Goginava's paper as a basis. Assume the condition (1) is not satisfied. As an example, we construct a function from $\Lambda BV^{(p)}$ that is not in H_q^{ω}. Since condition (1) is not satisfied, there exists a sequence of integers $\{\gamma_k, k\ge 1\}$ such that

$$\lim_{k\to\infty} \frac{1}{\omega(1/\gamma_k)\gamma_k^{1/(pq)}} \max_{1\le m\le \gamma_k} \frac{m^{1/(pq)}}{\left(\sum_{i=1}^{m} 1/\lambda_i\right)^{1/p}} = \infty.$$

Let $\{\gamma'_k, k \geq 1\}$ be a sequence of integers for which $2^{\gamma'_k - 1} \leq \gamma_k < 2^{\gamma'_k}$. Since $\omega(\delta)$ is nondecreasing, we have

$$\frac{2^{1/(pq)}}{\omega(2^{-\gamma'_k})2^{\gamma'_k/(pq)}} \max_{1\leq m\leq 2^{\gamma'_k}} \frac{m^{1/(pq)}}{\left(\sum_{i=1}^{m} 1/\lambda_i\right)^{1/p}} \geq \frac{1}{\omega(1/\gamma_k)\gamma_k^{1/(pq)}} \max_{1\leq m\leq \gamma_k} \frac{m^{1/(pq)}}{\left(\sum_{i=1}^{m} 1/\lambda_i\right)^{1/p}},$$

whence

$$\lim_{k\to\infty} \frac{1}{\omega(2^{-\gamma'_k})2^{-\gamma'_k/(pq)}} \max_{1\leq m\leq 2^{\gamma'_k}} \frac{m^{1/(pq)}}{\left(\sum_{i=1}^{m} 1/\lambda_i\right)^{1/p}} = +\infty.$$

Then there exists a sequence of integers $\{n'_k\colon k \geq 1\} \subset \{\gamma'_k\colon k \geq 1\}$ such that

$$\lim_{k\to\infty} \frac{1}{\omega(2^{-n'_k})} \frac{1}{\left(\sum_{i=1}^{m(n'_k)} 1/\lambda_i\right)^{1/p}} \left(\frac{m(n'_k)}{2^{n'_k}}\right)^{1/(pq)} = +\infty, \tag{2}$$

where

$$\max_{1\leq m\leq 2^{n'_k}} \frac{m^{1/(pq)}}{\left(\sum_{i=1}^{m} 1/\lambda_i\right)^{1/p}} = \frac{(m(n'_k))^{1/(pq)}}{\left(\sum_{i=1}^{m(n'_k)} 1/\lambda_i\right)^{1/p}}.$$

The following three cases are possible:

(a) there exists a sequence of integers $\{s'_k\colon k \geq 1\} \subset \{n'_k\colon k \geq 1\}$ such that

$$m(s'_k) < 2^{2s'_{k-1}};$$

(b) there exists a sequence of integers $\{z'_k\colon k \geq 1\} \subset \{n'_k\colon k \geq 1\}$ such that

$$2^{2z'_{k-1}} \leq m(z'_k) < 2^{z'_k - z'_{k-1}};$$

(c) $2^{n'_k - n'_{k-1}} \leq m(n'_k) < 2^{n'_k}$ for all $k \geq k_0$.

First, consider case (a). We choose a sequence of integers $\{s_k\colon k \geq 1\} \subset \{s'_k\colon k \geq 1\}$ such that

$$\left(\sum_{i=1}^{m(s_k)} \frac{1}{\lambda_i}\right)^{1/p} \geq 2^{2s_{k-1}/(pq)}.$$

Then relation (2) yields

$$\lim_{k\to\infty} \omega\left(\frac{1}{2^{s_k}}\right) 2^{s_k/(pq)} = 0.$$

Let $\{r_k\colon k \geq 1\} \subset \{s_k\colon k \geq 1\}$ be such that

$$\omega\left(\frac{1}{2^{r_k}}\right) 2^{r_k/(pq)} \leq 4^{\frac{-k}{p}}. \tag{3}$$

Consider the function f defined as follows:

$$f(x) = \begin{cases} 2c_j(2^{r_j}x-1) & \text{if} \quad x \in [2^{-r_j}, 3.2^{-r_j-1}), \\ -2c_j(2^{r_j}x-2) & \text{if} \quad x \in [3.2^{-r_j-1}), 2.2^{-r_j}) \quad \text{for} \quad j = 1, 2, \ldots, \\ 0 & \text{otherwise}, \end{cases}$$

$$f(x+l) = f(x), \quad l = \pm 1, \pm 2, \ldots,$$

where

$$c_j = \sqrt{\omega\left(\frac{1}{2^{r_j}}\right) 2^{r_j/(pq)}}.$$

Relation (3) leads that $f \in \Lambda BV^{(p)}$.

Now consider case (b). Let $\{z_k \colon k \geq 1\} \subset \{z'_k \colon k \geq 1\}$ be such that

$$\frac{1}{\omega(2^{-z_k})} \frac{1}{\left(\sum_{i=1}^{m(z_k)} 1/\lambda_i\right)^{1/p}} \left(\frac{m(z_k)}{2^{z_k}}\right)^{1/(pq)} \geq 4^k. \tag{4}$$

Consider the function g_k defined as follows:

$$g_k(x) = \begin{cases} h_k(2^{z_k}x - 2j + 1), & x \in [(2j-1)/2^{z_k}, 2j/2^{z_k}), \\ -h_k(2^{z_k}x - 2j - 1), & x \in [2j/2^{z_k}, (2j+1)/2^{z_k}) \\ & \text{for} \quad j = m(z_{k-1}), \ldots, m(z_k) - 1, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$h_k = \frac{1}{2^k \sum_{j=1}^{m(z_k)} 1/\lambda_j}.$$

Let

$$\mathrm{g}(x) = \sum_{k=2}^{\infty} \mathrm{g}_k(x), \qquad \mathrm{g}(x+l) = \mathrm{g}(x), \qquad l = \pm 1, \pm 2, \ldots.$$

First, we prove that $\mathrm{g} \in \Lambda BV^{(p)}$. For every choice of nonoverlapping intervals $\{I_n \colon n \geq 1\}$, we get

$$\sum_{j=1}^{\infty} \frac{|\mathrm{g}(I_j)|^p}{\lambda_j} \leq 2^p \sum_{i=1}^{\infty} h_i^p \sum_{j=1}^{m(z_i)} \frac{1}{\lambda_j} \leq$$

$$\leq 2^p \sum_{i=1}^{\infty} h_i \sum_{j=1}^{m(z_i)} \frac{1}{\lambda_j} = 2^p \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Hence g $\in \Lambda BV^{(p)}$. Finally, consider case (c). Let $\{n_k\colon k \geq 1\} \subset \{n'_k\colon k \geq k_0\}$ be such that

$$n_k \geq 2n_{k-1} + 1,$$

$$\frac{1}{\omega\left(2^{-n_k}\right)} \frac{1}{\left(\sum_{i=1}^{m(n_k)} 1/\lambda_i\right)^{1/p}} \left(\frac{m(n_k)}{2^{n_k}}\right)^{1/(pq)} \geq 2^{2n_{k-1}/(pq)+k}.$$

Consider the function ϕ_k defined as follows:

$$\phi_k(x) = \begin{cases} d_k(2^{n_k}x - 2j + 1), & x \in [(2j-1)/2^{n_k}, 2j/2^{n_k}), \\ -d_k(2^{n_k}x - 2j - 1), & x \in [2j/2^{n_k}, (2j+1)/2^{n_k}) \\ & \text{for} \quad j = 2^{n_{k-1}-n_{k-2}}, \ldots, 2^{n_k - n_{k-1} - 1} - 1, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$d_k = \frac{1}{2^k \sum_{j=1}^{m(n_k)}} 1/\lambda_j.$$

Let

$$\phi(x) = \sum_{k=3}^{\infty} \phi_k(x), \quad \phi(x+l) = \phi(x), \quad l = \pm 1, \pm 2, \ldots.$$

For every choice of nonoverlapping intervals $\{I_n, n \geq 1\}$, we get

$$\sum_{j=1}^{\infty} \frac{|\phi(I_j)|^p}{\lambda_j} \leq 2^p \sum_{i=2}^{\infty} d_i^p \sum_{j=1}^{2^{n_i - n_{i-1} - 1}} \frac{1}{\lambda_j} \leq$$

$$\leq 2^p \sum_{i=2}^{\infty} d_i \sum_{j=1}^{2^{n_i - n_{i-1} - 1}} \frac{1}{\lambda_j} \leq$$

$$\leq 2^p \sum_{i=2}^{\infty} d_i \sum_{j=1}^{m(n_i)} \frac{1}{\lambda_j} \leq 2^p \sum_{i=2}^{\infty} \frac{1}{2^i} < \infty.$$

Hence $\phi \in \Lambda BV^{(p)}$, Now similar to [3] (Theorem 1) we have f, g and ϕ do not belong to H_ω^q. Therefore, the theorem is proved. For $p \geq q$ Theorem 2.1 can be simplified.

To achieve this, we need to prove the following lemma.

Lemma 2.2. *Whenever* $p \geq q$

$$\frac{n}{\left(\sum_{k=1}^{n} 1/\lambda_k\right)^{q/p}} \leq \max_{1 \leq m \leq n} \frac{m}{\left(\sum_{k=1}^{m} 1/\lambda_k\right)^{q/p}} \leq \frac{n}{\left(\sum_{k=2}^{n+1} 1/\lambda_k\right)^{q/p}}.$$

Proof. The left inequality is obvious, and the right inequality is proved below.

Let $\lambda\colon [1,\infty) \to \mathbb{R}$ be an increasing, continuous, piecewise-linear function defined by the values $\lambda(k) = \lambda_k$, $k \geq 1$, and let

$$\Phi(x) := \int\limits_1^x \frac{dt}{\lambda(t)}, \quad H(x) := \frac{\Phi(x+1)}{x^\delta}, \quad \delta := \frac{p}{q} \geq 1.$$

Since Φ' decreases, we conclude that

$$H(x) = \frac{1}{x^{\delta-1}} \int\limits_0^1 \Phi'(1+tx)dt$$

also decreases. If, in addition, we take into account that, for $m \geq 2$,

$$\sum_{k=2}^{n} \frac{1}{\lambda_k} \leq \sum_{k=2}^{m-1} \int\limits_k^{k+1} \frac{dt}{\lambda(t)} = \Phi(m) \leq \sum_{k=1}^{m-1} \frac{1}{\lambda_k},$$

then, for $m \leq n$, we get

$$\frac{m^\delta}{\sum_{k=1}^{m} 1/\lambda_k} \leq \frac{m^\delta}{\Phi(m+1)} = \frac{1}{H(m)} \leq \frac{1}{H(n)} = \frac{n^\delta}{\Phi(n+1)} \leq \frac{n^\delta}{\sum_{k=2}^{n+1} 1/\lambda_k}.$$

Now using Theorem 2.1 and Lemma 2.2 we have the following corollary.

***Corollary* 2.2.** *For some $p, q \in [1,\infty)$ such that $p \geq q$, the inclusion $\Lambda BV^{(p)} \subset H_q^\omega$ holds if and only if*

$$\limsup_{n\to\infty} \frac{1}{\omega(1/n)} \frac{1}{\left(\sum_{i=2}^{n+1} 1/\lambda_i\right)^{1/p}} < +\infty.$$

Applying Corollary 2.2, we see the following corollary.

***Corollary* 2.3.** *For some $p, q \in [1,\infty)$ such that $p \geq q$, the inclusion $\{k^\beta\}BV^{(p)} \subset H_q^\omega$ holds if and only if*

$$\limsup_{n\to\infty} \frac{1}{\omega(1/n)} \frac{1}{\left(\sum_{k=2}^{n+1} 1/k^\beta\right)^{1/p}} < +\infty.$$

Acknowledgment. The author is grateful to the referee for his valuable comments that helped to improve the presentation of the paper. The author also would like to thank Professor Grigori Rozenblum for comments on previous version.

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Received 16.07.11,
after revision — 13.02.12