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THE BLOCK BY BLOCK METHOD WITH ROMBERG QUADRATURE FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATIONS ON THE LARGE INTERVALS

ПОБЛОЧНИЙ МЕТОД ІЗ КВАДРАТУРОЮ РОМБЕРГА ДЛЯ РОЗВ'ЯЗУВАННЯ НЕЛІНІЙНИХ ІНТЕГРАЛЬНИХ РІВНЯНЬ ВОЛЬТЕРРА НА ВЕЛИКИХ ІНТЕРВАЛАХ

We investigate the numerical solution of nonlinear Volterra integral equations by block by block method, which is useful specially for solving integral equations on large-size intervals. A convergence theorem is proved that shows that the method has at least sixth order of convergence. Finally, the performance of the method is illustrated by some numerical examples.

Досліджено чисельний розв'язок нелінійних інтегральних рівнянь Вольтерра поблочним методом, який є особливо корисним при розв'язуванні інтегральних рівнянь на великих інтервалах. Доведено теорему про збіжність, яка показує, що цей метод має щонайменше шостий порядок збіжності. Дію методу проілюстровано на кількох числових прикладах.

1. Introduction. Consider nonlinear Volterra Integral Equations (VIEs) of the form

$$f(x) = g(x) + \int_0^x k(x,s,f(s))ds, \quad 0 \leq x \leq X, \tag{1.1}$$

where g and k are continuous respectively on $[0,\infty]$, and $D = \{(x,s,f)|0 < s < x < \mathbb{R}, -\infty < < f < \infty\}$. Suppose

(i) $\forall X > 0$ and $\psi\colon [0,X] \to R$ continuous, $k(x,s,\psi(s))$ is continuous for $s \in (0,x)$ and $\left|\int_0^x k(x,s,\psi(s))ds\right| < \infty$ for $x \in [0,X]$;

(ii) $\exists\, q(x,s)$ continuous on $0 < s < x < \infty$ satisfying

$$\int_0^x q(x,s)ds < \infty, \quad x \in (0,\infty),$$

and all $X > 0$, as positive $\delta \longrightarrow 0$,

$$\int_x^{x+\delta} q(x+\delta,\delta)ds \longrightarrow 0, \quad x \in [0,X],$$

uniformly in x, such that

$$|k(x,s,f_1) - k(x,s,f_2)| \leq q(x,s)\,|f_1 - f_2| \quad \forall d > 0$$

and $f_1, f_2 \in S_d = \{f \in \mathbb{R}|\,|f| < d\}$.

Under these conditions, there exists a constant $\alpha > 0$ such that on $[0,\alpha]$, Eq. (1.1) has a unique solution. Furthermore, if there exist $B > 0$, such that for $|f(x)| \leq B$ on the all intervals of the form $[0,\beta]$ $(\beta > 0)$, the Eq. (1.1) has a unique solution, then Eq. (1.1) has a unique solution on $[0,\infty)$ [5].

Numerical methods to approximate solution of Eq. (1.1) have been extensively studied in literature [2–4, 7, 8]. Except some low order methods, such as the trapezoidal rule, the other methods that are based on numerical integration require one or more starting values which must be found by an other method. A different class of approximating formulas for (1.1) is based on extensions of the Runge-Kutta methods which have been studied in details by Pouzet [10, 11]. The Runge-Kutta methods are self-starting, but tend to be complicated and inefficient and hence its practical use is limited.

There is another approach which uses numerical quadrature, and its computations are arranged in such a way that several values of the unknown function are obtained at the same time. This is generally called a block by block method and it is what we are concerned about. The block by block approach was first suggested by Young [13] in connection with product integration techniques. In fact, a block by block method is an extrapolation procedure which has advantages of being self-starting and producing a block of values at the same time [2, 3]. Linz [7] described a two blocks method and used it for solving nonlinear VIE of the second kind. Also AL-Asdi [1] used two and three blocks for solving Hammerstien VIE of the second kind and then Saify [12] used two, three and four blocks for solving a system of linear VIE of the second kind. In 2010, the authors used a block by block method for solving system of nonlinear VIEs [6]. In this paper, we extend this method by using Romberg quadrature rule to get a desired order of the error. In addition to the general advantages of the block by block methods such as having no need to start values, simple structure for application and computing several values of the unknown function at the same time, the presented method has the following advantages.

1. Most of the available methods for solving (1.1) are based on expansion of solution, for example the Taylor and Chebyshev expansion methods, the Tau method, the Adomian and homotopy methods and so on. These methods are efficient only for the intervals with small length (say $[0, 1]$ or $[-1, 1]$) and they are useless for the large intervals. The method of this paper is one of the most suitable methods for the large intervals. In the final section of this paper, we will compare numerical results between HPM (Homotopy Perturbation Method) [14], ADM (Adomian Decomposition Method) [4] and the given block by block method (Tables 1 and 3).

2. For the given step size h, the order of convergence for this method is at least h^6 while it is h^4 by using Simpson rule [7].

3. By increasing number of blocks, the order of convergence increases in such a way that it would be at least h^8 and h^{10} respectively for 8 and 16 blocks.

4. At the first step of Romberg rule, the Simpson rule can be used instead of trapezoidal rule for increasing order of convergence.

5. Compared to known methods, the computation time for this method is low.

The rest of the paper is organized as follows. In Section 2, the general process is presented. In Section 3, the method for the large interval is described. The sixth-order convergence is proved in Section 4. Finally, the paper is closed by giving numerical experiments in order to test reliability of the method in Section 5.

2. The general process. Let $0 = x_0 < x_1 < \ldots < x_N = X$ be a partition of $[0, X]$ with the step size h, such that $x_i = x_0 + ih$, $i = 1, 2, \ldots, N$, and let F_i be the approximate value of $f(x)$ at the mesh point $x = x_i$ and $F_0 = g(x_0)$. To simplify formulation, let the number of blocks to be 4 (for $8, 16, \ldots$ blocks the process will be similar).

Putting $x = x_{4m+p}$ in (1.1), we have

$$F_{4m+p} \simeq f(x_{4m+p}) = g(x_{4m+p}) + \int\limits_0^{x_{4m+p}} k(x_{4m+p}, s, f(s))ds = g(x_{4m+p}) +$$

$$+ \int\limits_0^{x_{4m}} k(x_{4m+p}, s, f(s))ds + \int\limits_{x_{4m}}^{x_{4m+p}} k(x_{4m+p}, s, f(s))ds, \quad m = 0, 1, \ldots, N/4 - 1, \quad p = 1, 2, 3, 4. \tag{2.1}$$

Assume that $F_0, F_1, \ldots, F_{4m}$ are known, then the first integral can be approximated by standard quadrature rules and the second one can be estimated by Romberg quadrature rule at the points x_{4m}, x_{4m+1}, x_{4m+2}, x_{4m+3} and x_{4m+4}. Therefore we obtain a system containing four simultaneous equations that is solved for a block of four values of F. To simplify notation, we set $k_i := k(x_{4m+p}, x_i, F_i)$ and use the trapezoidal rule for $\int_{x_v}^{x_u} k(x_{4m+p}, s, f(s))ds$, thus we obtain

$$T_{u,v}^{(0)} := \frac{x_{u-v}}{2}\left[k_v + k_u\right],$$

$$T_{u,v}^{(1)} := \frac{1}{2}T_{u,v}^{(0)} + \frac{x_{u-v}}{2}k_{\frac{u+v}{2}},$$

$$T_{u,v}^{(2)} := \frac{1}{2}T_{u,v}^{(1)} + \frac{x_{u-v}}{4}\left[k_{\frac{u+3v}{4}} + k_{\frac{3u+v}{4}}\right],$$

$$T_{u,v}^{(3)} := \frac{1}{2}T_{u,v}^{(2)} + \frac{x_{u-v}}{8}\left[k_{\frac{u+7v}{8}} + k_{\frac{3u+5v}{8}} + k_{\frac{5u+3v}{8}} + k_{\frac{7u+v}{8}}\right].$$

It is easy to get

$$\int\limits_{x_{4m}}^{x_{4m+p}} k(x_{4m+p}, s, f(s))ds \simeq \frac{64}{45}T_{4m+p,4m}^{(2)} - \frac{20}{45}T_{4m+p,4m}^{(1)} + \frac{1}{45}T_{4m+p,4m}^{(0)} =$$

$$= \frac{x_p}{90}\left[7(k_{4m} + k_{4m+p}) + 12k_{4m+p/2} + 32(k_{4m+p/4} + k_{4m+3p/4})\right], \quad p = 1, 2, 3, 4, \tag{2.2}$$

by using Romberg rule. If $\frac{ip}{4}$, $i = 1, 2, 3$, are not integers, the points $x_{4m+\frac{ip}{4}}$ will not belong to the mesh points and $F_{4m+\frac{ip}{4}}$ will be unknown which leads to a difficulty in computing (2.2). In this case, we use the Lagrange interpolation polynomial at the points x_{4m}, x_{4m+1}, x_{4m+2}, x_{4m+3} and x_{4m+4} to approximate $F_{4m+\frac{ip}{4}}$, i.e.,

$$F_{4m+\frac{ip}{4}} \approx \mathcal{P}\left(x_{4m} + \frac{ip}{4}h\right) = \sum_{j=0}^{4} L_j\left(\frac{ip}{4}\right)F_{4m+j}, \quad i = 1, 2, 3,$$

where

$$L_j\left(\frac{ip}{4}\right) := \prod_{\substack{ii=0\\ ii\neq j}}^{4} \frac{ip/4 - ii}{j - ii}.$$

Then we find

$$\begin{aligned}
F_{4m+\frac{3}{2}} &\approx \frac{-5}{128}F_{4m} + \frac{15}{32}F_{4m+1} + \frac{45}{64}F_{4m+2} - \frac{5}{32}F_{4m+3} + \frac{3}{128}F_{4m+4},\\
F_{4m+\frac{3}{4}} &\approx \frac{195}{2048}F_{4m} + \frac{585}{512}F_{4m+1} - \frac{351}{1024}F_{4m+2} + \frac{65}{512}F_{4m+3} - \frac{45}{2048}F_{4m+4},\\
F_{4m+\frac{9}{4}} &\approx \frac{35}{2048}F_{4m} - \frac{63}{512}F_{4m+1} + \frac{945}{1024}F_{4m+2} + \frac{105}{512}F_{4m+3} - \frac{45}{2048}F_{4m+4},\\
F_{4m+\frac{1}{2}} &\approx \frac{35}{128}F_{4m} + \frac{35}{32}F_{4m+1} - \frac{35}{64}F_{4m+2} + \frac{7}{32}F_{4m+3} - \frac{5}{128}F_{4m+4},\\
F_{4m+\frac{1}{4}} &\approx \frac{1155}{2048}F_{4m} + \frac{385}{512}F_{4m+1} - \frac{495}{1024}F_{4m+2} + \frac{105}{512}F_{4m+3} - \frac{77}{2048}F_{4m+4}.
\end{aligned} \tag{2.3}$$

The first integral in (2.1) can be approximated by similar Romberg rule without any difficulty.

If $4m$ is a multiple of 8, then by using 3-stage Romberg quadrature rule we define

$$A := \int_0^{x_{4m}} k(x_{4m+p}, s, f(s))ds \approx (4096T^{(3)}_{4m,0} - 1344T^{(2)}_{4m,0} + 84T^{(1)}_{4m,0} - T^{(0)}_{4m,0})/2835 =$$

$$= \frac{x_{4m}}{2835}\left[108.5(k_0 + k_{4m}) + 218k_{2m} + 176(k_m + k_{3m}) + 512(k_{m/2} + k_{7m/2} + k_{3m/2} + k_{5m/2})\right], \tag{2.4}$$

otherwise

$$A := \int_0^{x_{4m}} k(x_{4m+p}, s, f(s))ds = \int_0^{x_4} k(x_{4m+p}, s, f(s))ds + \int_{x_4}^{x_{4m}} k(x_{4m+p}, s, f(s))ds \approx$$

$$\approx (64T^{(2)}_{4,0} - 20T^{(1)}_{4,0} + T^{(0)}_{4,0})/45 + (4096T^{(3)}_{4m,4} - 1344T^{(2)}_{4m,4} + 84T^{(1)}_{4m,4} - T^{(0)}_{4m,4})/2835 =$$

$$= \frac{x_4}{90}\left[7(k_0 + k_4) + 12k_2 + 32(k_1 + k_3)\right] + \frac{x_{4m} - x_4}{2835}\left[\frac{217}{2}(k_4 + k_{4m}) + 218k_{2m+2} + \right.$$

$$\left. +176(k_{m+3} + k_{3m+1}) + 512\left(k_{\frac{m+7}{2}} + k_{\frac{7m+1}{2}} + k_{\frac{3m+5}{2}} + k_{\frac{5m+3}{2}}\right)\right]. \tag{2.5}$$

Consequently, substituting in (2.1), yields for $p = 4$

$$F_{4m+4} = g(x_{4m+4}) + A + \frac{x_4}{90}\left[7(k_{4m} + k_{4m+4}) + 12k_{4m+2} + 32(k_{4m+1} + k_{4m+3})\right], \tag{2.6}$$

and $p = 1$, $p = 3$

$$F_{4m+p} = g(x_{4m+p}) + A + \frac{x_p}{90}\Big[7(k_{4m} + k_{4m+p}) + 12k\left(x_{4m+p}, x_{4m+p/2}, \mathcal{P}\left(x_{4m} + \frac{p}{2}h\right)\right) +$$

$$+32\left(k\left(x_{4m+p}, x_{4m+p/4}, \mathcal{P}\left(x_{4m} + \frac{p}{4}h\right)\right) + k(x_{4m+p}, x_{4m+3p/4}, \mathcal{P}\left(x_{4m} + \frac{3p}{4}h)\right)\right)\Big]. \quad (2.7)$$

For $p = 2$ need not use the Lagrange interpolation for $F_{4m+\frac{p}{2}}$, so

$$F_{4m+2} = g(x_{4m+2}) + A + \frac{x_2}{90}\Big[7(k_{4m} + k_{4m+2}) + 12k_{4m+1}+$$

$$+32\left(k\left(x_{4m+2}, x_{4m+1/2}, \mathcal{P}\left(x_{4m} + \frac{1}{2}h\right)\right) + k\left(x_{4m+2}, x_{4m+3/2}, \mathcal{P}\left(x_{4m} + \frac{3}{2}h\right)\right)\right)\Big]. \quad (2.8)$$

Therefore in each step (for different values of m) (2.6), (2.7) and (2.8) for $p = 1, 2, 3, 4$, forms a system of equations with four unknowns F_{4m+1}, F_{4m+2}, F_{4m+3} and F_{4m+4}, which will be linear and nonlinear respectively for linear and nonlinear integral equations. The linear case the system can be solved via a direct method, but in the nonlinear case the system may be solved by iterative methods or by a suitable software package such as Maple.

3. Large intervals. We noticed that the method of previous section gives four values of the unknown function in each step, but these values are computed approximately and we use some of them for the next steps. For instant $F_0, F_1, \ldots, F_4$ for $m = 1$ are applied from previous step to approximate the first integral in (2.1). In order to reduce the effect of accumulated errors, we need more accurate approximation for this integral in evaluating the unknown function at the points near to the end of interval, when x changes in a large interval or when the step size is very small. Thus for approximating the integral $\int_{x_v}^{x_u} k(x_{4m+p}, s, f(s))ds$, under above strategy, we define

$$T^{(4)}_{u,v} := \frac{1}{2}T^{(3)}_{u,v}+$$

$$+\frac{x_{u-v}}{16}\left[k_{\frac{u+15v}{16}} + k_{\frac{3u+13v}{16}} + k_{\frac{5u+11v}{16}} + k_{\frac{7u+9v}{16}} + k_{\frac{9u+7v}{16}} + k_{\frac{11u+5v}{16}} + k_{\frac{13u+3v}{16}} + k_{\frac{15u+v}{16}}\right]$$

then by using Romberg quadrature rule with 2, 3 and 4 stages, we have, respectively,

$$\int_{x_v}^{x_u} k(x_{4m+p}, s, f(s))ds \approx \frac{64}{45}T^{(2)}_{u,v} - \frac{20}{45}T^{(1)}_{u,v} + \frac{1}{45}T^{(0)}_{u,v} =$$

$$= \frac{u-v}{90}h\left[7(k_v + k_u) + 12k_{\frac{u+v}{2}} + 32\left(k_{\frac{u+3v}{4}} + k_{\frac{3u+v}{4}}\right)\right], \quad (3.1)$$

$$\int_{x_v}^{x_u} k(x_{4m+p}, s, f(s))ds \approx (4096T^{(3)}_{u,v} - 1344T^{(2)}_{u,v} + 84T^{(1)}_{u,v} - T^{(0)}_{u,v})/2835 =$$

$$= \frac{u-v}{2835}h\left[108.5(k_v+k_u)+218k_{\frac{u+v}{2}}+176\left(k_{\frac{u+3v}{4}}+k_{\frac{3u+v}{4}}\right)+\right.$$

$$\left.+\,512\left(k_{\frac{u+7v}{8}}+k_{\frac{7u+v}{8}}+k_{\frac{3u+5v}{8}}+k_{\frac{5u+3v}{8}}\right)\right] \qquad (3.2)$$

and

$$\int\limits_{x_v}^{x_u} k(x_{4m+p}, s, f(s))ds \approx$$

$$\approx (1048576T^{(4)}_{u,v} - 348160T^{(3)}_{u,v} + 22848T^{(2)}_{u,v} - 340T^{(1)}_{u,v} + T^{(0)}_{u,v})/722925 =$$

$$= \frac{u-v}{722925}h\left[13779.5(k_v+k_u)+27559k_{\frac{u+v}{2}}+27728\left(k_{\frac{u+3v}{4}}+k_{\frac{3u+v}{4}}\right)+\right.$$

$$+22016\left(k_{\frac{u+7v}{8}}+k_{\frac{7u+v}{8}}+k_{\frac{3u+5v}{8}}+k_{\frac{5u+3v}{8}}\right)+65536\left(k_{\frac{u+15v}{16}}+k_{\frac{15u+v}{16}}+\right.$$

$$\left.\left.+\,k_{\frac{3u+13v}{16}}+k_{\frac{13u+3v}{16}}+k_{\frac{5u+11v}{16}}+k_{\frac{11u+5v}{16}}+k_{\frac{7u+9v}{16}}+k_{\frac{9u+7v}{16}}\right)\right]. \qquad (3.3)$$

Now, if $4m$ is multiple of 16, then we use the 4-stages Romberg rule (3.3) to approximate the first integral in (2.1) and if remaining of $\frac{4m}{16}$ is 4, then we write

$$A := \int\limits_0^{x_{4m}} k(x_{4m+p}, s, f(s))ds = \int\limits_0^{x_4} k(x_{4m+p}, s, f(s))ds + \int\limits_{x_4}^{x_{4m}} k(x_{4m+p}, s, f(s))ds$$

and approximate the first integral by (3.1) and the second one by (3.3). Similarly, if remaining of $\frac{4m}{16}$ is 8, then we write

$$A := \int\limits_0^{x_{4m}} k(x_{4m+p}, s, f(s))ds = \int\limits_0^{x_8} k(x_{4m+p}, s, f(s))ds + \int\limits_{x_8}^{x_{4m}} k(x_{4m+p}, s, f(s))ds$$

and use (3.2) and (3.3) respectively for the first and second integrals. Otherwise we write

$$A := \int\limits_0^{x_{4m}} k(x_{4m+p}, s, f(s))ds =$$

$$= \int\limits_0^{x_4} k(x_{4m+p}, s, f(s))ds + \int\limits_{x_4}^{x_{12}} k(x_{4m+p}, s, f(s))ds + \int\limits_{x_{12}}^{x_{4m}} k(x_{4m+p}, s, f(s))ds$$

and approximate the integrals respectively by (3.1), (3.2) and (3.3).

Therefore, having larger interval or very small step size, one can use more accurate integration rules to approximate the first integral in (2.1). We conclude that, this method has not any restriction on the given intervals.

4. Convergence analysis.

Theorem 4.1. *The approximation method given by the system* (2.6), (2.7) *and* (2.8) *is convergent and its order of convergence is at least* 6 *for the functions k and f with at least six order derivatives.*

Proof. It can be written from (2.4) and (2.5)

$$A := \int\limits_{0}^{x_{4m}} k(x_{4m+p}, s, f(s))ds \approx h \sum_{i=0}^{4m} w_i k(x_{4m+p}, x_i, F_i).$$

Let $p = 1$ or $p = 3$ (for other values of p the process is similar), then from (2.1) and (2.7) we obtain

$$|\varepsilon_{4m+p}| := |f(x_{4m+p}) - F_{4m+p}| =$$

$$= \left| \int\limits_{0}^{x_{4m}} k(x_{4m+p}, s, f(s))ds + \int\limits_{x_{4m}}^{x_{4m+p}} k(x_{4m+p}, s, f(s))ds - \right.$$

$$-h \sum_{i=0}^{4m} w_i k(x_{4m+p}, x_i, F_i) - \frac{x_p}{90} \left[7k_{4m} + 7k_{4m+p} + 12k\left(x_{4m+p}, x_{4m+p/2}, \mathcal{P}\left(x_{4m} + \frac{p}{2}h\right)\right) + \right.$$

$$\left. \left. +32k\left(x_{4m+p}, x_{4m+p/4}, \mathcal{P}\left(x_{4m} + \frac{p}{4}h\right)\right) +32k\left(x_{4m+p}, x_{4m+3p/4}, \mathcal{P}\left(x_{4m} + \frac{3p}{4}h\right)\right) \right] \right|.$$

By adding and diminishing the terms

$$h \sum_{i=0}^{4m} w_i k(x_{4m+p}, x_i, f(x_i)),\ \frac{7x_p}{90} k\left(x_{4m+p}, x_{4m}, f(x_{4m})\right), \ldots$$

$$\ldots, \frac{32x_p}{90} k(x_{4m+p}, x_{4m+\frac{3p}{4}}, \sum_{j=0}^{4} L_j\left(\frac{3p}{4}\right) f(x_{4m+j}))$$

and using (ii) for $k(x, s, f(s))$, one obtains

$$|\varepsilon_{4m+p}| \le h \sum_{i=0}^{4m} w_i q(x_{4m+p}, x_i)|\varepsilon_i| +$$

$$+\frac{7x_p}{90} q(x_{4m+p}, x_{4m})|\varepsilon_{4m}| + \frac{7x_p}{90} q(x_{4m+p}, x_{4m+p})|\varepsilon_{4m+p}| +$$

$$+\frac{12x_p}{90} q(x_{4m+4}, x_{4m+p/2}) \left| \mathcal{P}\left(x_{4m} + \frac{p}{2}h\right) - \sum_{j=0}^{4} L_j\left(\frac{p}{2}\right) f(x_{4m+j})) \right| +$$

$$+\frac{32x_p}{90}q(x_{4m+p},x_{4m+p/4})\left|\mathcal{P}\left(x_{4m}+\frac{p}{4}h\right)-\sum_{j=0}^{4}L_j\left(\frac{p}{4}\right)f(x_{4m+j}))\right|+$$

$$+\frac{32x_p}{90}q(x_{4m+p},x_{4m+3p/4})\left|\mathcal{P}\left(x_{4m}+\frac{3p}{4}h\right)-\sum_{j=0}^{4}L_j\left(\frac{3p}{4}\right)f(x_{4m+j}))\right|+R_1+R_2,$$

where R_1 and R_2 are errors of the numerical integrations. Since $q(x,s)$ was supposed continuous, it is bounded on $[0,X]$. Therefore

$$|\varepsilon_{4m+p}|\leq h\sum_{i=0}^{4m}w_il_i\,|\varepsilon_i|+\frac{7p}{90}hl_{4m}\,|\varepsilon_{4m}|+\frac{7p}{90}hl_{4m+p}\,|\varepsilon_{4m+p}|+$$

$$+\frac{12p}{90}hl_{4m+\frac{p}{2}}\max_j\left\{L_j\left(\frac{p}{2}\right)\right\}\sum_{j=0}^{4}|\varepsilon_{4m+j}|+$$

$$+\frac{32p}{90}hl_{4m+\frac{p}{4}}\max_j\left\{L_j\left(\frac{p}{4}\right)\right\}\sum_{j=0}^{4}|\varepsilon_{4m+j}|+$$

$$+\frac{32p}{90}hl_{4m+\frac{3p}{4}}\max_j\left\{L_j\left(\frac{3p}{4}\right)\right\}\sum_{j=0}^{4}|\varepsilon_{4m+j}|+R_1+R_2\leq$$

$$\leq hc\sum_{i=0}^{4m}|\varepsilon_i|+hc_1\,|\varepsilon_{4m+1}|+hc_2\,|\varepsilon_{4m+2}|+hc_3\,|\varepsilon_{4m+3}|+hc_4\,|\varepsilon_{4m+4}|\,.$$

Without lose of generality, let $\|\varepsilon_j\|_\infty=\max_{j=1,2,3,4}|\varepsilon_{4m+j}|=|\varepsilon_{4m+p}|$, then it is easy to get

$$|\varepsilon_{4m+p}|\leq hc'\sum_{i=0}^{4m+p-1}|\varepsilon_i|+hc''\,|\varepsilon_{4m+p}|+R_1+R_2, \tag{4.1}$$

where c' and c'' are constants or equivalently

$$\|\varepsilon_j\|_\infty\leq\frac{hc'}{1-hc''}\sum_{i=0}^{4m+p-1}|\varepsilon_i|+\frac{R_1+R_2}{1-hc''}.$$

Then from the Gronwall inequality [8], we conclude that

$$\|\varepsilon_j\|_\infty\leq\frac{R_1+R_2}{1-hc''}e^{\frac{c'}{1-hc''}x_n},$$

which implies

$$\|\varepsilon_j\|_\infty\leq\frac{R_1+R_2}{1-hc''}e^{\frac{c'}{1-hc''}x_n}\underset{\text{as } h\to0}{\to}0.$$

For the functions k and f with at least sixth order derivatives, the orders for R_1 and R_2 will be at least $O(h^6)$ and so

$$\|\varepsilon_j\| = O(h^6).$$

5. Numerical results. In this section, some examples are given to illustrate convergence and error bound of the presented method. These examples have chosen from [9] and the results computed by programming in Maple 10.

I. Equation with exponential nonlinearity

$$f(x) + A\int_a^x e^{\lambda f(s)}ds = Bx + C, \quad 0 \le x \le X,$$

with the exact solution

$$f(x) = \begin{cases} \dfrac{-1}{\lambda}\ln[A\lambda(x-a) + e^{-C\lambda}], & B = 0, \\ \dfrac{-1}{\lambda}\ln\left[\dfrac{A}{B} + \left(e^{-\lambda f_0} - \dfrac{A}{B}\right)e^{\lambda B(a-x)}\right], & f_0 = aB + C, \quad B \ne 0. \end{cases}$$

II. Equation with power-low nonlinearity

$$f(x) + A\int_a^x f^2(s)ds = Bx + C, \quad 0 \le x \le X,$$

with the exact solution

$$f(x) = \begin{cases} K\dfrac{(K+f_a)e^{2AK(x-a)} + f_a - K}{(K+f_a)e^{2AK(x-a)} - f_a + K}, \quad K = \sqrt{\dfrac{B}{A}}, \quad f_a = aB + C, & AB > 0, \\ \dfrac{C}{AC(x-a)+1}, & AB = 0, \\ K\tan\left[AK(a-x) + \arctan\dfrac{f_a}{K}\right], \quad K = \sqrt{\dfrac{-B}{A}}, \quad f_a = aB + C, & AB < 0. \end{cases}$$

III. A linear equation with trigonometric kernel

$$f(x) - A\int_a^x \frac{\cos(\lambda x)}{\cos(\lambda s)}f(s)ds = g(x), \quad 0 \le x \le X,$$

with the exact solution

$$f(x) = g(x) + A\int_a^x e^{A(x-s)}\frac{\cos(\lambda x)}{\cos(\lambda s)}g(s)ds.$$

The results in Tables 1 – 5, show the absolute error $|f(x_i) - F_i|$, for $i = 1, 2, \ldots, N$, at the selected grid points, where $f(x_i)$ and F_i are the exact and corresponding approximate solutions at $x = x_i$.

In Tables 1 and 3 the results of the HPM [14], ADM [4] and block by block method are compared, where m denotes number of iterations for HPM and ADM and N denotes number of the mesh points for the block by block method.

The results of Examples 1 and 2 (see Tables 1 and 3) show that the HPM and ADM behave worse than the presented method even in the interval $[0, 1]$. For the Example 1, HPM and ADM does not work respectively for more than 3 and 5 iterations (probably because of exponential nonlinearity).

Table 3, shows that the above mentioned behavior occur again for the large intervals (say $X = 10$) by the HPM and ADM, although it does not occur for $[0, 1]$.

The last rows of Tables 1 – 5 compar the computing time for the HPM, ADM and block by block method which for the last one is less than two others, where programming for all methods have been done using Maple package.

***Remark* 5.1.** Let $E(h) = Ch^q$ be error of the block by block method, where C and q are respectively a constant and order of the error, then $q = \dfrac{\ln(E(h)/C)}{\ln(h)}$.

By computing the order q from this formula for the reported errors in Tables 1 – 5, we conclude that $q \geq 6$. This result confirms the claim that was stated in introduction and was proved in Section 4.

For example, we get $q = 7.68$ in Table 2 for $x_i = \dfrac{1}{20}$ $(X = 1, N = 20)$, error $= 7.664e^{-10}$, $h = 0.05$, and $q = 9$ in Table 5 for $x_i = 10$ $(X = 10, N = 1000)$, error $= 1.604e^{-18}$, $h = 0.01$.

Table 1. *Numerical results of example* I ($\lambda = 1/2$, $A = 4$, $B = 3$, $C = 1/8$, $a = 0$)

x_i	HPM		ADM		block by block	
	$X = 1$	$X = 10$	$X = 1$	$X = 10$	$X = 1$	$X = 10$
$N = 40$	$m = 3$	$m = 3$	$m = 5$	$m = 3$		
$4X/N$	$9.037e^{-01}$	$7.561e^{+00}$	$2.725e^{-05}$	$6.812e^{-01}$	$9.709e^{-10}$	$1.222e^{-04}$
$8X/N$	$1.770e^{+00}$	$4.875e^{+01}$	$1.255e^{-05}$	$1.158e^{+01}$	$5.386e^{-10}$	$3.729e^{-05}$
$12X/N$	$2.621e^{+00}$	$2.442e^{+01}$	$1.275e^{-04}$	$8.093e^{+01}$	$5.840e^{-10}$	$1.477e^{-05}$
$16X/N$	$3.473e^{+00}$	$1.131e^{+03}$	$6.470e^{-04}$	$4.047e^{+04}$	$5.415e^{-10}$	$7.495e^{-06}$
$20X/N$	$4.341e^{+00}$	$5.115e^{+03}$	$2.252e^{-03}$	$1.867e^{+03}$	$2.286e^{-10}$	$1.150e^{-05}$
$24X/N$	$5.238e^{+00}$	$2.298e^{+04}$	$6.187e^{-03}$	$8.439e^{+03}$	$2.285e^{-09}$	$1.848e^{-05}$
$28X/N$	$6.180e^{+00}$	$1.031e^{+05}$	$1.446e^{-02}$	$3.790e^{+04}$	$2.105e^{-09}$	$8.723e^{-04}$
$32X/N$	$7.182e^{+00}$	$4.620e^{+05}$	$3.004e^{-02}$	$1.699e^{+05}$	$6.388e^{-10}$	$7.938e^{-04}$
$36X/N$	$8.265e^{+00}$	$2.076e^{+06}$	$5.706e^{-02}$	$7.615e^{+05}$	$5.781e^{-10}$	$3.607e^{-04}$
X	$9.459e^{+00}$	$9.280e^{+06}$	$1.010e^{-01}$	$3.410e^{+06}$	$5.781e^{-10}$	$3.607e^{-04}$
time	$1.703''$	$3.375''$	$172.921''$		$0.609''$	$1.078''$

Table 2. *Numerical results of example* I $(\lambda = 10,\ A = 1/10,\ B = 0,\ C = 1/100,\ a = 0)$

x_i	$X = 1$		$X = 10$		$X = 30$	
	$N = 20$	$N = 60$	$N = 20$	$N = 60$	$N = 60$	$N = 100$
X/N	$7.664e^{-10}$	$1.469e^{-12}$	$3.907e^{-05}$	$3.904e^{-07}$	$3.907e^{-05}$	$5.377e^{-06}$
$2X/N$	$4.450e^{-10}$	$8.644e^{-13}$	$2.130e^{-05}$	$2.194e^{-07}$	$2.130e^{-05}$	$2.961e^{-06}$
$5X/N$	$5.842e^{-10}$	$1.224e^{-12}$	$5.199e^{-05}$	$3.792e^{-07}$	$5.199e^{-05}$	$6.261e^{-06}$
$7X/N$	$5.472e^{-10}$	$1.199e^{-12}$	$4.022e^{-05}$	$3.203e^{-07}$	$4.022e^{-05}$	$5.023e^{-06}$
$X/2$	$5.471e^{-11}$	$1.966e^{-13}$	$1.551e^{-05}$	$1.956e^{-07}$	$1.791e^{-05}$	$6.662e^{-05}$
$(N-7)X/N$	$3.182e^{-10}$	$2.498e^{-11}$	$2.841e^{-05}$	$1.693e^{-06}$	$2.825e^{-04}$	$2.686e^{-04}$
$(N-5)X/N$	$3.001e^{-10}$	$2.453e^{-11}$	$2.502e^{-05}$	$1.624e^{-06}$	$2.719e^{-04}$	$2.631e^{-04}$
$(N-2)X/N$	$1.769e^{-10}$	$1.342e^{-11}$	$2.349e^{-06}$	$1.888e^{-06}$	$1.336e^{-04}$	$5.289e^{-04}$
X	$1.737e^{-10}$	$1.317e^{-11}$	$2.144e^{-06}$	$1.806e^{-06}$	$1.293e^{-04}$	$5.188e^{-04}$
time	$3.891''$	$5.343''$	$3.813''$	$6.297''$	$6.172''$	$8.188''$

Table 3. *Numerical results of example* II $(A = 1/2,\ B = 2,\ C = 1,\ a = 0)$

x_i	HPM		ADM		block by block	
	$X = 1$	$X = 10$	$X = 1$	$X = 10$	$X = 1$	$X = 10$
	$m = 10$	$m = 10$	$m = 9$	$m = 9$	$N = 60$	$N = 100$
X/N	$7.218e^{-05}$	$3.128e^{-03}$	0	$1.000e^{-09}$	$4.518e^{-12}$	$2.709e^{-07}$
$5X/N$	$2.095e^{-03}$	$1.666e^{-01}$	$1.000e^{-09}$	$3.000e^{-09}$	$4.912e^{-13}$	$3.524e^{-08}$
$10X/N$	$9.995e^{-03}$	$1.481e^{+00}$	$2.000e^{-09}$	$1.050e^{-05}$	$3.559e^{-12}$	$4.986e^{-08}$
$15X/N$	$2.655e^{-02}$	$8.021e^{+00}$	$1.000e^{-09}$	$1.431e^{-03}$	$5.771e^{-12}$	$1.492e^{-07}$
$25X/N$	$1.002e^{-01}$	$1.121e^{+70}$	$1.000e^{-09}$	$2.824e^{+67}$	$4.888e^{-12}$	$8.361e^{-08}$
$X/2$	$1.666e^{-01}$	$3.078e^{+340}$	$4.000e^{-09}$	$1.000e^{+320}$	$2.543e^{-13}$	$2.160e^{-06}$
$(N-25)X/N$	$2.606e^{-01}$	$1.283e^{+507}$	$1.900e^{-08}$	$2.308e^{+491}$	$6.600e^{-12}$	$1.794e^{-06}$
$(N-15)X/N$	$5.631e^{-01}$	$3.875e^{+559}$	$4.030e^{-07}$	$5.031e^{+551}$	$7.977e^{-13}$	$1.167e^{-06}$
$(N-10)X/N$	$7.924e^{-01}$	$3.072e^{+583}$	$1.431e^{-06}$	$3.508e^{+576}$	$1.561e^{-11}$	$1.046e^{-06}$
$(N-5)X/N$	$1.092e^{+00}$	$1.916e^{+606}$	$4.510e^{-06}$	$1.075e^{+600}$	$8.741e^{-11}$	$8.601e^{-07}$
X	$1.481e^{+00}$	$9.205e^{+627}$	$1.050e^{-05}$	$2.021e^{+622}$	$4.768e^{-11}$	$5.353e^{-06}$
time	$75.235''$	$125.858''$	$24.609''$	$26.329''$	$0.329''$	$0.626''$

Table 4. *Numerical results of example* II $(A = 1/10,\ B = 0,\ C = 1,\ a = 0)$

x_i	$X = 1$		$X = 10$		$X = 30$	
	$N = 100$	$N = 1000$	$N = 100$	$N = 300$	$N = 100$	$N = 300$
X/N	$4.444e^{-18}$	$4.494e^{-24}$	$3.977e^{-12}$	$5.922e^{-15}$	$2.287e^{-09}$	$3.977e^{-12}$
$10X/N$	$2.430e^{-18}$	$2.642e^{-24}$	$9.448e^{-13}$	$2.666e^{-15}$	$1.938e^{-10}$	$9.448e^{-13}$
$20X/N$	$9.411e^{-20}$	$9.993e^{-27}$	$5.535e^{-13}$	$3.706e^{-16}$	$5.214e^{-10}$	$5.535e^{-13}$
$30X/N$	$2.033e^{-18}$	$2.596e^{-24}$	$9.575e^{-14}$	$1.450e^{-15}$	$1.104e^{-10}$	$9.575e^{-14}$
$40X/N$	$7.571e^{-20}$	$9.316e^{-27}$	$2.554e^{-13}$	$1.929e^{-16}$	$2.029e^{-09}$	$2.554e^{-13}$
$X/2$	$1.618e^{-18}$	$3.712e^{-21}$	$2.146e^{-11}$	$1.875e^{-11}$	$1.567e^{-07}$	$1.065e^{-07}$
$(N-40)X/N$	$3.002e^{-19}$	$2.375e^{-18}$	$7.913e^{-12}$	$2.938e^{-09}$	$1.414e^{-08}$	$6.862e^{-06}$
$(N-30)X/N$	$1.726e^{-18}$	$2.850e^{-18}$	$1.833e^{-10}$	$1.923e^{-09}$	$4.355e^{-07}$	$2.524e^{-06}$
$(N-20)X/N$	$1.722e^{-20}$	$3.420e^{-18}$	$6.947e^{-11}$	$3.895e^{-09}$	$3.494e^{-09}$	$5.948e^{-06}$
$(N-10)X/N$	$1.409e^{-18}$	$3.370e^{-18}$	$7.000e^{-10}$	$7.443e^{-09}$	$3.929e^{-07}$	$1.345e^{-06}$
X	$2.661e^{-18}$	$4.031e^{-18}$	$7.436e^{-09}$	$5.407e^{-09}$	$1.343e^{-05}$	$6.929e^{-06}$
time	$0.64''$	$6.484''$	$0.672''$	$2.046''$	$0.702''$	$2.046''$

Table 5. *Numerical results of example* III $(A = 1/100,\ \lambda = 2,\ g(x) = \cos(2x),\ a = 0)$

x_i	$X = 1$		$X = 10$		$X = 30$	
	$N = 100$	$N = 1000$	$N = 100$	$N = 1000$	$N = 100$	$N = 1000$
X/N	$5.869e^{-17}$	$1.291e^{-22}$	$1.869e^{-09}$	$5.869e^{-17}$	$3.072e^{-06}$	$3.463e^{-13}$
$10X/N$	$5.887e^{-16}$	$2.045e^{-23}$	$1.576e^{-08}$	$5.887e^{-16}$	$4.132e^{-06}$	$1.256e^{-12}$
$20X/N$	$1.539e^{-18}$	$5.205e^{-27}$	$6.651e^{-11}$	$1.539e^{-18}$	$3.372e^{-07}$	$5.532e^{-15}$
$30X/N$	$1.864e^{-15}$	$1.218e^{-22}$	$1.821e^{-09}$	$1.864e^{-15}$	$1.773e^{-05}$	$5.187e^{-12}$
$40X/N$	$3.505e^{-18}$	$1.728e^{-26}$	$1.691e^{-11}$	$3.505e^{-18}$	$2.031e^{-07}$	$1.527e^{-14}$
$X/2$	$2.756e^{-15}$	$7.623e^{-24}$	$1.188e^{-09}$	$1.744e^{-16}$	$5.367e^{-07}$	$6.250e^{-13}$
$(N-40)X/N$	$8.240e^{-18}$	$1.132e^{-22}$	$1.313e^{-10}$	$4.937e^{-16}$	$7.443e^{-08}$	$2.100e^{-13}$
$(N-30)X/N$	$2.744e^{-15}$	$3.420e^{-21}$	$4.674e^{-10}$	$1.904e^{-15}$	$7.642e^{-07}$	$5.577e^{-11}$
$(N-20)X/N$	$2.820e^{-18}$	$1.370e^{-22}$	$1.061e^{-10}$	$1.711e^{-15}$	$2.074e^{-07}$	$2.430e^{-12}$
$(N-10)X/N$	$4.316e^{-15}$	$3.410e^{-21}$	$6.186e^{-09}$	$2.209e^{-15}$	$2.076e^{-06}$	$2.528e^{-12}$
X	$9.055e^{-18}$	$2.221e^{-23}$	$6.585e^{-11}$	$1.604e^{-18}$	$3.491e^{-06}$	$1.080e^{-13}$
time	$0.265''$	$3.077''$	$0.311''$	$3.765''$	$0.312''$	$3.844''$

6. Conclusion. In this paper, we presented a block by block method for solving Volterra integral equations on the large intervals with at least 6 order of convergence. The method can be improved by using accurate Romberg rule or even other suitable integration methods. Numerical results given in Tables 1 – 5 show high accuracy of the method. The last rows of these tables show that the computing time of the presented method is less than two other methods (HPM and ADM).

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