We show that a hyperquadric $M$ in $\mathbb{R}^4_2$ is a Lie group by using the bicomplex number product. For our purpose, we change the definition of tensor product. We define a new tensor product by considering the tensor product surface in the hyperquadric $M$. By using this new tensor product, we classify totally real tensor product surfaces and complex tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve.

By means of the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve, we determine a special subgroup of the Lie group $M$. Thus, we obtain the Lie group structure of tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. Moreover, we obtain left invariant vector fields of these Lie groups. We consider the left invariant vector fields on these groups, which constitute a pseudo-Hermitian structure. By using this, we characterize these Lie groups as totally real or slant in $\mathbb{R}^4_2$. 

1. Introduction. In the Euclidean space $\mathbb{E}^n$, the tensor product immersion of two immersions of a given Riemannian manifold was firstly defined and studied by Chen in [3]. In particular, the direct sum and the tensor product maps of two immersions of two different Riemannian manifolds are defined by Decruyenaere and coauthors in [4] in the following way:

Let $M$ and $N$ be two differentiable manifolds and assume that $f: M \rightarrow \mathbb{E}^m$ and $h: N \rightarrow \mathbb{E}^n$ are two immersions. The direct sum map and tensor product map are defined respectively by

$$f \oplus h: M \times N \rightarrow \mathbb{E}^{m+n}, \quad (f \oplus h)(p, q) = (f_1(p), \ldots, f_m(p), h_1(q), \ldots, h_n(q)),$$

and

$$f \otimes h: M \times N \rightarrow \mathbb{E}^{mn}, \quad (f \otimes h)(p, q) = (f_1(p)h_1(q), \ldots, f_1(p)h_n(q), \ldots, f_m(p)h_n(q)).$$

Under certain conditions obtained in [4], the tensor product map $f \otimes h$ is an immersion in the space $\mathbb{E}^{mn}$.

The simplest examples of the tensor product immersions are tensor product surfaces. In the Euclidean space $\mathbb{E}^n$, the tensor product surfaces of two Euclidean planar curves, as well as of a Euclidean space curve and a Euclidean plane curve are investigated in [8] and [1], respectively. Moreover, in the semi-Euclidean space $\mathbb{E}^n_\nu$, the tensor product surfaces of two Lorentzian planar surfaces are studied in [2].
curves, as well as of a Lorentzian plane curve and a Euclidean plane curve are studied in [9] and [10], respectively. Also, the tensor product surfaces of a Lorentzian space curve and a Euclidean plane curve as well as of a Euclidean space curve and a Lorentzian plane curve are studied in [5] and [6], respectively.

It is often quite difficult to decide if a manifold is parallelizable. \( S^n \) is parallelizable if and only if \( n = 1, 3, 7 \). Is it possible to make parallelization of any surface? The answer is yes. If \( M \) is a Lie group then \( M \) is parallelizable.

In [12], the authors showed that a hyperquadric \( M \) in \( \mathbb{R}^4 \) is a Lie group by using bicomplex number product. Also, in the same paper, Lie group structure of tensor product surfaces of Euclidean planar curves was obtained.

In this paper, we obtain Lie group structure of some special hypersurface in \( \mathbb{R}^{4,2} \). By changing the tensor product rule given in \( \mathbb{R}^{4,2} \) in [3] we give a new tensor product definition. As a result, the tensor product surface is obtained as a subset of the hyperquadric \( M \). Hence, we investigate the tensor product surface as a Lie group. For our aim, if we change the definition of tensor product given in above as;

\[
(\alpha \otimes \beta)(t,s) = (\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), -\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s))
\]

we can easily obtain the same results which given in [10]. By using the new tensor product, we classify totally real tensor product surfaces and complex tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. Furthermore we give some theorems for tensor product surfaces to be Lie groups and one parameter Lie subgroups. Finally, we give the necessary conditions for Lie group structures of tensor product surfaces to be totally real or slant in \( \mathbb{R}^{4,2} \), respectively.

2. Preliminary. A bicomplex number is defined by the basis \( \{1, i, j, ij\} \) where \( i, j, ij \) satisfy \( i^2 = -1, j^2 = -1, ij = ji \). Thus any bicomplex number \( x \) can be expressed as \( x = x_11 + x_2i + x_3j + x_4ij \) \( \forall x_1, x_2, x_3, x_4 \in \mathbb{R} \). We denote the set of bicomplex numbers by \( C_2 \). For any \( x = x_11 + x_2i + x_3j + x_4ij \) and \( y = y_11 + y_2i + y_3j + y_4ij \) in \( C_2 \), the bicomplex number addition is defined as

\[
x + y = (x_1 + y_1)1 + (x_2 + y_2)i + (x_3 + y_3)j + (x_4 + y_4)ij.
\]

The multiplication of a bicomplex number \( x = x_11 + x_2i + x_3j + x_4ij \) by a real scalar \( \lambda \) is defined as

\[
\lambda x = \lambda x_11 + \lambda x_2i + \lambda x_3j + \lambda x_4ij.
\]

With this addition and scalar multiplication operations, \( C_2 \) is a real vector space.

Bicomplex number product, denoted by \( \times \), over the set of bicomplex numbers \( C_2 \) is given by the following table:

<table>
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<th>1</th>
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<td>ij</td>
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</table>
The vector space $C_2$ together with the bicomplex product $\times$ is an real algebra [13].
Since bicomplex number algebra is associative it can be considered in terms of matrices. Consider the set of matrices

$$
Q = \left\{ \begin{bmatrix}
    x_1 & -x_2 & -x_3 & x_4 \\
x_2 & x_1 & -x_4 & -x_3 \\
x_3 & -x_4 & x_1 & -x_2 \\
x_4 & x_3 & x_2 & x_1
\end{bmatrix} : x_i \in \mathbb{R}, 1 \leq i \leq 4 \right\}.
$$

The set $Q$ together with matrix addition and scalar matrix multiplication is a real vector space. Furthermore, the vector space together with matrix product is an algebra.

The transformation $h : C_2 \rightarrow Q$ given by

$$
\frac{\partial f}{\partial s} = \left( \alpha_1'(t)\beta_1(s), \alpha_1'(t)\beta_2(s), -\alpha_2'(t)\beta_2(s), \alpha_2'(t)\beta_1(s) \right),
$$

where $\alpha'$ means the derivative of $\alpha$. 

ISSN 1027-3190. Укр. мат. журн., 2012, т. 64, № 3
In the following, we will assume that $\alpha$ is a spacelike or a timelike curve with spacelike or a timelike position vector and we will assume that $\beta$ is a regular curve. We shall also assume that the tensor product surface $f(t,s)$ is a regular surface, i.e., $g_{11}g_{22} - g_{12}^2 \neq 0$.

Hence relations (3.1) and (3.2) imply that the coefficients of the pseudo-Riemannian metric, induced on $f(t,s)$ by the pseudo-Euclidian metric $g$ of $\mathbb{R}_4^2$ is given by

$$
g_{11} = g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) = g_1(\alpha', \alpha'),
g_{12} = g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) = g_1(\alpha, \alpha'),
g_{22} = g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) = g_1(\alpha, \alpha'),
$$

where $g_1 = dx_1^2 - dx_2^2$ and $g_2 = dx_1^2 + dx_2^2$ are the metrics of $\mathbb{R}^2_1$ and $\mathbb{R}^2$, respectively. Consequently, an orthonormal basis for the tangent space of $f(t,s)$ is given by

$$
e_1 = \frac{1}{\sqrt{|g_{11}|}} \frac{\partial f}{\partial t},
e_2 = \frac{1}{\sqrt{|g_{11}(g_{11}g_{22} - g_{12}^2)|}} \left(g_{11} \frac{\partial f}{\partial s} - g_{12} \frac{\partial f}{\partial t}\right).
$$

Recall that the mean curvature vector field $H$ is defined by

$$H = \frac{1}{2}(\varepsilon_1 h(e_1, e_1) + \varepsilon_2 h(e_2, e_2)),
$$

where $h$ is a second fundamental form of $\alpha \otimes \beta$ and $\varepsilon_i = g(e_i, e_i)$, $i = 1, 2$. In particular by Beltrami’s formula we have

$$H = -\frac{1}{2}\Delta f.
$$

Next, recall that a surface $M$ in $\mathbb{R}_4^2$ is said to be minimal, if its mean curvature vector field $H$ vanishes identically.

A basis of the normal space of $f(t,s)$ can be calculated as follows. Let $J_1: \mathbb{R}_1^2 \to \mathbb{R}_1^2$ and $J_2: \mathbb{R}^2 \to \mathbb{R}^2$ be the following maps:

$$J_1(x, y) = (y, x),
J_2(x, y) = (-y, x).
$$

Observe that $g_1(X, J_1(X)) = 0$ for $X \in \mathbb{R}_1^2$ and $g_2(Y, J_2(Y)) = 0$ for $Y \in \mathbb{R}^2$.

Then the normal space is spanned by

$$n_1(t,s) = J_1(\alpha(t)) \otimes J_2(\beta(s)) = (\alpha_2(t), \alpha_1(t)) \otimes (-\beta_2(s), \beta_1(s)) =
\begin{pmatrix}
-\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s), -\alpha_1(t)\beta_1(s), -\alpha_1(t)\beta_2(s)
\end{pmatrix},
$$

ISSN 1027-3190. Укр. мат. журн., 2012, т. 64, № 3
\[ n_2(t, s) = J_1(\alpha'(t)) \otimes J_2(\beta'(s)) = (\alpha'_2(t), \alpha'_1(t)) \otimes (-\beta'_2(s), \beta'_1(s)) = \]
\[ = \left( -\alpha'_2(t)\beta'_2(s), \alpha'_2(t)\beta'_1(s), -\alpha'_1(t)\beta'_1(s), -\alpha'_1(t)\beta'_2(s) \right). \]

4. Totally real and complex Lorentzian immersion and slant tensor product surface. Let \( \alpha : \mathbb{R} \rightarrow \mathbb{R}^2_1(+-) \) and \( \beta : \mathbb{R} \rightarrow \mathbb{R}^2 \) be respectively a Lorentzian planar curve and a Euclidean planar curve and let \( f = \alpha \otimes \beta \) be their tensor product. We consider the pseudo-Hermitian structure \( J \) given by
\[ J(u, v, z, w) = (-v, u, -w, z), \quad u, v, z, w \in \mathbb{R}. \]

In the next theorem by using the new product we classify totally real tensor product surface in the semi-Euclidean space \( \mathbb{R}^4_2 \), i.e., the pseudo-Hermitian structure \( J \) at each point transforms the tangent space to the surface into the normal space.

**Theorem 1.** The tensor product immersion \( f = \alpha \otimes \beta \) of a Lorentzian plane curve and a Euclidean plane curve is a totally real Lorentzian immersion with respect to the pseudo-Hermitian structure \( J \) on \( \mathbb{R}^4_2 \) if and only if \( \alpha \) is a Lorentzian circle centered at 0 or \( \beta \) is a straight line passing through origin.

**Proof.** \( \text{Im} f \) is a totally real surface if and only if \( J \left( \frac{\partial f}{\partial t} \right) \) is orthogonal to \( \frac{\partial f}{\partial s} \) and \( J \left( \frac{\partial f}{\partial s} \right) \) is orthogonal to \( \frac{\partial f}{\partial t} \).

We have
\[ J \left( \frac{\partial f}{\partial t} \right) = \left( -\alpha'_1(t)\beta'_2(s), \alpha'_1(t)\beta'_1(s), -\alpha'_2(t)\beta'_1(s), -\alpha'_2(t)\beta'_2(s) \right). \]

By a straightforward calculation, we obtain
\[ g \left( J \left( \frac{\partial f}{\partial t} \right), \frac{\partial f}{\partial s} \right) = -g \left( J \left( \frac{\partial f}{\partial s} \right), \frac{\partial f}{\partial t} \right), \]
\[ g \left( J \left( \frac{\partial f}{\partial t} \right), \frac{\partial f}{\partial t} \right) = 0 \]
if and only if \( (\alpha_1\alpha'_1 - \alpha_2\alpha'_2) = 0 \) or \( (\beta_1\beta'_2 - \beta'_1\beta_2) = 0 \). Integrating these equations, we find that either \( \beta \) is a straight line passing through origin, or \( \alpha \) is a Lorentzian circle centered at 0.

Theorem 1 is proved.

**Theorem 2.** The tensor product immersion \( f = \alpha \otimes \beta \) of a Lorentzian plane curve and a Euclidean plane curve is a complex Lorentzian immersion with respect to the pseudo-Hermitian structure \( J \) on \( \mathbb{R}^4_2 \) if and only if \( \alpha \) is a straight line passing through origin and \( \beta \) is an Euclidean planar curve.

**Proof.** By definition, the following equations are satisfied:
\[ g \left( J \left( \frac{\partial f}{\partial t} \right), n_i \right) = 0, \quad g \left( J \left( \frac{\partial f}{\partial s} \right), n_i \right) = 0, \quad i = 1, 2. \quad (4.1) \]

We have

ISSN 1027-3190. Укр. мат. журн., 2012, т. 64, № 3
By a straightforward calculation, we obtain
\[ g \left( J \left( \frac{\partial f}{\partial t} \right), n_2 \right) = g \left( J \left( \frac{\partial f}{\partial s} \right), n_1 \right) = 0, \]
\[ g \left( J \left( \frac{\partial f}{\partial t} \right), n_1 \right) = (\alpha_1(t)\alpha_2(t) - \alpha_1'(t)\alpha_2'(t)) \left( \beta_1^2(s) + \beta_2^2(s) \right), \quad (4.2) \]
\[ g \left( J \left( \frac{\partial f}{\partial s} \right), n_2 \right) = (\alpha_1(t)\alpha_2(t) - \alpha_1(t)\alpha_2'(t)) \left( \beta_1^2(s) + \beta_2^2(s) \right). \]
From equations (4.1) and (4.2) we have
\[ \alpha_1'(t)\alpha_2(t) - \alpha_1(t)\alpha_2'(t) = 0. \]
It follows that \( \alpha \) is straight line passing through the origin.

Theorem 2 is proved.

Recall the definition of a slant surface with respect to pseudo-Hermitian structure \( J \) on \( \mathbb{R}^3_2 \). Let \( M \) be a surface with respect to the pseudo-Hermitian structure \( J \) on \( \mathbb{R}^4_2 \). Then \( M \) is said to be a proper slant if
\[ g \left( J (e_1), e_2 \right) = \lambda, \quad \lambda \in \mathbb{R}, \]
along \( M \) for a given orthonormal basis \( \{ e_1, e_2 \} \) of \( T_p M \) \((p \in M)\) which is independent of the choice of \( \{ e_1, e_2 \} \) [3].

Let \( \alpha : \mathbb{R} \to \mathbb{R}^2_2 (+) \) and \( \beta : \mathbb{R} \to \mathbb{R}^2 \) be respectively a Lorentzian planar curve and a Euclidean planar curve. We consider polar coordinates on \( \alpha \) and \( \beta \). Then,
\[ \alpha(t) = \rho_1(t) \left( \cosh t, \sinh t \right), \]
\[ \beta(s) = \rho_2(s) \left( \cos s, \sin s \right). \]
A straightforward computation leads to
\[ g \left( J (e_1), e_2 \right) = \frac{\rho_1'\rho_2}{\sqrt{\left| \rho_1'^2 - \rho_1^2 \right| \left( \rho_2^2 + \rho_2'^2 \right) - \rho_1'^2 \rho_2'^2}}. \]
If \( \rho_2 = \text{constant} \), it follows that \( \rho_1 = a_1 e^{b_1 t}, a_1 \in \mathbb{R}, b_1 \in \mathbb{R} \). Hence \( \alpha \) is a hyperbolic spiral and \( \beta \) is a circle centered at the origin. If \( \rho_2 \neq \text{constant} \), let us put \( \frac{\rho_k}{\rho_{k'}} = c_k, k = 1, 2 \). Then
\[ g \left( J (e_1), e_2 \right) = \frac{c_2}{\sqrt{\left| (c_2^2 + 1) \left( 1 - c_1^2 \right) - 1 \right|}}. \]
Therefore \( \text{Im} \ f \) is a proper slant surface if and only if
\[
\frac{c_2}{(c_2 + 1) (1 - c_1^2)} - 1 = \lambda^2.
\]
It follows that
\[
\frac{c_2^2 + \lambda^2}{c_2 + 1} = \lambda^2 (1 - c_1^2).
\]
This means that \( c_1(t) \) and \( c_2(s) \) must be constant functions, which implies that \( \rho_1(t) = a_1 e^{b_1 t} \), \( \rho_2(s) = a_2 e^{b_2 s} \), \( a_1, b_1, a_2, b_2 \in \mathbb{R} \). Consequently, \( \alpha \) is a hyperbolic spiral and \( \beta \) is a logarithmic spiral. In this way, we proved the following theorem.

**Theorem 3.** The tensor product immersion \( f = \alpha \otimes \beta \) of a Lorentzian plane curve and a Euclidean plane curve is a slant surface with respect to pseudo-Hermitian structure \( J \) on \( \mathbb{R}^2 \) if and only if \( \alpha \) is a hyperbolic spiral and \( \beta \) is either a circle centered at \( O \) or a spiral curve.

**5. Lie groups and some special subgroup.** In this section, we deal with the hyperquadric
\[
M = \{ x = (x_1, x_2, x_3, x_4) : x_1 x_3 + x_2 x_4 = 0, \ g(x, x) \neq 0 \},
\]
\[
M = \{ x = (x_1, x_2, x_3, x_4) : x_1 x_3 + x_2 x_4 = 0, \ x_1^2 + x_2^2 - x_3^2 - x_4^2 \neq 0 \}.
\]
We consider \( M \) as the set of bicomplex numbers,
\[
M = \{ x = x_1 + x_2 i + x_3 j + x_4 i j \in C_2 : x_1 x_3 + x_2 x_4 = 0, \ g(x, x) \neq 0 \}.
\]
The components of \( M \) are easily obtained by representing bicomplex number multiplication in matrix form:
\[
\tilde{M} = \left\{ x = \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ -x_4 & x_1 & x_2 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \end{bmatrix} : x_1 x_3 + x_2 x_4 = 0, \ g(x, x) \neq 0 \right\}.
\]

**Theorem 4.** The set of \( M \) together with the bicomplex number product is a Lie group.

**Proof.** \( \tilde{M} \) is a differentiable manifold and at the same time a group with group operation given by matrix multiplication. The group function
\[
: \tilde{M} \times \tilde{M} \to \tilde{M}
\]
defined by \( (x, y) \to x y \) is differentiable. So, \( (M, \times) \) can be made a Lie group so that \( h \) is a isomorphism.

Theorem 4 is proved.

Consider the group \( M_1 \) of all unit bicomplex numbers \( x = x_1 1 + x_2 i + x_3 j + x_4 i j \) on \( M \) with the group operation of bicomplex multiplication. That is
\[
M_1 = \{ x \in M : g(x, x) = 1 \},
\]
\[
M_1 = \{ x \in M : x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1 \}.
\]
Lemma 1. \( M_1 \) is 2-dimensional Lie subgroup of \( M \).

6. Lie algebra of Lie group \( M \) and \( M_1 \). \( M \) is a Lie group of dimension three. Let us find its Lie algebra. Thus, let

\[
\alpha(t) = \alpha_1(t) + \alpha_3(t) + \alpha_4(t) + \alpha_3(t) + \alpha_4(t)
\]

be a curve on \( M \) such that \( \alpha(0) = 1 \), i.e., \( \alpha_1(0) = 1 \), \( \alpha_m(0) = 0 \) for \( m = 2, 3, 4 \). Differentiation of the equation \( \alpha_1(t) + \alpha_3(t) + \alpha_4(t) + \alpha_3(t) + \alpha_4(t) = 0 \) yields the equation \( \alpha_1'(t) + \alpha_3'(t) + \alpha_4'(t) + \alpha_3(t) + \alpha_4(t) = 0 \). Substituting \( t = 0 \), we obtain \( \alpha_1'(0) = 0 \). The Lie algebra is thus constituted by vectors of the form \( \alpha = \alpha_m \left( \frac{\partial}{\partial \alpha_m} \right) |_{\alpha=1} \), where \( m = 1, 2, 4 \). The vector \( \alpha \) is formally written in the form \( \alpha = \alpha_1 + \alpha_2 i + \alpha_4 ij \). Let us find the left invariant vector field \( X \) on \( M \) for which \( X |_{\alpha=1} = \alpha \). Let \( \beta(t) \) be a curve on \( M \) such that \( \beta(0) = 1 \). \( \beta'(0) = \alpha \). Then \( L_x (\beta(t)) = x \beta(t) \) is the left translation of the curve \( \beta(t) \) by the bicomplex number \( x \), its tangent vector is \( x \beta'(0) = x \alpha \). In particular, denote by \( X_m \) those left invariant vector fields on \( M \) for which, \( X_m |_{\alpha=1} = \frac{\partial}{\partial \alpha_m} \bigg|_{\alpha=1} \),

where \( m = 1, 2, 4 \). These three vector fields are represented at the point \( \alpha = 1 \), in bicomplex notation, by the bicomplex units \( 1, i, ij \). For the components of these vector fields at the point \( x = x_1 + x_2 i + x_3 j + x_4 ij \) we have \( (X_1)_x = x_1, (X_2)_x = x_2, (X_3)_x = x_3, (X_4)_x = x_4 \):

\[
X_1 = (x_1, x_2, x_3, x_4),
X_2 = (-x_2, x_1, -x_4, x_3),
X_4 = (x_4, -x_3, -x_2, x_1),
\]

where all the partial derivatives are at the point \( x \).

\( M_1 \) is a Lie group of dimension two. Its Lie algebra can be easily found that

\[
X_2 = (-x_2, x_1, -x_4, x_3),
X_4 = (x_4, -x_3, -x_2, x_1).
\]

Theorem 5. \( M \) is paralelizable.

Proof. If we put

\[
x_1 = \rho_1 \cos \phi,
\]
\[
x_2 = \rho_1 \sin \phi,
\]
\[
x_3 = \rho_2 \cos \theta,
\]
\[
x_4 = \rho_2 \sin \theta,
\]

then from \( x_1 x_3 + x_2 x_4 = 0 \) we have

\[ ISSN 1027-3190. Укр. мат. журн., 2012, т. 63, № 3 \]
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$$\cos \phi = \sin \theta, \quad \sin \phi = - \cos \theta,$$

or

$$\cos \phi = - \sin \theta, \quad \sin \phi = \cos \theta.$$ 

Besides, the condition $x_1^2 + x_2^2 - x_3^2 - x_4^2 \neq 0$ gives $\rho_1^2 - \rho_2^2 \neq 0$. We have the parametric representation of one component of $M$ with vector position $r(\rho_1, \rho_2, \phi)$

$$r = (\rho_1 \cos \phi, \rho_1 \sin \phi, - \rho_2 \sin \phi, \rho_2 \cos \phi).$$

Hence, we have three vectors tangent to coordinate curves

$$r_{\rho_1} = (\cos \phi, \sin \phi, 0, 0),$$

$$r_{\rho_2} = (0, 0, - \sin \phi, \cos \phi),$$

$$r_{\phi} = (- \rho_1 \sin \phi, \rho_1 \cos \phi, - \rho_2 \cos \phi, - \rho_2 \sin \phi).$$

Evidently these vectors are orthogonal each other and put together a parallelization of $M$.

Theorem 5 is proved.

7. Tensor product surfaces and Lie groups. In this section, by using the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve, we determine some special subgroup of this Lie group $M$. Thus, we obtain Lie group structure of tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. Also, we obtain left invariant vector fields of these Lie groups.

Theorem 6. Let $\alpha: \mathbb{R} \to \mathbb{R}^2_1$ be a hyperbolic spiral, and $\beta: \mathbb{R} \to \mathbb{R}^2$ be a spiral with the same parameter, i.e., $\alpha(t) = e^{at}(\cosh t, \sinh t)$ and $\beta(t) = e^{bt}(\cos t, \sin t)$, $a, b \in \mathbb{R}$. Then their tensor product is a one-parameter subgroup in a Lie group $M$.

Proof. We obtain

$$\gamma(t) = \alpha(t) \otimes \beta(t) = e^{(a+b)t}(\cosh t \cos t, \cosh t \sin t, - \sinh t \sin t, \sinh t \cos t).$$

It can be easily seen that

$$\gamma(t_1) \times \gamma(t_2) = \gamma(t_1 + t_2)$$

for all $t_1, t_2$. Hence, $(\gamma(t), \times)$ is a one-parameter Lie subgroup of $(M, \times)$.

Theorem 6 is proved.

Corollary 1. Let $\alpha: \mathbb{R} \to \mathbb{R}^2_1$ be a hyperbolic spiral and $\beta: \mathbb{R} \to \mathbb{R}^2$ be a circle centered at $O$ with the same parameter, i.e., $\alpha(t) = e^{at}(\cosh t, \sinh t)$, $a \in \mathbb{R}$, and $\beta(t) = (\cos t, \sin t)$. Then their tensor product is a one-parameter subgroup in a Lie group $M$.

Proof. In Theorem 6 taking $b = 0$, we find that $\beta$ is a circle centered at $O$. Then their tensor product is a one-parameter subgroup in a Lie group $M$.

Corollary 2. Let $\alpha: \mathbb{R} \to \mathbb{R}^2_1$ be a Lorentzian circle centered at $O$ and $\beta: \mathbb{R} \to \mathbb{R}^2$ be circle centered at $O$ with the same parameter, i.e., $\alpha(t) = (\cosh t, \sinh t)$ and $\beta(t) = (\cos t, \sin t)$. Then their tensor product is a one-parameter subgroup in a Lie group $M_1$. 

ISSN 1027-3190. Укр. мат. журн., 2012, т. 64, № 3
Proof. Since \(\|\alpha(t) \otimes \beta(t)\|_1 = 1\), it follows that \(\alpha(t) \otimes \beta(t) \subset M_1\). By taking \(a = b = 0\), in Theorem 6, we find that \(\alpha\) is a Lorentzian circle centered at \(O\) and \(\beta\) is a circle centered at \(O\). Then their tensor product is a one-parameter subgroup in a Lie group \(M_1\).

Theorem 7. Let \(\alpha : \mathbb{R} \to \mathbb{R}^4_1\) be a Lorentzian circle centered at \(O\) and \(\beta : \mathbb{R} \to \mathbb{R}^2\) be circle centered at \(O\) with the same parameter, i.e., \(\alpha(t) = (\cosh t, \sinh t)\), \(\beta(t) = (\cos t, \sin t)\), and \(\gamma(t) = \alpha(t) \otimes \beta(t)\) be their tensor product. Then, the left invariant vector field on \(\gamma(t)\) is \(X = X_2 + X_4\), where \(X_2\) and \(X_4\) are left invariant vector fields on \(M_1\).

Proof. Let us find the left invariant vector field on \(\gamma(t)\) to the vector,

\[ u = \frac{d}{dt} \bigg|_{t=0} \]

\(\eta(t) = (1, t, 0, t)\) is a curve with tangent vector \(u\). Its image under \(L_g\) is the curve,

\[ L_g(\eta(t)) = g\eta(t) = (x_1 + x_2 + x_3 + x_4) \times (1 + ti + tij) = (x_1 - x_2t + x_4t) + i(x_1t + x_2 - x_3t) + j(-x_2t + x_3 - x_4t) + ij(x_1t + x_3t + x_4). \]

Its tangent vector is,

\[ L_g(\eta(t))(t) = (-x_2 + x_4) + i(x_1 - x_3) + j(-x_2 - x_4) + ij(x_1 + x_3). \]

For the left invariant vector field \(X\) we have

\[ X = (-x_2 + x_4) \frac{\partial}{\partial x_1} + (x_1 - x_3) \frac{\partial}{\partial x_2} + (-x_2 - x_4) \frac{\partial}{\partial x_3} + (x_1 + x_3) \frac{\partial}{\partial x_4}. \]

Theorem 7 is proved.

Conclusion 1. Let \(\alpha : \mathbb{R} \to \mathbb{R}^4_1\) be a hyperbolic spiral (or a Lorentzian circle centered at \(O\)) and \(\beta : \mathbb{R} \to \mathbb{R}^2\) be a spiral (or circle centered at \(O\)) with the same parameter. Then their tensor product is the maximal integral curve.

Now, we want to classify these Lie groups as totally real or slant in semi-Euclidean space \(\mathbb{R}^4_2\).

In order to do so, consider the left invariant vector field on these groups which constitute pseudo-Hermitian structure which is given by \(J = X_2\).

Corollary 3. Let \(\alpha : \mathbb{R} \to \mathbb{R}^4_1\) be a Lorentzian circle centered at \(O\), \(\beta : \mathbb{R} \to \mathbb{R}^2\) be either a spiral or a circle centered at \(O\), and \(f = \alpha \otimes \beta\) be their tensor product immersion. Then the Lie group \(f(t, s)\) is totally real Lorentzian immersion with respect to the pseudo-Hermitian structure \(J\).

Proof. From Theorem 1 we know that, if \(\alpha\) is a Lorentzian circle centered at \(O\) then \(f = \alpha \otimes \beta\) is totally real surface with respect to the pseudo-Hermitian structure \(J\).

Corollary 4. Let \(\alpha : \mathbb{R} \to \mathbb{R}^4_1\) be a hyperbolic spiral and \(\beta : \mathbb{R} \to \mathbb{R}^2\) be either a circle centered at \(O\) or a spiral and \(f = \alpha \otimes \beta\) be their tensor product. Then the Lie group \(f(t, s)\) is a proper slant surface with respect to pseudo-Hermitian structure \(J\) on \(\mathbb{R}^4_2\).

Proof. From Theorem 3 we know that, if \(\alpha\) is a hyperbolic spiral and \(\beta\) is either a circle centered at \(O\) or a spiral curve then \(f = \alpha \otimes \beta\) is proper slant surface with respect to the pseudo-Hermitian structure \(J\).

BICOMPLEX NUMBER AND TENSOR PRODUCT SURFACES IN $\mathbb{R}^2_2$


Received 15.10.09, after revision — 24.02.12