UDC 517.5

F. G. Abdullayev, N. P. Özkartepe (Mersin Univ., Turkey)

ON THE IMPROVEMENT OF THE RATE OF CONVERGENCE OF THE GENERALIZED BIEBERBACH POLYNOMIALS IN DOMAINS WITH ZERO ANGLES

ПРО ПОКРАЩЕННЯ ШВИДКОСТІ ЗБІЖНОСТІ УЗАГАЛЬНЕНИХ ПОЛІНОМІВ БІБЕРБАХА В ОБЛАСТЯХ З НУЛЬОВИМИ КУТАМИ

Let $\mathbb{C}$ be the complex plane, let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, let $G \subset \mathbb{C}$ be a finite Jordan domain with $0 \in G$, let $L := \partial G$, let $\Omega := \overline{\mathbb{C}} \backslash \overline{G}$, and let $w = \varphi(z)$ be the conformal mapping of G onto a disk $B(0, \rho_0) := \{w \colon |w| < \rho_0\}$ normalized by $\varphi(0) = 0$, $\varphi'(0) = 1$, where $\rho_0 = \rho_0\,(0, G)$ is the conformal radius of G with respect to 0. Let $\varphi_p(z) := \int_0^z \left[\varphi'(\zeta)\right]^{2/p} d\zeta$ and let $\pi_{n,p}(z)$ be the generalized Bieberbach polynomial of degree n for the pair $(G, 0)$ that minimizes the integral $\iint_G \left|\varphi_p'(z) - P_n'(z)\right|^p d\sigma_z$ in the class of all polynomials of degree $\deg P_n \leq n$ such that $P_n(0) = 0$ and $P_n'(0) = 1$.

We study the uniform convergence of the generalized Bieberbach polynomials $\pi_{n,p}(z)$ to $\varphi_p(z)$ on $\overline{G}$ with interior and exterior zero angles determined depending on properties of boundary arcs and the degree of their tangency. In particular, for Bieberbach polynomials, we obtain better estimates for the rate of convergence in these domains.

Нехай $\mathbb{C}$ — комплексна площина, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $G \subset \mathbb{C}$ — скінченна жорданова область із $0 \in G$, $L := \partial G$, $\Omega := \overline{\mathbb{C}} \backslash \overline{G}$ і $w = \varphi(z)$ — конформне відображення G на круг $B(0, \rho_0) := \{w \colon |w| < \rho_0\}$, нормоване умовами $\varphi(0) = 0$, $\varphi'(0) = 1$, де $\rho_0 = \rho_0\,(0, G)$ — конформний радіус G відносно 0. Покладемо $\varphi_p(z) := \int_0^z [\varphi'(\zeta)]^{2/p} d\zeta$. Нехай $\pi_{n,p}(z)$ — узагальнений поліном Бібербаха степеня n для пари $(G, 0)$, що мінімізує інтеграл $\iint_G \left|\varphi_p'(z) - P_n'(z)\right|^p d\sigma_z$ у класі всіх поліномів степеня $\deg P_n \leq n$ таких, що $P_n(0) = 0$, $P_n'(0) = 1$.

Вивчається рівномірна збіжність узагальнених поліномів Бібербаха $\pi_{n,p}(z)$ до $\varphi_p(z)$ у $\overline{G}$ із внутрішніми та зовнішніми нульовими кутами, що визначаються в залежності від властивостей граничних дуг та степеня їхнього дотику. Зокрема, для поліномів Бібербаха отримано покращені оцінки швидкості збіжності у цих областях.

1. Introduction and main result. Let $\mathbb{C}$ be the complex plane, let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, let $G \subset \mathbb{C}$ be a finite Jordan domain with $0 \in G$, let $L := \partial G$, let $\Omega := \overline{\mathbb{C}} \backslash \overline{G}$, and let $w = \varphi(z)$ be the conformal mapping of G onto a disk $B(0, \rho_0) := \{w \colon |w| < \rho_0\}$ normalized by the conditions $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\rho_0 = \rho_0\,(0, G)$ is the conformal radius of G with respect to 0.

For $p > 0$, let $A_p^1(G)$ denote the set of functions $f(z)$ analytic in G, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, and such that

$$\|f\|_p := \|f\|_{A_p^1(G)} := \left(\iint\limits_G \left|f'(z)\right|^p d\sigma_z \right)^{1/p} < \infty,$$

where σ denotes the two-dimensional Lebesgue measure.

Consider the extremal problem

$$\left\{ \|f\|_p \,, f \in A_p^1(G) \right\} \to \inf\,. \tag{1.1}$$

It is well known [22, p. 426] that the function

$$\varphi_p(z) := \int\limits_0^z \left[\varphi'(\zeta)\right]^{2/p} d\zeta, \quad z \in G, \tag{1.2}$$

is the unique solution of the extremal problem (1.1). This function is well known in the geometric theory of functions and is of great interest (see, e.g., [21]).

Let $\wp_n$ denote the class of all polynomials $P_n(z)$, $\deg P_n(z) \leq n$, satisfying the conditions $P_n(0) = 0$ and $P_n'(0) = 1$. For each $p > 0$, we consider the following extremal problem:

$$\left\{\left\|\varphi_p - P_n\right\|_p, P_n \in \wp_n\right\} \to \inf. \tag{1.3}$$

Using a method similar to that given in [13, p. 137], one can see that, for any $p > 0$, there exists a polynomial $P^*_{n,p}(z)$ that realizes a minimum of the integral $\left\|\varphi_p - P_n\right\|_p$ in the class $\wp_n$, and for $p > 1$ this polynomial is uniquely determined [13, p. 142]. We call this polynomial the *n-th generalized Bieberbach polynomial* for the pair $(G, 0)$ and denote it by $\pi_{n,p}(z)$. In the case $p = 2$, the polynomial $\pi_{n,2}(z)$ coincides with the Bieberbach polynomial for the pair $(G, 0)$ (see, e.g., [14]).

If G is a Carathéodory domain, then $\left\|\varphi_p - \pi_{n,p}\right\|_p \to 0$ as $n \to \infty$ [27, p. 63], and so the sequence $\{\pi_{n,p}(z)\}_{n=0}^{\infty}$ converges uniformly to $\varphi_p(z)$ on compact subsets of G. Our purpose is to extend the uniform convergence of the sequence $\{\pi_{n,p}(z)\}_{n=0}^{\infty}$ to $\varphi_p(z)$ on $\overline{G}$. Moreover, we investigate the estimate

$$\left\|\varphi_p - \pi_{n,p}\right\|_{C(\overline{G})} := \max\left\{\left|\varphi_p(z) - \pi_{n,p}(z)\right|, z \in \overline{G}\right\} \leq \text{const} \cdot \varepsilon_{n,p}, \tag{1.4}$$

where $\varepsilon_{n,p} = \varepsilon_{n,p}(\varphi, G) \to 0$, $n \to \infty$, and its dependence on the geometric properties of G.

For $p = 2$, estimate (1.4) was studied in [18, 20, 26] in the case where L satisfies certain smoothness conditions and in [2, 5, 8 – 10, 14, 15, 24], etc., in the case where L has some zero or nonzero angles.

In the case $p \neq 2$, the existence of a sequence $\{\varepsilon_{n,p}\} \to 0$, $n \to \infty$, that satisfies (1.4) for some domains with quasiconformal and piecewise-smooth (without cusps) boundary was investigated in [17, 3, 6], etc. It is well known that quasiconformal curves have many properties, but they do not have zero angles. Similar problems for domains of the class $PQ\,(K, \alpha, \beta)$ with piecewise-quasiconformal boundaries having interior and exterior zero angles with "power tangency" (of the type $cx^{1+\alpha}$ and $cx^{1+\beta}$ for some $\alpha > 0$ and $\beta > 0$) were investigated in [4].

Prior to introducing the class $PQ\,(K, \alpha, \beta)$, we give several definitions.

Definition 1.1 [19, p. 97; 23]. *A Jordan curve L is called K-quasiconformal $(K \geq 1)$ if there is a K-quasiconformal mapping f of a domain $D \supset L$ such that $f(L)$ is a circle.*

Let $F(L)$ denote the set of all sense-preserving plane homeomorphisms f of domains $D \supset L$ such that $f(L)$ is a circle and let

$$K\,(L) = \inf\left\{K(f) \colon f \in F(L)\right\},$$

where $K(f)$ is the maximal dilatation of a mapping f of this type. The curve L is K-quasiconformal if and only if $K\,(L) < \infty$. If L is K-quasiconformal, then $K\,(L) \leq K$.

In this paper, we consider the case $D \equiv \overline{\mathbb{C}}$, i.e., we use the global definition of K-quasiconformal curve.

Definition 1.2. *A Jordan arc ℓ is called K-quasiconformal if ℓ is a part of some closed K-quasiconformal curve.*

We can now define the class $PQ\,(K,\alpha,\beta)$.

Note that, throughout this paper, $c, c_1, c_2, \ldots$ are positive constants and $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants that depend, in general, on G.

Definition 1.3 [2]**.** *We say that $G \in PQ\,(K,\alpha,\beta)$, $K > 1$, $\alpha > 0$, $\beta > 0$, if $L := \partial G$ is the union of a finite number of K-quasiconformal arcs ($K = \max_{1\le j\le m}\{K_j\}$) connected at the points $z_0, z_1, \ldots, z_m$ and such that L is a locally K-quasiconformal curve at z_0 and the following conditions are satisfied in the local coordinate system (x,y) with origin at z_j, $1 \le j \le m$:*

a) *for $1 \le j \le p$, one has*

$$\left\{z = x + iy\colon\ |z| \le \varepsilon_1,\ c_1 x^{1+\alpha} \le y \le c_2 x^{1+\alpha}\right\} \subset \overline{\Omega},$$

$$\{z = x + iy\colon\ |z| \le \varepsilon_1,\ |y| \ge \varepsilon_2 x\} \subset \overline{G};$$

b) *for $p+1 \le j \le m$, one has*

$$\left\{z = x + iy\colon\ |z| \le \varepsilon_3,\, c_3 x^{1+\beta} \le y \le c_4 x^{1+\beta}\right\} \subset \overline{G},$$

$$\{z = x + iy\colon\ |z| \le \varepsilon_3,\, |y| \ge \varepsilon_4 x\} \subset \overline{\Omega}.$$

Here, $-\infty < c_1 < c_2 < \infty$, $-\infty < c_3 < c_4 < \infty$ and $\varepsilon_i > 0$, $i = \overline{1,4}$, are some constants.

It is clear from Definition 1.3 that each domain $G \in PQ\,(K,\alpha,\beta)$ may have p exterior and $m-p$ interior zero angles. If a domain G does not have exterior $(p=0)$ (interior $(p=m)$) zero angles, then we write $G \in PQ\,(K,0,\beta)$ ($G \in PQ\,(K,\alpha,0)$). If a domain G does not have these angles $(\alpha=\beta=0)$, then G is bounded by a K-quasiconformal curve. Further, $PQ\,(K,\alpha_1,\beta) \subset \subset PQ\,(K,\alpha_2,\beta)$ ($PQ\,(K,\alpha,\beta_1) \subset PQ\,(K,\alpha,\beta_2)$) for $\alpha_2 > \alpha_1$ ($\beta_2 > \beta_1$) and every fixed $\beta > 0$ and $K > 1$ ($\alpha > 0$ and $K > 1$).

In this paper, we study the convergence of generalized Bieberbach polynomials in the closed domains $\overline{G} \in PQ\,(K,\alpha,\beta)$ and estimate an upper bound $\varepsilon_{n,p} = \varepsilon_{n,p}(\varphi, G) \to 0$, $n \to \infty$, and its dependence on the geometric properties of G.

Prior to giving the main results, we introduce the following notation:

$$p_0 := \min\left\{p-1;\ \frac{2}{2+p}\right\}, \quad p_1 := \frac{\sqrt{17}-1}{2}, \quad \widetilde{p}_2 := \frac{\sqrt{20K^4+4K^2+1}-2K^2-1}{2K^2},$$

$$\widetilde{p}_3 := \frac{\sqrt{33K^4+2K^2+1}-K^2-1}{2K^2},$$

and

$$\widetilde{\beta}_2(p,K) :=$$

$$:= \frac{\sqrt{(8K^2+10+2pK^2-p)^2-16(K^2+1)\left[4(K^2+1)-2p(K^2+1)-2\right]}-8K^2+10-2pK^2-p}{8(K^2+1)}.$$

Theorem 1.1. *Suppose that* $p > 2$, $G \in PQ\,(K, \alpha, \beta)$ *for some* $K > 1$, $\alpha < \dfrac{2}{p}$, *and* $0 < \beta < \min\left\{\dfrac{p}{2} - 1; \dfrac{2}{p+2}\right\}$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leq c_1 \left(\frac{1}{\ln n}\right)^{\frac{2-\alpha p}{2\alpha p}}.$$

Theorem 1.2. *Suppose that* $2 < p < 2\sqrt{2}$ *and* $G \in PQ\left(K, \alpha, \dfrac{p}{2} - 1\right)$ *for some* $K > 1$, $\alpha < \dfrac{2}{3p}$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leq c_2 \left(\frac{1}{\ln n}\right)^{\frac{2-3\alpha p}{2\alpha p}}.$$

***Corollary* 1.1.** *Suppose that* $p = 2$ *and* $G \in PQ(K, \alpha, 0)$ *for some* $K > 1$, $0 < \alpha < \dfrac{1}{2}$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi - \pi_n\|_{C(\overline{G})} \leq c_3 \left(\frac{1}{\ln n}\right)^{\frac{1-2\alpha}{2\alpha}}. \tag{1.5}$$

In approximation problems, it is well known that if a domain has an exterior zero angle, then the rate of approximation is “slower” than in its absence. Therefore, the right-hand sides of estimates for such rates usually involve quantities of the type $\left(\dfrac{1}{\ln n}\right)^{\lambda}$. In the presence of an exterior zero angle, quantities of the type $\left(\dfrac{1}{\ln n}\right)^{\lambda}$, $\lambda > 0$, cannot be replaced by $\left(\dfrac{1}{n}\right)^{\mu}$ for any $\mu > 0$. Moreover, since $PQ(K, \alpha_1, \beta) \subset PQ(K, \alpha_2, \beta)$ for $\alpha_2 > \alpha_1$, we may claim that the rate of approximation improves as the class $PQ(K, \alpha, \beta)$ becomes narrower with respect to α (for the same K and β). In other words, as the exterior zero angle of a domain becomes “wider” (for the same K and β), the degree of approximation improves.

In particular, for $p = 2$ and a domain $G \in PQ(K, \alpha, 0)$, a result corresponding to Corollary 1.1 was obtained by Andrievskii in [8] (see also [9] (Th. 2)), in which the right-hand side of (1.5) contains the additional multiplier $\sqrt{\ln \ln n}$. However, Corollary 1.1 shows that this multiplier can be omitted.

Now assume that there is no exterior zero angle ($\alpha = 0$). Then, in theory, the rate of approximation must increase. The theorems presented below confirm this: the degree of approximation is measured along the scale of $\left(\dfrac{1}{n}\right)^{\mu}$ but not the scale of $\left(\dfrac{1}{\ln n}\right)^{\lambda}$.

Theorem 1.3. *Let* $p > 2$ *and* $G \in PQ\,(K, 0, \beta)$ *for some* $K > 1$ *and* $0 < \beta < \min\left\{\dfrac{p}{2} - 1; \dfrac{K^2 - 1}{1+pK^2 + 3K^2}\right\}$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leq c_4 n^{-\gamma},$$

for every γ *such that* $0 < \gamma < \dfrac{1}{pK^2}$.

Theorem 1.4. *Let* $2 < p < \widetilde{p}_3$ *and* $G \in PQ\left(K, 0, \frac{p}{2} - 1\right)$ *for some* $K > 1$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leq c_5 \frac{\ln n}{n^\gamma},$$

for every γ *such that* $0 < \gamma < \frac{1}{pK^2}$.

Theorem 1.5. *Let* $2 - \frac{1}{2K^2} < p < \widetilde{p}_3$ *and* $G \in PQ\,(K, 0, \beta)$ *for some* $K > 1$ *and* $\max\left\{\frac{p}{2} - 1; 0\right\} < \beta < \min\left\{\frac{K^2-1}{1+pK^2+3K^2}; \frac{1}{4K^2} + \frac{p}{2} - 1\right\}$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leq c_6 n^{-\gamma},$$

for every γ *such that* $0 < \gamma < \frac{1 - 2K^2(2\beta + 2 - p)}{pK^2}$.

***Corollary* 1.2.** *Let* $p = 2$ *and* $G \in PQ\,(K, 0, \beta)$ *for some* $K > 1$ *and* $0 \leq \beta < \min\left\{\frac{K^2-1}{1+5K^2}; \frac{1}{4K^2}\right\}$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi - \pi_n\|_{C(\overline{G})} \leq c_7 n^{-\gamma},$$

for every γ *such that* $0 < \gamma < \frac{1}{2K^2} - 2\beta$.

Theorem 1.6. *Let* $p > \widetilde{p}_3$ *and* $G \in PQ\,(K, 0, \beta)$ *for some* $K > 1$ *and* $\frac{K^2-1}{1+pK^2+3K^2} < \beta < \min\left\{\frac{p}{2} - 1; \frac{2}{2+p}\right\}$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leq c_8 n^{-\gamma},$$

for every γ *such that* $0 < \gamma < \frac{2 - (p+2)\beta}{p(1+\beta)(K^2+1)}$.

Theorem 1.7. *Let* $\widetilde{p}_3 < p < 2\sqrt{2}$ *and* $G \in PQ\left(K, 0, \frac{p}{2} - 1\right)$ *for some* $K > 1$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leq c_9 \frac{\ln n}{n^\gamma},$$

for every γ *such that* $0 < \gamma < \frac{2 - (p+2)\beta}{p(1+\beta)(K^2+1)}$.

Theorem 1.8. *Let* $\frac{3}{2} < p < 2\sqrt{2}$ *and* $G \in PQ\,(K, 0, \beta)$ *for some* $1 < K < \widetilde{K}_1$, *where* $\widetilde{K}_1 := \max\left\{K \colon \frac{K^2-1}{1+pK^2+3K^2} < \widetilde{\beta}_2(p, K)\right\}$, *and* $\max\left\{\frac{p}{2} - 1; \frac{K^2-1}{1+pK^2+3K^2}\right\} < \beta < \min\left\{p_0; \widetilde{\beta}_2(p, K)\right\}$. *Then, for any* $n \geq 2$, *one has*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leq c_{10} n^{-\gamma},$$

for every γ *such that* $0 < \gamma < \frac{2 - (p+2)\beta}{p(1+\beta)(K^2+1)} - \frac{2}{p}(2\beta + 2 - p)$.

***Remark* 1.1.** a) Theorems 1.1 – 1.8 are unimprovable, and, in some cases, they improve the corresponding theorems of [4],

b) Theorems 1.1 – 1.8 extend the results of [3, 4, 6, 18, 20] to domains with zero angles.

c) Corollary 1.2 improves the corresponding theorems of [4].

d) Theorems 1.3 – 1.8 show that, as indicated above, the rate of approximation changes in this case according to the state of β, i.e., since $PQ(K, \alpha, \beta_1) \subset PQ(K, \alpha, \beta_2)$ for $\beta_2 > \beta_1$, we may claim that the rate of approximation improves as the class $PQ(K, \alpha, \beta)$ becomes narrower with respect to β (for the same K and α).

2. Some auxiliary facts. Throughout this paper, the notation "$a \prec b$" means that $a \leq c_1 b$ for a constant c_1 that does not depend on a and b. The relation "$a \asymp b$" indicates that $c_2 b \leq a \leq c_3 b$, where c_2 and c_3 are independent of a and b.

Let $G \subset \mathbb{C}$ be a finite domain bounded by a Jordan curve L and let $w = \Phi(z)$ $(w = \widehat{\varphi}(z))$ be a conformal mapping of $\Omega := \operatorname{ext} \overline{G}$ (G) onto $\Delta = \{w\colon |w| > 1\}$ $(B(0,1))$ normalized by the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$ $\big(\widehat{\varphi}(0) = 0$ and $\widehat{\varphi}'(0) > 0\big)$.

The (exterior or interior) level curve can be defined for $t > 0$ as follows:

$$L_t := \{z\colon |\widehat{\varphi}(z)| = t \text{ if } t < 1;\ |\Phi(z)| = t \text{ if } t > 1\}, \quad L_1 \equiv L.$$

Denote $G_t := \operatorname{int} L_t$, $\Omega_t := \operatorname{ext} L_t$, and $d(z, L) := \inf\{|\varsigma - z| : \varsigma \in L\}$.

Let L be a K-quasiconformal curve. Then there exists a quasiconformal reflection $y(\cdot)$ across L such that $y(G) = \Omega$, $y(\Omega) = G$, and $y(\cdot)$ fixes the points of L. By using the results of [7, p. 76] (see also [14], Lemma 1), we can find a $C(K)$-quasiconformal reflection $\alpha(\cdot)$ across L such that

$$|z_1 - \alpha(z)| \asymp |z_1 - z|, \quad z_1 \in L, \quad \varepsilon < |z| < \frac{1}{\varepsilon},$$

$$|\alpha_{\overline{z}}| \asymp |\alpha_z| \asymp 1, \quad \varepsilon < |z| < \frac{1}{\varepsilon},$$

$$|\alpha_{\overline{z}}| \asymp |\alpha_z|^2, \quad |z| < \varepsilon, \quad |\alpha_{\overline{z}}| \asymp |z|^{-2}, \quad |z| > \frac{1}{\varepsilon}, \tag{2.1}$$

$$\big|\alpha(z) - z'\big| \asymp \big|z - z'\big|, \quad z' \in L;$$

$$J_\alpha := |\alpha_z|^2 - |\alpha_{\overline{z}}|^2 : J_\alpha \asymp 1$$

in a certain neighborhood of L.

Lemma 2.1 [1]. *Suppose that L is a K-quasiconformal curve,* $z_1 \in L$, $z_2, z_3 \in G \cap \{z\colon |z - z_1| \leq c_1 d(z_1, L_{R_0})\}$, $w_j = \widehat{\varphi}(z_j)$ $(z_2, z_3 \in \Omega \cap \{z\colon |z - z_1| \leq c_2 d(z_1, L_{r_0})\}$, *and* $w_j = \Phi(z_j))$, $j = 1, 2, 3$. *Then the following assertions are true*:

1) *the statements* $|z_1 - z_2| \prec |z_1 - z_3|$ *and* $|w_1 - w_2| \prec |w_1 - w_3|$ *are equivalent, and so are* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$;

2) *if* $|z_1 - z_2| \prec |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2}.$$

Lemma 2.2. *Suppose that L is a K-quasiconformal curve, $z_1 \in L$, and $z_2 \in G$, $w_2 = \widehat{\varphi}(z_2)$ $(z_2 \in \Omega,\, w_2 = \Phi(z_2))$. Then*

$$|w_1 - w_2|^{\frac{2K^2}{K^2+1}} \prec |z_1 - z_2| \prec\ |w_1 - w_2|^{\frac{2}{K^2+1}}. \tag{2.2}$$

Proof. Let L be a K-quasiconformal curve. Then there exists a K^2-quasiconformal reflection $y(\cdot)$ across L. Therefore, L is a k-quasicircle with $k = \dfrac{K^2-1}{K^2+1}$. According to [21, p. 287] and the estimate for Ψ' [11] (Theorem 2.8), we have

$$|w_1 - w_2|^{1+k} \prec |\Psi(w_1) - \Psi(w_2)| \prec |w_1 - w_2|^{1-k}. \tag{2.3}$$

Lemma 2.3 [3, 12]. *Let L be a K-quasiconformal curve. Then, for every $z \in L$ and $z_0 \in G$, there exists an arc $\ell(z, z_0)$ in G that joins z and z_0 and possesses the following properties*:

i) $d(\zeta, L) \asymp |\zeta - z|$ *for every* $\zeta \in \ell(z, z_0)$;

ii) *for every* $\zeta_1, \zeta_2 \in \ell(z, z_0)$, *if* $\tilde{\ell}(\zeta_1, \zeta_2)$ *is a subarc of* $\ell(z, z_0)$, *then* $\operatorname{mes}\tilde{\ell}(\zeta_1, \zeta_2) \prec |\zeta_1 - \zeta_2|$.

Let $G^{\varepsilon} := \left\{ z\colon z \in G \cap D, d(z, L) < \varepsilon < \frac{1}{2} d(\partial D, L) \right\}$.

Lemma 2.4 [9]. *Let L be a K-quasiconformal curve. Then, for every rectifiable arc $\ell \subset G^{\varepsilon}$, one has* $\operatorname{mes}\ell \asymp \operatorname{mes}\alpha(\ell)$.

Lemma 2.5. *Let L be a K-quasiconformal curve. Then*

a) $\operatorname{mes} G^{\varepsilon} \asymp \operatorname{mes}\alpha(G^{\varepsilon})$;

b) $\operatorname{mes}\widehat{\varphi}(G^{\varepsilon}) \prec \varepsilon^{\delta}$, $\delta = \dfrac{K^2+1}{2K^2}$.

Proof. a) Let $J_{\alpha}(z) := |\alpha_z(z)|^2 - |\alpha_{\overline{z}}(z)|^2$ be a Jacobian of an antiquasiconformal mapping $\alpha(\cdot)$ across L. Then, according to (2.1), we obtain

$$\operatorname{mes}\alpha(G^{\varepsilon}) = \iint\limits_{G^{\varepsilon}} (-J_{\alpha}(z)) d\sigma_z \asymp \iint\limits_{G^{\varepsilon}} d\sigma_z \asymp \operatorname{mes} G^{\varepsilon}.$$

b) It is obvious that

$$\operatorname{mes}\widehat{\varphi}(G^{\varepsilon}) \leq \sup_{z \in \overline{G^{\varepsilon}}} \pi(1 - |\widehat{\varphi}(z)|^2) \prec \sup_{z \in \overline{G^{\varepsilon}}} (1 - |\widehat{\varphi}(z)|). \tag{2.4}$$

According to (2.2) we get

$$d(z, L) \succ (1 - |\widehat{\varphi}(z)|)^{\frac{2K^2}{K^2+1}}. \tag{2.5}$$

Using (2.4), (2.5), we complete the proof.

Lemma 2.6. *Let L be a K-quasiconformal curve. Then, for every u, $0 < u < R_0 - 1$, one has*

$$\operatorname{mes}\alpha(G_{1+u} \backslash G) \prec u^{\frac{1}{K^2}}.$$

Proof. The required statement follows from Lemma 2.5 and [16].

3. Some properties of domains $G \in PQ(K, \alpha, \beta)$. Suppose that a domain $G \in PQ(K, \alpha, \beta)$ is given. Then, for simplicity but without loss of generality, we can assume that $\alpha > 0$, $\beta > 0$, $p = 1$, $m = 2$, $z_1 = 1$, $z_2 = -1$, $(-1, 1) \subset G$, the local coordinate axes in Definition 1.3 are parallel to OX

and OY in the coordinate system, $L^1 := \{z\colon z \in L, \operatorname{Im} z \geq 0\}$, and $L^2 := \{z\colon z \in L,\ \operatorname{Im} z \leq 0\}$. Then z_0 is taken as an arbitrary point on L^2 (or on L^1, depending on the chosen direction).

Recall that the domain $G \in PQ(K, \alpha, \beta)$ has interior and exterior zero angles in the nearest neighborhoods of each of the points $z_1 = 1$ and $z_2 = -1$ respectively. Therefore, following the argument presented in [9], we can say that, for the domain $G \in PQ(K, \alpha, \beta)$, the function $w = \Phi(z)$ $(w = \widehat{\varphi}(z))$ satisfies the conditions given in Lemma 2.2 in the nearest neighborhood of the point $z_2 = -1$ $(z_1 = 1)$. Thus, using Lemma 2.2, we can easily get

$$d(z, L) \prec (|\widehat{\varphi}(z)| - 1)^{\frac{2}{K^2+1}}, \quad |z - 1| \prec |\widehat{\varphi}(z) - \widehat{\varphi}(1)|^{\frac{2}{K^2+1}}$$

$$\forall z \in M_1 := \{z \in G\colon |z + 1| > \varepsilon_1\},$$

$$d(z, L) \prec (|\Phi(z)| - 1)^{\frac{2}{K^2+1}}; \quad |z + 1| \prec |\Phi(z) - \Phi(-1)|^{\frac{2}{K^2+1}} \tag{3.1}$$

$$\forall z \in M_2 := \{z \in \Omega\colon |z - 1| > \varepsilon_2\}.$$

On the other hand, if $G \in PQ\,(K, \alpha, \beta)$, then, for points $z \in \Omega \setminus M_2$ and $z \in G \setminus M_1$, using the properties of the functions $w = \Phi(z)$ and $w = \widehat{\varphi}(z)$ in the nearest neighborhoods of the points $z_1 = 1$ and $z_2 = -1$, respectively, we obtain (see [9])

$$|z - 1| \prec [-\ln|\Phi(z) - \Phi(1)|]^{-\alpha^{-1}}, \quad |z + 1| \prec [-\ln|\widehat{\varphi}(z) - \widehat{\varphi}(-1)|]^{-\beta^{-1}}. \tag{3.2}$$

Lemma 3.1 [4]. *Let G be Jordan domain such that, for every $z \in L$, there exists an arc $\gamma(z, 0)$ in G that joins 0 and z and possesses the following properties*:

i) $\operatorname{mes} \gamma(\zeta_1, \zeta_2) \prec |\zeta_1 - \zeta_2|$ *for every* $\zeta_1, \zeta_2 \in \gamma(z, 0)$;

ii) *there exists a monotonically increasing function $f(t)$ such that $d(\zeta, L) \succ f(|\zeta - z|)$ for every $\zeta \in \gamma(z, 0)$.*

Then, for all polynomials $P_n(z)$, $\deg P_n \leq n$, $P_n(0) = 0$, one has

$$\|P_n\|_{C(\overline{G})} \prec \left\{ \int\limits_{\varepsilon n^{-2}}^{c} f^{-2/p}(t)dt \right\} \|P_n\|_p, \quad p > 0.$$

***Corollary* 3.1.** *Let $G \in PQ(K, \alpha, \beta)$ for some $K > 1$, $\alpha \geq 0$, and $\beta > 0$. Then*

$$\|P_n\|_{C(\overline{G})} \prec A_n \|P_n\|_p, \tag{3.3}$$

where

$$A_n = \begin{cases} n^{\frac{2}{p}(2\beta+2-p)}, & \beta > \dfrac{p}{2} - 1, \\ \ln n, & \beta = \dfrac{p}{2} - 1, \\ c, & \beta < \dfrac{p}{2} - 1. \end{cases} \tag{3.4}$$

***Remark* 3.1.** If $p = 2$, then $A_n = \sqrt{\ln n}$ [9].

Let G be an arbitrary Jordan domain and let $\gamma \in \Omega$ be a rectifiable arc, except for one endpoint $z_0 \in L$, that satisfies the following conditions:

i) $\operatorname{mes} \gamma(\zeta_1, \zeta_2) \prec |\zeta_1 - \zeta_2|$ for all $\zeta_1, \zeta_2 \in \gamma$;

ii) there exists a monotonically increasing function $g(t)$ such that $d(\zeta, L) \succ g(|\zeta - z_0|)$ for all $\zeta \in \gamma$.

Lemma 3.2 [4]. *Suppose that a measurable function $f(z)$ is given on the arc γ and there exists a monotonically increasing function $\nu(t)$, $\nu(0) = 0$, such that $|f(\zeta)| \prec \nu(|\zeta - z_0|)$ for all $\zeta \in \gamma$. Then the function*

$$F_\gamma(z) = \int\limits_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \notin \gamma,$$

satisfies the following relation:

$$\|F_\gamma\|_p^2 \prec \ell^{\frac{4(1-p)}{p}} \left[\int\limits_0^\ell \nu(t) dt\right]^2 +$$

$$+\begin{cases} \ell^{\frac{2(2-p)}{p}} \int\limits_0^{c\ell} \nu^2(t) \left[\dfrac{1}{t} + \dfrac{h_{0,1}(t)}{t^2} + h_{2,1}(t)\right] dt, & 1 < p < 2, \\ \int\limits_0^{c\ell} \nu^2(t) \left[t^{\frac{4-3p}{p}} + \dfrac{1}{t^2} h_{0,\frac{p}{2}}^{\frac{2}{p}}(t) + h_{p,\frac{p}{2}}^{\frac{2}{p}}(t)\right] dt, & p \geq 2, \end{cases}$$

where

$$h_{\lambda,\mu}(t) := \int\limits_0^t \frac{r^{1-\lambda} dr}{g^\mu(r)}.$$

Corollary **3.2.** *Let $G \in PQ(K, \alpha, 0)$ for some $K > 1$ and $\alpha > 0$ and let $\nu(t) = t^{1-\frac{1}{p}}$. Then, for any $p > 1$, one has*

$$\|F_\gamma\|_p^2 \prec \ell^{\frac{2-\alpha p}{2p}}, \quad \alpha < \min\left\{2\left(1 - \frac{1}{p}\right); \frac{2}{p}\right\}.$$

Corollary **3.3.** *Let $G \in PQ(K, 0, \beta)$ for some $K > 1$ and $\beta > 0$ and let $\nu(t) = t^{1-\frac{1+\beta}{p}}$. Then, for any $p > 1$, one has*

$$\|F_\gamma\|_p^2 \prec \ell^{\frac{2-(2+p)\beta}{2p}}, \quad \beta < \frac{2}{p+2}.$$

We now give conditions under which the function φ_p admits a continuous extension to $\overline{G}$.

Lemma 3.3 [4]. *Let $p > 1$ and $G \in PQ(K, \alpha, \beta)$ for some $K > 1$, $\alpha \geq 0$, and $\beta < p - 1$. Then the function $\varphi_p(z)$ can be extended to $\overline{G}$ by continuity.*

Corollary **3.4.** *Let $p > 1$ and $G \in PQ(K, \alpha, \beta)$. Then, for all $z \in L$ and $\zeta \in G$, one has*

$$|\varphi_p(z) - \varphi_p(\zeta)| \prec |z - \zeta|^{1-(1+\beta)\frac{1}{p}}.$$

4. Polynomials approximation in the A_p-norm. Let a domain $G \in PQ(K,\alpha,\beta)$, $\alpha > 0$, $\beta > 0$, be given. For simplicity but without loss of generality, we can take the domain G as at the beginning of Sec. 3.

Each L^j, $i,j = 1,2$, is a K_j-quasiconformal arc. Let $\alpha_j(\cdot)$ be a quasiconformal reflection across L^j. We also set

$$\gamma_1^1 := \left\{ z = x + iy \colon y = \frac{2c_1 + c_2}{3} (x-1)^{1+\alpha} \right\},$$

$$\gamma_1^2 := \left\{ z = x + iy \colon y = \frac{c_1 + 2c_2}{3} (x-1)^{1+\alpha} \right\},$$

$$\gamma_2^1 := \alpha_j \left\{ z = x + iy \colon y = \frac{2c_3 + c_4}{3} (x+1)^{1+\beta} \right\},$$

$$\gamma_2^2 := \alpha_j \left\{ z = x + iy \colon y = \frac{c_3 + 2c_4}{3} (x+1)^{1+\beta} \right\},$$

where the constants c_j, $j = \overline{1,4}$, are taken from the definition of the class $PQ(K,\alpha,\beta)$. It is easy to check that $\operatorname{mes} \gamma_j^i(\zeta_1,\zeta_2) \prec |\zeta_1 - \zeta_2|$ for all $\zeta_1, \zeta_2 \in \gamma_j^i$, $i,j = 1,2$, from Lemma 2.4.

Let $0 < \varepsilon < 1$ be sufficiently small and let $R := 1 + cn^{\varepsilon - 1}$. We choose points z_j^i, $i,j = 1,2$, so that they are the intersections of L_R and γ_i^j and the first points in $\tilde{L}_R^1 := \{z \colon z \in L_R, \operatorname{Im} z \geq 0\}$ or $\tilde{L}_R^2 := L_R \setminus \tilde{L}_R^1$ (according to the motion on L_R). These points divide L_R into four parts: $L_R^1 := L_R^1(z_1^1, z_2^1)$ (connecting the points z_1^1 and z_2^1), $L_R^2 := L_R^2(z_2^2, z_1^2)$, $L_R^3 := L_R^3(z_1^2, z_1^1)$, and $L_R^4 := L_R^4(z_2^1, z_2^1)$. We have $L_R := \bigcup_{j=1}^4 L_R^j$, $\gamma_i^j(R) = \gamma_i^j \cap \overline{\operatorname{int} L_R}$, $\Gamma_R^j := \gamma_1^j(R) \cup \gamma_2^j(R) \cup L_R^j$, and $U_j := \operatorname{int}\left(\Gamma_R^j \cup L^j\right)$, $i,j = 1,2$.

We extend the function φ_p to $U_1 \cup U_2$ as follows:

$$\tilde{\varphi}_p(z) := \begin{cases} \varphi_p(z), & z \in \overline{G}, \\ (\varphi_p \circ \alpha_j)(z), & z \in U_j. \end{cases} \tag{4.1}$$

Then

$$\tilde{\varphi}_{p,\overline{z}}(z) = \begin{cases} 0, & z \in G, \\ (\varphi_p' \circ \alpha_j)(z)\alpha_{j,\overline{z}}, & z \in U_j. \end{cases} \tag{4.2}$$

Using the Cauchy – Pompeiu formula [19, p. 148], we get

$$\varphi_p(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_R^1 \cup \Gamma_R^2} \frac{\tilde{\varphi}_p(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \iint\limits_{U_1 \cup U_2} \frac{\tilde{\varphi}_{p,\overline{\zeta}}(\zeta)}{\zeta - z} d\sigma_\zeta, \quad z \in G.$$

Then, using the notation introduced above, we obtain

$$\varphi_p(z) = \frac{1}{2\pi i} \int\limits_{L_R} \frac{f_p(\zeta)}{\zeta - z} d\zeta + \sum_{i,j=1}^{2} \frac{1}{2\pi i} \int\limits_{\gamma_i^j(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p\left((-1)^i\right)}{\zeta - z} d\zeta - \frac{1}{\pi} \iint\limits_{U_1 \cup U_2} \frac{\tilde{\varphi}_{p,\overline{\zeta}}(\zeta)}{\zeta - z} d\sigma_\zeta, \tag{4.3}$$

where

$$f_p(\zeta) := \begin{cases} \tilde{\varphi}_p(\zeta), & \zeta \in L_R^1 \cup L_R^2, \\ \tilde{\varphi}_p(1), & \zeta \in L_R^3, \\ \tilde{\varphi}_p(-1), & \zeta \in L_R^4. \end{cases}$$

Lemma 4.1. *Let $p > 1$ and $G \in PQ(K, \alpha, \beta)$ for some $K > 1$ and $0 < \alpha < \min\left\{2\left(1 - \frac{1}{p}\right); \frac{2}{p}\right\}$, $0 \le \beta < p_0$. Then, for any $n \ge 2$, one has*

$$\left\|\varphi_p - \pi_{n,p}\right\|_p \prec \left(\frac{1}{\ln n}\right)^{\frac{2-\alpha p}{2\alpha p}}. \tag{4.4}$$

Lemma 4.2. *Let $p > 1$ and $G \in PQ(K, 0, \beta)$ for some $K > 1$ and $0 < \beta < \min\left\{p - 1; \frac{K^2 - 1}{1 + pK^2 + 3K^2}\right\}$. Then, for any $n \ge 2$ and arbitrary small $\varepsilon > 0$, one has*

$$\left\|\varphi_p - \pi_{n,p}\right\|_p \prec \left(\frac{1}{n}\right)^{\frac{1-\varepsilon}{pK^2}}. \tag{4.5}$$

Lemma 4.3. *Let $p > \widetilde{p}_2$ and $G \in PQ(K, 0, \beta)$ for some $K > 1$ and $\frac{K^2 - 1}{1 + pK^2 + 3K^2} < \beta < p_0$. Then, for any $n \ge 2$ and arbitrary small $\varepsilon > 0$, one has*

$$\left\|\varphi_p - \pi_{n,p}\right\|_p \prec \left(\frac{1}{n}\right)^{\frac{2-(p+2)\beta-\varepsilon}{p(1+\beta)(K^2+1)}}.$$

Proof. The proofs of Lemmas 4.1 – 4.3 an similar, and we present them together. Since the first term in (4.3) is analytic in $\overline{G}$, there is a polynomial $P_n(z)$ of degree not higher than n [24, p. 142] such that

$$\left|\frac{1}{2\pi i}\int\limits_{L_R} \frac{f_p(\zeta)}{(\zeta - z)^2} d\zeta - P_n'(z)\right| \prec \frac{1}{n}, \quad z \in \overline{G}. \tag{4.6}$$

Hence, using (4.3), we get

$$\left\|\varphi_p' - P_n'\right\|_p \prec$$

$$\prec \frac{1}{n} + \sum_{i,j=1}^{2} \left\| \int\limits_{\gamma_i^j(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p\left((-1)^i\right)}{\zeta - z} d\zeta \right\|_p + \left\| \iint\limits_{U_1 \cup U_2} \frac{\tilde{\varphi}_{p,\bar{\zeta}}(\zeta)}{\zeta - z} d\sigma_\zeta \right\|_p =: \frac{1}{n} + \sum_{k=1}^{5} J_k. \tag{4.7}$$

For all $p > 1$ and $\beta < p - 1$, we have

$$|\tilde{\varphi}_p(\zeta) - \varphi_p(-1)| = |\varphi_p(\alpha_j(\zeta)) - \varphi_p(-1)| \prec |\zeta + 1|^{1-\frac{1+\beta}{p}}, \quad \zeta \in \gamma_j^1(R), \tag{4.8}$$

$$|\tilde{\varphi}_p(\zeta) - \varphi_p(1)| = |\varphi_p(\alpha_j(\zeta)) - \varphi_p(1)| \prec |\zeta - 1|^{1-\frac{1}{p}}, \quad \zeta \in \gamma_j^2(R), \tag{4.9}$$

by virtue of relation (2.1) and Corollary 3.4. Therefore, for every $\alpha < \min\left\{2\left(1 - \frac{1}{p}\right); \frac{2}{p}\right\}$ and $\beta < p_0$, we obtain

$$\left\| \int\limits_{\gamma_i^1(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p(-1)}{\zeta - z} d\zeta \right\|_p \prec \ell_{i,2}^{\frac{2-(2+p)\beta}{2p}}, \tag{4.10}$$

$$\left\| \int\limits_{\gamma_i^2(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p(1)}{\zeta - z} d\zeta \right\|_p \prec \ell_{i,1}^{\frac{2-\alpha p}{2p}} \tag{4.11}$$

by virtue of Corollaries 3.2 and 3.3 and the fact that $\ell_{i,j} = \operatorname{mes} \gamma_i^j(R), \ i, j = 1, 2$. On the other hand, according to [21] (Lemma 9), we have

$$d(z_j, L^j) \prec \left(\frac{1}{n}\right)^{\frac{2-\varepsilon}{K^2+1}}.$$

Then, using (2.1), (3.1), and (3.2), we get

$$\ell_{i,j} \prec \left|z_j^i - (-1)^i\right| \prec \begin{cases} d(z_2^i, L^i)^{\frac{1}{1+\beta}} \prec \left(\frac{1}{n}\right)^{\frac{2-\varepsilon}{(1+\beta)(K^2+1)}} & \forall \varepsilon > 0, \quad i = 1, 2, \\ d(z_1^i, L^i) \prec (\ln n)^{-\alpha^{-1}}, & i = 1, 2. \end{cases}$$

Thus, it follows from (4.10) and (4.11) that

$$\left\| \int\limits_{\gamma_i^1(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p(-1)}{\zeta - z} d\zeta \right\|_p \prec \left(\frac{1}{n}\right)^{\frac{2-(p+2)\beta-\varepsilon}{p(1+\beta)(K^2+1)}}, \quad \beta < \min\left\{p - 1; \ \frac{2}{p+2}\right\}, \tag{4.12}$$

$$\left\| \int\limits_{\gamma_i^2(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p(1)}{\zeta - z} d\zeta \right\|_p \prec \left(\frac{1}{\ln n}\right)^{\frac{2-\alpha p}{2\alpha p}}, \quad 0 < \alpha < \min\left\{2\left(1 - \frac{1}{p}\right); \frac{2}{p}\right\}. \tag{4.13}$$

Since the Hilbert transformation

$$(Tf)(z) := -\frac{1}{\pi} \int\int \frac{f(\zeta)}{(\zeta - z)^2} d\sigma_\zeta$$

is a bounded linear operator from L_p into itself for $p > 1$, we have

$$\iint\limits_{U_1 \cup U_2} \left|\tilde{\varphi}_{p,\bar{\zeta}}(\zeta)\right|^p d\sigma_\zeta \asymp \iint\limits_{U_1 \cup U_2} \left|\varphi'(\alpha_j(\zeta))\right|^2 d\sigma_\zeta \prec$$

$$\prec \sum_{j=1}^{2} \iint\limits_{\alpha(U_j)} \left|\varphi'(\zeta)\right|^2 d\sigma_\zeta \prec \sum_{j=1}^{2} \mathrm{mes}\ \varphi\left(\alpha_j(U_j)\right).$$

According to (4.2) and (2.1), the Calderon – Zygmund inequality [7, p. 89] yields

$$J_5 \prec \left(\sum_{j=1}^{2} \mathrm{mes}\, \varphi\left(\alpha_j(U_j)\right)\right)^{\frac{1}{p}}. \tag{4.14}$$

For sufficiently large c and small $\varepsilon_0 < \dfrac{1}{2}$, we set

$$V_1^j := \left\{\zeta \colon \zeta \in \alpha_j(U_j), |\zeta - 1| \le c(\ln n)^{-\alpha^{-1}}\right\},$$

$$V_2^j := \alpha_j(U_j) \backslash V_1^j, \quad j = 1, 2, \quad \alpha > 0,$$

$$U_{\varepsilon_0} := \{\zeta : |\zeta + 1| \le \varepsilon_0\}; \quad \tilde{V}_j^1 := U_j \cap U_{\varepsilon_0}, \quad j = 1, 2, \quad \alpha = 0.$$

Then, by virtue of Lemma 2.6, we obtain

$$\mathrm{mes}\, \varphi(V_1^{j)} \prec (\ln n)^{-\alpha^{-1}},$$

$$\mathrm{mes}\, \varphi(\alpha_j(\tilde{V}_j^1) \prec n^{\frac{\varepsilon-2}{K^2+1}\delta} = n^{\frac{\varepsilon-1}{K^2}},$$

$$\mathrm{mes}\, \varphi(\alpha_j(U_j \backslash \tilde{V}_j^1)) \prec n^{\frac{\varepsilon-1}{K^2}} \quad \forall \varepsilon > 0,$$

and

$$J_5 \prec \begin{cases} \left(\dfrac{1}{\ln n}\right)^{\frac{1}{\alpha p}}, & \alpha > 0, \\ \left(\dfrac{1}{n}\right)^{\dfrac{1-\varepsilon}{pK^2}}, & \forall \varepsilon > 0, \quad \alpha = 0. \end{cases} \tag{4.15}$$

Using (4.8), (4.9), (4.12), (4.13), and (4.15), we get

$$\|\varphi_p - P_n\|_p \prec \frac{1}{n} + \begin{cases} \left(\dfrac{1}{\ln n}\right)^{\frac{2-\alpha p}{2\alpha p}} \\ \left(\dfrac{1}{n}\right)^{\frac{2-(p+2)\beta-\varepsilon}{p(1+\beta)(K^2+1)}} \end{cases} +$$

$$+ \begin{cases} \left(\dfrac{1}{\ln n}\right)^{\frac{1}{\alpha p}}, & 0 < \alpha < \min\left\{2\left(1 - \dfrac{1}{p}\right); \dfrac{2}{p}\right\}, \quad \beta \ge 0, \\ \left(\dfrac{1}{n}\right)^{\frac{1-\varepsilon}{pK^2}} & \forall \varepsilon > 0, \quad \alpha = 0, \quad \beta > 0. \end{cases} \tag{4.16}$$

Case 1. Let $p > 1,\ 0 < \alpha < \min\left\{2\left(1 - \dfrac{1}{p}\right);\ \dfrac{2}{p}\right\}$, $\beta \ge 0$. Then

$$\|\varphi_p - P_n\|_p \prec \frac{1}{n} + \left(\frac{1}{\ln n}\right)^{\frac{2-\alpha p}{2\alpha p}} + \left(\frac{1}{\ln n}\right)^{\frac{1}{\alpha p}} \prec \left(\frac{1}{\ln n}\right)^{\frac{2-\alpha p}{2\alpha p}}. \tag{4.17}$$

Case 2. Let $p > 1,\ \alpha = 0,\ \beta > 0$. Then

$$\|\varphi_p - P_n\|_p \prec \frac{1}{n} + \left(\frac{1}{n}\right)^{\frac{2-(p+2)\beta-\varepsilon}{p(1+\beta)(K^2+1)}} + \left(\frac{1}{n}\right)^{\frac{1-\varepsilon}{pK^2}} \prec$$

$$\prec \begin{cases} \left(\frac{1}{n}\right)^{\frac{1-\varepsilon}{pK^2}}, & \beta < \min\left\{p_0, \dfrac{K^2-1}{1+pK^2+3K^2}\right\}, \\ \left(\frac{1}{n}\right)^{\frac{2-(p+2)\beta-\varepsilon}{p(1+\beta)(K^2+1)}}, & \dfrac{K^2-1}{1+pK^2+3K^2} \le \beta < p_0, \end{cases} \tag{4.18}$$

for any $p > 1$ and arbitrary small $\varepsilon > 0$.

Case 3. Let $p > 1,\ \alpha = 0,\ 0 < \beta < \min\left\{p_0, \dfrac{K^2-1}{1+pK^2+3K^2}\right\}$. Then

$$\|\varphi_p - P_n\|_p \prec \left(\frac{1}{n}\right)^{\frac{1-\varepsilon}{pK^2}}. \tag{4.19}$$

Case 4. It is clear that

$$\min\left\{p-1;\ \frac{2}{p+2}\right\} = \begin{cases} p-1 & \text{if} \quad p < p_1 := \dfrac{\sqrt{17}-1}{2}, \\ \dfrac{2}{p+2} & \text{if} \quad p \ge p_1. \end{cases}$$

Let $p > \widetilde{p}_2,\ K > 1,\ \alpha = 0,\ \dfrac{K^2-1}{1+pK^2+3K^2} < \beta < p_0$. Then

$$\|\varphi_p - P_n\|_p \prec \left(\frac{1}{n}\right)^{\frac{2-(p+2)\beta-\varepsilon}{p(1+\beta)(K^2+1)}}, \tag{4.20}$$

for arbitrary small $\varepsilon > 0$. If $\tilde{P}_n(z) := P_n(z) - P_n(0) + z[1 - P_n'(0)]$, then it is easy to see that relations (4.17) – (4.20) are also satisfied for $\tilde{P}_n(z)$, $\tilde{P}_n(0) = 0$, and $\tilde{P}_n'(0) = 1$. Thus, we can complete the proof of Lemmas 4.1 – 4.3 considering the extremal properties of $\pi_{n,p}(z)$.

5. Proof of Theorems 1.1 – 1.8. We use the known method given in [3, 4, 9].

Lemma 5.1. *Suppose that* G *is a Jordan domain such that, for* $\{\alpha_n\}\downarrow$, $\{\beta_n\}\uparrow$, $\{\gamma_n := \alpha_n\beta_n\}\downarrow$, *and* $n \to \infty$, *under the condition*

$$\|\varphi_p - \pi_{n,p}\|_p \prec \alpha_n, \quad n = 2, 3, \ldots,$$

one has

$$\|P_n\|_{C(\overline{G})} \prec \beta_n \|P_n'\|_p, \quad n = 1, 2, \ldots,$$

for all polynomials $P_n(z)$ *of degree not higher than* n *with* $P_n(0) = 0$. *Also assume that there exists a sequence of indices* $\{n_k\}_{k=1}^{\infty}$ *such that* $\beta_{n_{k+1}} \le c\beta_{n_k}$ *and* $\gamma_{n_{k+1}} \le \varepsilon\gamma_{n_k}$ *for all* $k = 1, 2, \ldots$ *and some* $c \ge 1$ *and* $0 < \varepsilon < 1$. *Then*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \prec \gamma_n.$$

The proof of this lemma is similar to that of [9] (Lemma 15). Therefore, by taking α_n from Lemmas 4.1 – 4.3 and β_n from Corollary 3.1 and combining the results for $G \in PQ(K, \alpha, \beta)$ in the case $\alpha = 0$ or $\beta = 0$, we prove Theorems 1.1 – 1.8.

1. *Abdullayev F. G.* On orthogonal polynomials in domains with quasiconformal boundary (in Russian): Dissertation. – Donetsk, 1986.
2. *Abdullayev F. G.* On the convergence of Bieberbach polynomials in domains with interior zero angles (in Russian) // Dokl. Akad. Nauk. Ukr. SSR. Ser. A. – 1989. – № 12. – P. 3 – 5.
3. *Abdullayev F. G.* Uniform convergence of the generalized Bieberbach polynomials in regions with nonzero angles // Acta math. hung. – 1987. – **77**, № 3. – P. 223 – 246.
4. *Abdullayev F. G.* Uniform convergence of the generalized Bieberbach polynomials in regions with zero angles // Czech. Math. J. – 2001. – **51(126)**. – P. 643 – 660.
5. *Abdullayev F. G., Baki A.* On the convergence of Bieberbach polynomials in domains with interior zero angles // Complex Anal. Theor. & Appl. – 2001. – **34**, № 2. – P. 131 – 143.
6. *Abdullayev F. G., Çavuş A.* On the uniform convergence of the generalized Bieberbach polynomials in regions with quasiconformal boundary // Approxim. Theory and Appl. – 2001. – **17**, № 1. – P. 97 – 105.
7. *Ahlfors L. V.* Lectures on quasiconformal mappings. – Princeton, NJ: Van Nostrand, 1966.
8. *Andrievskii V. V.* Uniform convergence of Bieberbach polynomials in domains with zero angles (in Russian) // Dokl. Akad. Nauk. Ukr. SSR. Ser. A. – 1982. – № 4. – P. 3 – 5.
9. *Andrievskii V. V.* Uniform convergence of Bieberbach polynomials in domains with piecewise-quasiconformal boundary (in Russian) // Theory Mappings and Approxim. Functions. – Kiev: Naukova Dumka, 1983. – P. 3 – 18.
10. *Andrievskii V. V.* Convergence of Bieberbach polynomials in domains with quasiconformal boundary // Ukr. Math. J. – 1984. – **35**, № 2. – P. 233 – 236.
11. *Andrievskii V. V., Belyi V. I., Dzjadyk V. K.* Conformal invariants in constructive theory of functions of complex variable. – Atlanta, Georgia: World Federation Publ., 1995.
12. *Belyi V. I.* Conformal mappings and the approximation of analytic functions in domains with a quasiconformal boundary // Math. USSR-Sb. – 1977. – **31**. – P. 289 – 317.
13. *Davis P. J.* Interpolation an approximation. – Blaisdell Publ. Co., 1963.
14. *Gaier D.* On the convergence of the Bieberbach polynomials in regions with corners // Constr. Approxim. – 1988. – **4**. – P. 289 – 305.
15. *Gaier D.* On the convergence of the Bieberbach polynomials in regions with piecewise-analytic boundary // Arch. Math. – 1992. – **58**. – P. 462 – 470.
16. *Goldstein V. M.* The degree of summability of generalized derivatives of plane quasiconformal homeomorphisms // Soviet Math. Dokl. – 1980. – **21**, № 1. – P. 10 – 13.
17. *Israfilov D. M.* On approximation properties of extremal polynomials (in Russian). – Dep. VINITI, № 5461, 1981.
18. *Keldysh M. V.* Sur l’approximation en moyenne quadratique des fonctions analytiques // Math. Sb. – 1939. – **5(47)**. – P. 391 – 401.
19. *Lehto O., Virtanen K. I.* Quasiconformal mappings in the plane. – Berlin: Springer, 1973.
20. *Mergelyan S. N.* Certain questions of the constructive theory of functions (in Russian) // Trudy Math. Inst. Steklov. – 1951. – **37**.
21. *Pommerenke C.* Univalent functions. – Göttingen, 1975.
22. *Privalov I. I.* Introduction to the theory of functions of a complex variable. – Moscow: Nauka, 1984.
23. *Rickman S.* Characterization of quasiconformal Arcs // Ann. Acad. Sc. Fenn. Ser. A. I. Math. – 1996. – **395**.
24. *Smirnov V. I., Lebedev N. A.* Functions of a complex variable // Constructive Theory. – The M. I. T. Press, 1968.
25. *Simonenko I. B.* On the convergence of Bieberbach polynomials in the case of a Lipschitz domain // Math. USSR-Izv. – 1980. – **13**. – P. 166 – 174.
26. *Suetin P. K.* Polynomials orthogonal over a region and Bieberbach polynomials // Proc. Steklov Inst. Math. – 1971. – **100**.
27. *Walsh J. L.* Interpolation and approximation by rational functions in the complex domain (in Russian). – Moscow, 1961.

Received 02.12.11