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AN ADMISSIBLE ESTIMATOR OF THE rth POWER OF A BOUNDED SCALE-PARAMETER IN A SUBCLASS OF THE EXPONENTIAL FAMILY UNDER ENTROPY LOSS FUNCTION

ДОПУСТИМА ОЦІНКА ДЛЯ r-ГО СТЕПЕНЯ ОБМЕЖЕНОГО ПАРАМЕТРА МАСШТАБУ У ПІДКЛАСІ ЕКСПОНЕНЦІАЛЬНОЇ СІМ'Ї З ЕНТРОПІЙНОЮ ФУНКЦІЄЮ ВТРАТ

We consider an admissible estimator for the rth power of a scale parameter that is lower or upper bounded in a subclass of the scale-parameter exponential family under entropy loss function. An admissible estimator of a bounded parameter in the family of transformed chi-square distributions is also given.

Розглянуто допустиму оцінку для r-го степеня параметра масштабу, обмеженого зверху або знизу у підкласі експоненціальної сім'ї параметрів масштабу з ентропійною функцією втрат. Наведено також допустиму оцінку обмеженого параметра у сім'ї трансформованих розподілів хі-квадрат.

1. Introduction. The first result in the case of truncated parameter space for minimax estimation of the scale parameter λ and the recioprocal of the scale parameter λ^{-1}, in gamma distribution, was obtained by Zubrzycki [19] who applied the well-known method of Lehmann (cf. Lehmann [9]).

Using Karlin method (cf. Karlin [7]), Ghosh and Singh [3] proved admissibility of the estimator $(s-2)\,X^{-1}$ of the gamma parameter λ. They also gave the minimax estimator of λ, but this result is contained in that of Zubrzycki [19]. Singh [16] showed that $\dfrac{\Gamma(s-r)}{\Gamma(s-2r)}X^{-r}$ is an admissible estimator of λ^r under squared error loss, where r is an integer, $r<\dfrac{s}{2}$. Ghosh and Meeden [4] and Ralescu and Ralescu [14] have found admissible estimators of λ and λ^{-1} in the gamma distribution. Also, Kaluszka [6] obtained an admissible minimax estimator of the parameter λ^r under the scale-invariant squared error loss, where $r\neq 0$ is an integer and $\lambda\in(\lambda_0,\infty)$ or $\lambda\in(-\infty,\lambda_0)$ with given constant λ_0.

Minimaxity and admissibility results for lower-bounded parameters can be found in Katz [8], Berry [1] and van Eeden [17, 18]. Using scale-invariant squared-error loss, van Eeden [17] gives an admissible minimax estimator of the scale-parameter θ of the gamma distribution with known shape parameter, where $\theta\in[a,\infty)$.

Jafari Jozani et al. [5] extended the results of van Eeden [17]. They obtained an admissible minimax estimator of a bounded scale parameter in a subclass of the exponential family under scale-invariant squared error loss. Also, they studied the admissibility and minimaxity in the family of transformed chi-square distributions due to Rahman and Gupta [13].

Recently, Mahmoudi and Zakerzadeh [10] obtained an admissible estimator of a lower bounded scale-parameter under squared-log error loss function. Also Mahmoudi [11] studied an admissible minimax estimator of θ^r in a subclass of the exponential family with truncated parameter space under squared-log error loss function.

Assuming the entropy error loss, of the form

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$$L(\delta, \theta^r) = \frac{\delta}{\theta^r} - \ln \frac{\delta}{\theta^r} - 1. \tag{1.1}$$

Sanjari Farsipour [15] obtained an admissible estimator of the parameter λ^r, where $\lambda \in (0, \lambda_0)$ or $\lambda \in (\lambda_0, \infty)$ with given constant λ_0 and $r \neq 0$ in the gamma distribution under entropy loss function (1.1).

In this paper we consider a subclass of the scale-parameter exponential family. We obtain an admissible estimator of the rth power of a lower or upper bounded scale-parameter (say θ^r), using Karlin's method, under entropy loss function (1.1). We show that the admissible estimator obtained by Sanjari Farsipour [15] is a special case of our estimator. In fact, our paper generalizes the results of Sanjari Farsipour [15]. The rest of the paper is as follow:

A subclass of the scale-parameter exponential family is introduced in Section 2. We give the admissibility results in Section 3. An admissible estimator of the bounded parameter, for the family of transformed chi-square distributions, introduced by Rahman and Gupta [13], is presented in Section 4.

2. A subclass of the exponential family. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a distribution with density $(1/\eta)\, g(x/\eta)$, where g is known and η is an unknown scale parameter. The joint density of $X_1, X_2, \ldots, X_n$ is denoted by $f(\mathbf{x}; \eta) = \dfrac{1}{\eta^n} f\left(\dfrac{\mathbf{x}}{\eta}\right)$. In some cases the above model reduces to

$$f(\mathbf{x}; \theta) = c(\mathbf{x}, n)\theta^{\nu} e^{-\theta T(\mathbf{x})}, \tag{2.1}$$

where $c(\mathbf{x}, n)$ is a function of $\mathbf{x} = (x_1, \ldots, x_n)'$ and n, $\theta = \eta^{-r}$ $(r > 0)$, ν is a function of n and $T(\mathbf{X})$ is a complete sufficient statistic for θ with a $\Gamma(\nu, \theta)$ distribution.

Jafari Jozani et al. [5] have listed some distributions belonging to this subclass of the scale-parameter exponential family such as gamma, inverse Gaussian with zero drift, normal, Weibull and Rayleigh distribution. For example $\Gamma(\alpha, \beta)$ with known α belongs to this subclass of distributions with:

$$\theta = \beta^{-1} (\eta = \beta,\ r = 1), \qquad \nu = n\alpha,\ T(\mathbf{X}) = \sum_{i=1}^{n} X_i, \qquad c(\mathbf{x}, n) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}}{\Gamma(\alpha)}, \tag{2.2}$$

where the joint density is given by

$$f(\mathbf{x}, \beta) = \left(\prod_{i=1}^{n} \frac{x_i^{\alpha-1}}{\Gamma(\alpha)}\right) \beta^{-n\alpha} e^{-\sum_{i=1}^{n} x_i/\beta}, \qquad x_i > 0, \quad i = 1, 2, \ldots, n.$$

Some properties of the family of distributions (2.1) along an admissible linear estimator of $\theta = \eta^r$, under the entropy loss function, can be found in Parsian and Nematollahi [12].

In Section 3 an admissible estimator of θ^r, where $r \neq 0$ is constant, is given where θ is restricted to $(0, \theta_0)$ or (θ_0, ∞).

3. Admissible estimator of $\theta^{\mathbf{r}}$ with truncated parameter space. Let us denote by $\gamma(.,.)$, $\Gamma(.,.)$, the incomplete gamma functions, i.e.,

$$\gamma(x, y) = \int_0^y t^{x-1} \exp(-t)\, dt, \qquad \Gamma(x, y) = \int_y^{\infty} t^{x-1} \exp(-t)\, dt, \quad x, y > 0.$$

We now give an admissible estimator of θ^r in the scale-parameter exponential family (2.1), where θ is unknown parameter to satisfy the restrictions $\theta \in (0,\theta_0)$ or $\theta \in (\theta_0,\infty)$, for some known $\theta_0 > 0$ and integer $r \neq 0$ with $r < \dfrac{\nu}{2}$. Consider the following two lemmas.

Lemma 3.1. *We have*

$$\lim_{b\to 0}\int\limits_b^{\theta_0} \theta^{\nu-\rho-1}e^{-(T(\mathbf{x})+k)\theta}d\theta = (T(\mathbf{x})+k)^{\rho-\nu}\,\gamma\,(\nu-\rho,\theta_0\,(T(\mathbf{x})+k)).$$

Proof. Using the integration by parts we have

$$\lim_{b\to 0}\int\limits_b^{\theta_0} \theta^{\nu-\rho-1}e^{-(T(\mathbf{x})+k)\theta}d\theta = \lim_{b\to 0}\frac{1}{(T(\mathbf{x})+k)^{\nu-\rho}}\int\limits_{b(T(\mathbf{x})+k)}^{\theta_0(T(\mathbf{x})+k)} t^{\nu-\rho-1}e^{-t}dt =$$

$$= \frac{1}{(T(\mathbf{x})+k)^{\nu-\rho}}\int\limits_0^{\theta_0(T(\mathbf{x})+k)} t^{\nu-\rho-1}e^{-t}dt =$$

$$= (T(\mathbf{x})+k)^{\rho-\nu}\,\gamma\,(\nu-\rho,\theta_0\,(T(\mathbf{x})+k)).$$

Thus, the proof is completed.

Lemma 3.2. *We have*

$$\lim_{b\to \infty}\int\limits_{\theta_0}^{b} \theta^{\nu-\rho-1}e^{-(T(\mathbf{x})+k)\theta}d\theta = (T(\mathbf{x})+k)^{\rho-\nu}\,\Gamma\,(\nu-\rho,\theta_0\,(T(\mathbf{x})+k)).$$

Proof. Proof of this lemma is similar to the proof of Lemma 3.1.

Remark **3.1.** Setting $\rho = r,\ 2r$ in Lemma 3.1 and Lemma 3.2 gives the following results:

(i) $\displaystyle\lim_{b\to 0}\int_b^{\theta_0} \theta^{\nu-r-1}e^{-(T(\mathbf{x})+k)\theta}d\theta = (T(\mathbf{x})+k)^{r-\nu}\,\gamma\,(\nu-r,\theta_0\,(T(\mathbf{x})+k))\,,$

(ii) $\displaystyle\lim_{b\to 0}\int_b^{\theta_0} \theta^{\nu-2r-1}e^{-(T(\mathbf{x})+k)\theta}d\theta = (T(\mathbf{x})+k)^{2r-\nu}\,\gamma\,(\nu-2r,\theta_0\,(T(\mathbf{x})+k))\,,$

(iii) $\displaystyle\lim_{b\to \infty}\int_{\theta_0}^{b} \theta^{\nu-r-1}e^{-(T(\mathbf{x})+k)\theta}d\theta = (T(\mathbf{x})+k)^{r-\nu}\,\Gamma\,(\nu-r,\theta_0\,(T(\mathbf{x})+k))\,,$

(iv) $\displaystyle\lim_{b\to \infty}\int_{\theta_0}^{b} \theta^{\nu-2r-1}e^{-(T(\mathbf{x})+k)\theta}d\theta = (T(\mathbf{x})+k)^{2r-\nu}\,\Gamma\,(\nu-2r,\theta_0\,(T(\mathbf{x})+k))\,.$

Theorem 3.1. *The estimator*

$$\hat{\delta}\,(\mathbf{X}) = \begin{cases} \dfrac{\gamma\,(\nu-r,\ \theta_0\,(k+T\,(\mathbf{X})))}{\gamma\,(\nu-2r,\ \theta_0\,(k+T\,(\mathbf{X})))}\,(T\,(\mathbf{X})+k)^{-r}\,, & 0<\theta<\theta_0, \\ \dfrac{\Gamma\,(\nu-r,\ \theta_0\,(k+T\,(\mathbf{X})))}{\Gamma\,(\nu-2r,\ \theta_0\,(k+T\,(\mathbf{X})))}\,(T\,(\mathbf{X})+k)^{-r}\,, & \theta_0<\theta<\infty, \end{cases} \tag{3.1}$$

in which $k \geq 0$ is an arbitrary constant, is admissible for θ^r under the entropy loss function (1.1), *where θ has the improper prior density function*

$$\pi\,(\theta) = \theta^{-r-1}\exp\,(-k\theta)\,, \qquad \theta\in(0,\theta_0) \quad \textit{or} \quad \theta\in(\theta_0,\infty)\,. \tag{3.2}$$

Proof. In the proof of Theorem 2.1 of Sanjari Farsipour [15], if we replace the values s, λ, x and $\frac{x^{s-1}}{\Gamma(s)}$ with ν, θ, $T(\mathbf{x})$ and $c(\mathbf{x}, n)$ respectively, then using Remark 3.1, the proof is still established. Thus, the proof of Theorem 3.1 can be derived parallel to the proof of Theorem 2.1 of Sanjari Farsipour [15]. Here we only give the short proof of the first case of this theorem, where $0 < \theta < \theta_0$.

Suppose that there exist an estimator $\tilde{\delta}$ which is better than $\hat{\delta}$. This implies that the inequality

$$\int_0^\infty \left(\frac{\tilde{\delta}}{\theta^r} - \ln \frac{\tilde{\delta}}{\theta^r} - 1 \right) f(\mathbf{x}, \theta)\, d\mathbf{x} \le \int_0^\infty \left(\frac{\hat{\delta}}{\theta^r} - \ln \frac{\hat{\delta}}{\theta^r} - 1 \right) f(\mathbf{x}, \theta)\, d\mathbf{x}$$

holds for all $\theta \in (0, \theta_0)$ or $\theta \in (\theta_0, \infty)$ with strict inequality for some θ. Using above inequality, we get

$$\int_0^\infty \left(\frac{\tilde{\delta}}{\hat{\delta}} - \ln \frac{\tilde{\delta}}{\hat{\delta}} - 1 \right) f(\mathbf{x}, \theta)\, d\mathbf{x} \le \int_0^\infty \left(\frac{\tilde{\delta}}{\hat{\delta}} - \frac{\tilde{\delta}}{\theta^r} + \frac{\hat{\delta}}{\theta^r} - 1 \right) f(\mathbf{x}, \theta)\, d\mathbf{x}. \tag{3.3}$$

Integration both sides of (3.3) with respect to the improper prior density function (3.2) gives

$$\int_b^{\theta_0} \int_0^\infty \left(\frac{\tilde{\delta}}{\hat{\delta}} - \ln \frac{\tilde{\delta}}{\hat{\delta}} - 1 \right) f(\mathbf{x}, \theta)\, \pi(\theta)\, d\mathbf{x} d\theta \le$$

$$\le \int_b^{\theta_0} \int_0^\infty \left(\frac{\tilde{\delta}}{\hat{\delta}} - \frac{\tilde{\delta}}{\theta^r} + \frac{\hat{\delta}}{\theta^r} - 1 \right) f(\mathbf{x}, \theta)\, \pi(\theta)\, d\mathbf{x} d\theta. \tag{3.4}$$

By interchanging the order of integration in the right-hand side of (3.4) and substituting $\hat{\delta}$ from (3.1) into (3.4) we have

$$\int_b^{\theta_0} \int_0^\infty \left(\frac{\tilde{\delta}}{\hat{\delta}} - \frac{\tilde{\delta}}{\theta^r} + \frac{\hat{\delta}}{\theta^r} - 1 \right) f(\mathbf{x}, \theta)\, \pi(\theta)\, d\mathbf{x} d\theta =$$

$$= \int_0^\infty \frac{\tilde{\delta}\gamma(\nu - 2r,\ \theta_0 (T(\mathbf{x}) + k))}{\gamma(\nu - r,\ \theta_0 (T(\mathbf{x}) + k))} (T(\mathbf{x}) + k)^r\, c(\mathbf{x}, n) \int_b^{\theta_0} \theta^{\nu - r - 1} e^{-(T(\mathbf{x}) + k)\theta} d\theta d\mathbf{x} -$$

$$- \int_0^\infty \tilde{\delta} c(\mathbf{x}, n) \int_b^{\theta_0} \theta^{\nu - 2r - 1} e^{-(T(\mathbf{x}) + k)\theta} d\theta d\mathbf{x} +$$

$$+ \int_0^\infty \frac{\gamma(\nu - r,\ \theta_0 (T(\mathbf{x}) + k))}{\gamma(\nu - 2r,\ \theta_0 (T(\mathbf{x}) + k))} (T(\mathbf{x}) + k)^{-r}\, c(\mathbf{x}, n) \int_b^{\theta_0} \theta^{\nu - 2r - 1} e^{-(T(\mathbf{x}) + k)\theta} d\theta d\mathbf{x} -$$

$$- \int_0^\infty c(\mathbf{x}, n) \int_b^{\theta_0} \theta^{\nu - r - 1} e^{-(T(\mathbf{x}) + k)\theta} d\theta d\mathbf{x}. \tag{3.5}$$

Using Lemma 3.1 with $\rho = r$ for $\theta \in (0, \theta_0)$, (3.5) tends to zero and we have

$$\int\limits_0^\infty \left(\frac{\tilde{\delta}}{\hat{\delta}} - \ln \frac{\tilde{\delta}}{\hat{\delta}} - 1 \right) f(\mathbf{x}, \theta)\, d\mathbf{x} = 0,$$

i.e., $\tilde{\delta} = \hat{\delta}$ a.e., and the admissibility of $\hat{\delta}(\mathbf{X})$, in the first case is completed.

The proof of the second case, where $\theta \in (\theta_0, \infty)$, is quite similar. So according to the first case, $\int_0^\infty \left(\frac{\tilde{\delta}}{\hat{\delta}} - \ln \frac{\tilde{\delta}}{\hat{\delta}} - 1 \right) f(\mathbf{x}, \theta)\, d\mathbf{x} = 0$, i.e., $\tilde{\delta} = \hat{\delta}$ a.e., and the admissibility of $\hat{\delta}(\mathbf{X})$, in this case, is completed.

***Remark* 3.2.** For the untruncated case $\theta > 0$, it can be easily show that $\frac{\Gamma(\nu - r)}{\Gamma(\nu - 2r)}(T(\mathbf{X}) + k)^{-r}$ is an admissible estimator of θ^r under the entropy loss function (1.1) and the improper prior $\pi(\theta) = = \theta^{-r-1} \exp(-k\theta), \theta > 0$.

The following remark shows that the result of Sanjari Farsipour [15] is a special case of our result.

***Remark* 3.3.** In the special case where the random variable X has $\Gamma(s, \lambda)$ distribution, choosing

$$\theta = \lambda, \qquad \nu = s, \qquad T(\mathbf{X}) = X, \qquad c(\mathbf{x}, n) = \frac{x^{s-1}}{\Gamma(s)},$$

gives the admissible estimator

$$\hat{\delta}(X) = \begin{cases} \dfrac{\gamma(s - r,\ \lambda_0 (k + X))}{\gamma(s - 2r,\ \lambda_0 (k + X))} (X + k)^{-r}, & 0 < \lambda < \lambda_0, \\ \dfrac{\Gamma(s - r,\ \lambda_0 (k + X))}{\Gamma(s - 2r,\ \lambda_0 (k + X))} (X + k)^{-r}, & \lambda_0 < \lambda < \infty, \end{cases}$$

which is an admissible estimator of θ^r, $r < \frac{s}{2}$, obtained by Sanjari Farsipour [15].

According to Theorem 3.1 and Remark 3.2 we have the following examples.

***Example* 3.1.** Consider a sample of size n from gamma distribution with pdf

$$f(x, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\lambda x}, \quad x > 0,$$

where α is known and $\lambda \in (\lambda_0, \infty)$ is unknown.

(i) An admissible estimator of the scale parameter λ^r, $r < \frac{n\alpha}{2}$, under the loss function (1.1) and improper prior (3.2), is given by

$$\hat{\delta}(\mathbf{X}) = \frac{\Gamma\left(n\alpha - r, \lambda_0 \left(k + n\bar{X}_n\right)\right)}{\Gamma\left(n\alpha - 2r, \lambda_0 \left(k + n\bar{X}_n\right)\right)} \left(n\bar{X}_n + k\right)^{-r}.$$

(ii) An admissible estimator of the scale parameter λ^r, $r < \frac{n\alpha}{2}$, for the untruncated case $\lambda > 0$, is of the form

$$\hat{\delta}(\mathbf{X}) = \frac{\Gamma(n\alpha - r)}{\Gamma(n\alpha - 2r)} \left(n\bar{X}_n + k\right)^{-r}.$$

***Example* 3.2.** Suppose that $X_1, X_2, \ldots, X_n$ is a sample of size n from inverse Gaussian with zero drift, having the pdf

$$f(x,\lambda) = \left(2\pi x^3\right)^{-1/2} \lambda^{1/2} e^{-\lambda/2x}, \quad x > 0.$$

(i) An admissible estimator of λ^r, $r < \dfrac{n}{4}$ and $\lambda \in (\lambda_0, \infty)$, under the loss (1.1) is

$$\hat{\delta}(\mathbf{X}) = \frac{\Gamma\left(\frac{n}{2} - r, \lambda_0\left(k + \frac{1}{2}\sum_{i=1}^n \frac{1}{X_i}\right)\right)}{\Gamma\left(\frac{n}{2} - 2r, \lambda_0\left(k + \frac{1}{2}\sum_{i=1}^n \frac{1}{X_i}\right)\right)} \left(\frac{1}{2}\sum_{i=1}^n \frac{1}{X_i} + k\right)^{-r}.$$

(ii) For the untruncated case $\lambda > 0$, this estimator is replaced by

$$\hat{\delta}(\mathbf{X}) = \frac{\Gamma(n/2 - r)}{\Gamma(n/2 - 2r)} \left(\frac{1}{2}\sum_{i=1}^n \frac{1}{X_i} + k\right)^{-r}.$$

4. Admissibility results in the family of transformed chi-square distributions. In this section, we use a subfamily of the one-parameter exponential family of distributions called the transformed chi-square family of distributions, introduced by Rahman and Gupta [13], to derive an admissible estimator of the rth power of the unknown bounded parameter.

Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be a random vector whose joint density function belongs to the one-parameter exponential family, i.e.,

$$f(\mathbf{x}, \eta) = e^{a(\mathbf{x})b(\eta) + c(\eta) + h(\mathbf{x})}. \tag{4.1}$$

Rahman and Gupta [13] proved the following theorem for this family of distributions.

Theorem 4.1. *In a one-parameter exponential family* (4.1), *the function* $-2a(\mathbf{X})b(\eta)$ *has a* $\Gamma(j/2, 2)$*-distribution if and only if*

$$\frac{2c'(\eta)b(\eta)}{b'(\eta)} = j, \tag{4.2}$$

where j *is positive and free from* η*. In the case that* j *is an integer,* $-2a(\mathbf{X})b(\eta)$ *follows a central chi-square distribution with* j *degrees of freedom.*

The one-parameter exponential family (4.1) satisfying the condition (4.2) is called the family of transformed chi-square distributions, provided j is a positive integer. Note that if $a(\mathbf{x}) > 0$ then $b(\eta)$ must be a negative. From condition (4.2) we get

$$c(\eta) = \frac{j}{2}\ln|b(\eta)| + k_1. \tag{4.3}$$

Let $\theta = -b(\eta) > 0$, then (4.3) reduces to $e^{c(\eta)} = [-b(\eta)]^{j/2} e^{k_1} = \theta^{j/2} e^{k_1}$. So the family of distributions (4.1) can be written in the form

$$f(\mathbf{x}, \eta) = e^{-a(\mathbf{x})[-b(\eta)] + c(\eta) + h(\mathbf{x})} = c(\mathbf{x}, m)\, \theta^{j/2} e^{-\theta a(\mathbf{x})},$$

where $c(\mathbf{x}, m) = e^{h(\mathbf{x}) + k_1}$. Also note that $2\theta a(\mathbf{X}) \sim \Gamma(j/2, 2)$ or $a(\mathbf{X}) \sim \Gamma(j/2, \theta)$. Therefore, if condition (4.2) holds then the one-parameter exponential family (4.1) is in the form of the scale-parameter exponential family (2.1) with $\nu = \dfrac{j}{2}$, $T(\mathbf{X}) = a(\mathbf{X})$ and $\theta = -b(\eta)$.

Jafari Jozani et al. [5] listed some distributions belong to the family of transformed chi-square distributions in Table 1. This table contains normal, lognormal, exponential, gamma, Rayleigh, Weibull, Maxwell and inverse Gaussian distributions.

Pareto, Burr X, Burr XII, Laplace, generalized Laplace, generalized gamma and etc. are other distributions belonging to this family of distributions which did not list by Jafari Jozani et al. [5]. Some of these distributions with associated $a(\mathbf{x})$, $\theta = -b(\eta)$ and j are:

(i) *Pareto distribution*: $Pareto(\alpha, \beta)$ with α known,

$$a(\mathbf{x}) = \sum_{i=1}^{n} \ln \frac{x_i}{\alpha}, \qquad b(\beta) = -\beta, \qquad c(\beta) = n \ln \beta, \qquad \theta = \beta, \quad j = 2n,$$

and the joint density is given by

$$f(\mathbf{x}, \alpha, \beta) = \frac{\beta^n}{\prod_{i=1}^{n} x_i} \exp\left(-\beta \sum_{i=1}^{n} \ln(x_i/\alpha)\right), \qquad x_i > \alpha, \quad i = 1, 2, \ldots, n,$$

where $-2b(\beta)a(\mathbf{X}) = 2\beta \sum_{i=1}^{n} \ln \dfrac{X_i}{\alpha} \sim \Gamma(n, 2)$.

(ii) *Generalized Laplace distribution*: $GL(\alpha, \beta)$ with α known,

$$a(\mathbf{x}) = \sum_{i=1}^{n} |x_i|^{\alpha}, \qquad b(\beta) = -\beta^{-\alpha}, \qquad c(\beta) = -n \ln \beta, \qquad \theta = \beta^{-\alpha}, \quad j = 2n/\alpha,$$

and the joint density is given by

$$f(\mathbf{x}, \alpha, \beta) = \frac{\alpha^n}{(2\beta)^n \Gamma^n(1/\alpha)} \exp\left(-\frac{1}{\beta^{\alpha}} \sum_{i=1}^{n} |x_i|^{\alpha}\right), \qquad x_i \in \mathbb{R}, \quad i = 1, 2, \ldots, n,$$

where $-2b(\beta)a(\mathbf{X}) = 2 \sum_{i=1}^{n} |X_i/\beta|^{\alpha} \sim \Gamma(n/\alpha, 2)$.

Note that in a special case, $GL(2, \sqrt{2}\sigma) \sim N(0, \sigma^2)$ and $GL(1, \sigma)$ has Laplace distribution.

(iii) *Generalized gamma distribution*: $GG(p, \alpha, \lambda)$, in which p and α are known and $p\alpha > 0$,

$$a(\mathbf{x}) = \sum_{i=1}^{n} x_i^{\alpha}, \qquad b(\lambda) = -\lambda, \quad c(\lambda) = \frac{p}{\alpha} \ln \lambda, \qquad \theta = \lambda, \quad j = 2np/\alpha,$$

and the joint density is given by

$$f(\mathbf{x}, p, \alpha, \lambda) = \frac{|\alpha|^n \prod_{i=1}^{n} x_i^{p-1}}{\Gamma^n(p/\alpha)} \lambda^{np/\alpha} \exp\left(-\lambda \sum_{i=1}^{n} x_i^{\alpha}\right), \qquad x_i > 0, \quad i = 1, 2, \ldots, n,$$

where $-2b(\lambda)a(\mathbf{X}) = 2\lambda \sum_{i=1}^{n} X_i^{\alpha} \sim \Gamma(np/\alpha, 2)$.

Note that this distribution contains Maxwell, Weibull, Rayleigh and others in a special case.

For the above distributions which belong to the family of transformed chi-square distributions and have been applied by Sanjari Farsipour [15] in Section 3, the following theorem, which is the extended version of Theorem 3.1, gives an admissible estimator of parameter θ^r, in which $r \neq 0$ is an integer and $\theta \in (\theta_0, \infty)$ or $(0, \theta_0)$, under entropy loss function (1.1).

Theorem 4.2. *Let* $X = (X_1, \ldots, X_n)'$ *with joint density function* (4.1), *satisfies condition* (4.2). *An admissible estimator of* θ^r, $r < \frac{j}{4}$, *n the truncated parameter space* $\theta \in (\theta_0, \infty)$ *or* $\theta \in (0, \theta_0)$, *under entropy loss function* (1.1) *and the improper prior* (3.2), *is given by*

$$\hat{\delta}(\mathbf{X}) = \begin{cases} \dfrac{\gamma\left(j/2 - r,\ \theta_0\left(k + a\left(\mathbf{X}\right)\right)\right)}{\gamma\left(j/2 - 2r,\ \theta_0\left(k + a\left(\mathbf{X}\right)\right)\right)} \left(a\left(\mathbf{X}\right) + k\right)^{-r}, & 0 < \theta < \theta_0, \\ \dfrac{\Gamma\left(j/2 - r,\ \theta_0\left(k + a\left(\mathbf{X}\right)\right)\right)}{\Gamma\left(j/2 - 2r,\ \theta_0\left(k + a\left(\mathbf{X}\right)\right)\right)} \left(a\left(\mathbf{X}\right) + k\right)^{-r}, & \theta_0 < \theta < \infty. \end{cases} \tag{4.4}$$

Proof. The proof is quite similar to the proof of Theorem 3.1.

Example **4.1.** With a random sample of size n from $N(0, \sigma^2)$ distribution, the joint density is of the form (4.1), with

$$a(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \qquad b(\sigma^2) = -\frac{1}{\sigma^2}, \qquad c(\sigma^2) = -\frac{n}{2}\ln\sigma^2, \qquad \theta = \sigma^{-2}, \quad j = n.$$

and

$$-2b(\sigma^2)a(\mathbf{X}) = \frac{1}{\sigma^2}\sum_{i=1}^{m} X_i^2 \sim \Gamma\left(\frac{n}{2}, 2\right).$$

So, an admissible estimator of σ^{-2r}, $r < \frac{n}{4}$, is given by

$$\hat{\delta}(\mathbf{X}) = \begin{cases} \dfrac{\gamma\left(\frac{n}{2} - r,\ \sigma_0^2\left(k + \frac{n}{2}\bar{X}^2\right)\right)}{\gamma\left(\frac{n}{2} - 2r,\ \sigma_0^2\left(k + \frac{m}{2}\bar{X}^2\right)\right)} \left(\frac{n}{2}\bar{X}^2 + k\right)^{-r}, & \sigma_0^2 < \sigma^2 < \infty, \\ \dfrac{\Gamma\left(\frac{n}{2} - r,\ \sigma_0^2\left(k + \frac{n}{2}\bar{X}^2\right)\right)}{\Gamma\left(\frac{n}{2} - 2r,\ \sigma_0^2\left(k + \frac{n}{2}\bar{X}^2\right)\right)} \left(\frac{n}{2}\bar{X}^2 + k\right)^{-r}, & 0 < \sigma^2 < \sigma_0^2. \end{cases}$$

By choosing $r = -1$ an admissible estimator of σ^2 for the untrancated case $0 < \sigma^2 < \infty$, under entropy loss function (1.1) and the improper prior (3.2), has the form

$$\hat{\delta}(\mathbf{X}) = \frac{\Gamma(n/2 + 1)}{\Gamma(n/2 + 2)}\left(\frac{1}{2}\sum_{i=1}^{n} X_i^2 + k\right).$$

Putting $k = 0$ gives an admissible estimator

$$\hat{\delta}(\mathbf{X}) = \frac{\Gamma(n/2 + 1)}{\Gamma(n/2 + 2)}\left(\frac{1}{2}\sum_{i=1}^{n} X_i^2\right),$$

for σ^2 under the noninformative improper prior $\pi(\sigma^2) = 1,\ \sigma^2 > 0$.

Example **4.2.** Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from $BurrXII(c, \lambda)$-distribution with known c and unknown λ. The joint density function is

$$f(\mathbf{x}, c, \lambda) = \frac{c^n \lambda^n \left(\prod_{i=1}^{n} x_i\right)^{c-1}}{\prod_{i=1}^{n}(1 + x_i^c)} \prod_{i=1}^{n}(1 + x_i^c)^{-\lambda},$$

which is of form (4.1) with

$$a(\mathbf{x}) = \sum_{i=1}^{n} \ln(1 + x_i^c), \qquad b(\lambda) = -\lambda, \quad c(\lambda) = n \ln \lambda, \qquad \theta = \lambda, \quad j = 2n.$$

and

$$-2b(\lambda)a(\mathbf{X}) = 2\lambda \sum_{i=1}^{n} \ln(1 + X_i^c) \sim \Gamma\left(n, 2\right).$$

Therefore, an admissible estimator of λ^r, $r < \dfrac{n}{2}$, is given by

$$\hat{\delta}\left(\mathbf{X}\right) = \begin{cases} \dfrac{\gamma\left(n-r,\ \lambda_0\left(k + \sum_{i=1}^{n} \ln(1+X_i^c)\right)\right)}{\gamma\left(n-2r,\ \lambda_0\left(k + \sum_{i=1}^{n} \ln(1+X_i^c)\right)\right)} \left(\sum_{i=1}^{n} \ln(1+X_i^c) + k\right)^{-r}, & \\ & \lambda_0 < \lambda < \infty, \\ \dfrac{\Gamma\left(n-r,\ \lambda_0\left(k + \sum_{i=1}^{n} \ln(1+X_i^c)\right)\right)}{\Gamma\left(n-2r,\ \lambda_0\left(k + \sum_{i=1}^{n} \ln(1+X_i^c)\right)\right)} \left(\sum_{i=1}^{n} \ln(1+X_i^c) + k\right)^{-r}, & 0 < \lambda < \lambda_0. \end{cases}$$

For the untruncated case $0 < \lambda < \infty$, an admissible estimator of λ^r under the entropy loss function (1.1) and the noninformative improper prior $\pi(\lambda) = 1$, $\lambda > 0$, is given by

$$\hat{\delta}\left(\mathbf{X}\right) = \frac{\Gamma\left(n-r\right)}{\Gamma\left(n-2r\right)} \left(\sum_{i=1}^{n} \ln(1+X_i^c)\right)^{-r}.$$

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