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D-HOMOTHETIC DEFORMATION OF NORMAL ALMOST CONTACT METRIC MANIFOLDS

D-ГОМОТЕТИЧНА ДЕФОРМАЦІЯ НОРМАЛЬНИХ МАЙЖЕ КОНТАКТНИХ МНОГОВИДІВ

The object of the present paper is to study a transformation called the D-homothetic deformation of normal almost contact metric manifolds. In particular, it is shown that, in a $(2n+1)$-dimensional normal almost contact metric manifold, the Ricci operator Q commutes with the structure tensor ϕ under certain conditions, and the operator $Q\phi - \phi Q$ is invariant under a D-homothetic deformation. We also discuss the invariance of η-Einstein manifolds, ϕ-sectional curvature, and the local ϕ-Ricci symmetry under a D-homothetic deformation. Finally, we prove the existence of such manifolds by a concrete example.

Метою цієї статті є вивчення перетворення, що називається D-гомотетичною деформацією нормальних майже контактних многовидів. Зокрема, показано, що у $(2n+1)$-вимірному нормальному майже контактному многовиді оператор Річчі Q комутує за певних умов із структурним тензором ϕ, а оператор $Q\phi - \phi Q$ є інваріантним щодо D-гомотетичної деформації. Також розглянуто питання про інваріантність η-ейнштейнівських многовидів, ϕ-секційну кривину та локальну ϕ-симетрію Річчі при D-гомотетичній деформації. Існування таких многовидів доведено на конкретному прикладі.

1. Introduction. Let M be an almost contact metric manifold and (ϕ, ξ, η) its almost contact structure. This means, M is an odd-dimensional differentiable manifold and ϕ, ξ, η are tensor fields on M of types $(1,1)$, $(1,0)$ and $(0,1)$ respectively, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{1.1}$$

Let $\mathbb{R}$ be the real line and t a coordinate on $\mathbb{R}$. Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\phi X - \lambda\xi, \eta(X)\frac{d}{dt}\right), \tag{1.2}$$

where the pair $\left(X, \lambda\frac{d}{dt}\right)$ denotes a tangent vector on $M \times \mathbb{R}$, X and $\lambda\frac{d}{dt}$ being tangent to M and $\mathbb{R}$ respectively.

M and (ϕ, ξ, η) are said to be normal if the structure J is integrable [1, 2]. The necessary and sufficient condition for (ϕ, ξ, η) to be normal is

$$[\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{1.3}$$

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \tag{1.4}$$

for any $X, Y \in \chi(M)$; $\chi(M)$ being the Lie algebra of vector fields on M.

We say that the contact form η has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$ and has rank $r = 2s+1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. We also say r is rank of the structure (ϕ, ξ, η).

A Riemannian metric g on M satisfying the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.5}$$

for any $X, Y \in \chi(M)$, is said to be compatible with the structure (ϕ, ξ, η). If g is such a metric, then the quadruple (ϕ, ξ, η, g) is called an almost contact metric structure on M and M is an almost contact metric manifold. On such a manifold we also have

$$\eta(X) = g(X, \xi) \tag{1.6}$$

for any $X \in \chi(M)$ and we can always define the 2-form Φ by

$$\Phi(Y, Z) = g(Y, \phi Z), \tag{1.7}$$

where $Y, Z \in \chi(M)$.

A normal almost contact metric structure (ϕ, ξ, η, g) satisfying additionally the condition $d\eta = \Phi$ is called Sasakian. Of course, any such structure on M has rank 3. Also a normal almost contact metric structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian [3].

In the paper [8], Olszak studied the curvature properties of normal almost contact manifold of dimension three with several examples. Also in [4], U. C. De and A. K. Mondal studied three dimensional normal almost contact metric manifolds satisfying certain curvature conditions.

An almost contact metric manifold is said to be η-Einstein if its Ricci tensor S is of the form

$$S = \lambda g + \mu \eta \otimes \eta \tag{1.8}$$

where λ and μ are smooth functions on the manifold.

The notion of locally ϕ-symmetry first introduced by T. Takahashi [9] on a Sasakian manifold. Again in a recent paper [5] U. C. De and Avijit Sarkar introduced the notion of locally ϕ-Ricci symmetric Sasakian manifolds.

A three dimensional normal almost contact metric manifold is said to be locally ϕ-Ricci symmetric if

$$\phi^2(\nabla_X Q)(Y) = 0,$$

where Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and X, Y are orthogonal to ξ.

Let M (ϕ, ξ, η, g) be an almost contact metric manifold with dim $M = m = 2n + 1$. The equation $\eta = 0$ defines an $(m-1)$-dimensional distribution D on M [12]. By an $(m-1)$-homothetic deformation or D-homothetic deformation [10] we mean a change of structure tensors of the form

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant. If $M(\phi, \xi, \eta, g)$ is an almost contact metric structure with contact form η, then $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost contact metric structure [10]. Denoting by W^i_{jk} the difference $\bar{\Gamma}^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols we have in an almost contact metric manifold [10]

$$W(X, Y) = (1-a)[\eta(Y)\phi X + \eta(X)\phi Y] + \frac{1}{2}\left(1 - \frac{1}{a}\right)[(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X)]\xi \tag{1.9}$$

for all $X, Y \in \chi(M)$. If R and $\bar{R}$ denote respectively the curvature tensor of the manifold $M(\phi, \xi, \eta, g)$ and $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, then we have [10]

$$\bar{R}(X,Y)Z = R(X,Y)Z + (\nabla_X W)(Z,Y) - (\nabla_Y W)(Z,X) + \\ + W(W(Z,Y),X) - W(W(Z,X),Y) \tag{1.10}$$

for all $X, Y, Z \in \chi(M)$.

In [10, 13] the authors used D-homothetic deformation on a Sasakian and K-contact structures to get results on the first Betti number, second Betti number and harmonic forms. Hence the D-homothetic deformation can be used to get the results on the first Betti number, second Betti number and harmonic forms of the normal almost contact structure. A plane section in the tangent space $T_p(M)$ is called a ϕ-section if there exists a unit vector X in $T_p(M)$ orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature

$$K(X, \phi X) = g(R(X, \phi X)X, \phi X)$$

is called a ϕ-sectional curvature. A contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be of constant ϕ-sectional curvature if at any point $p \in M$, the sectional curvature $K(X, \phi X)$ is independent of the choice of non-zero $X \in D_p$, where D denotes the contact distribution of the contact metric manifold defined by $\eta = 0$.

The model spaces of contact metric structure are complete and simply connected Sasakian manifolds of constant ϕ-sectional curvature H. These Sasakian manifolds admit the maximal dimensional automorphism [14]. The Riemann curvature tensor R of Sasakian manifold of constant ϕ-sectional curvature is determined by Ogiue [7]. The geometry of contact Riemannian manifold of constant ϕ-sectional curvature is obtained by Tanno [15]. If the ϕ-sectional curvature H is constant on a K-contact Riemannian manifold $M(\phi, \xi, \eta, g)$, then H can be deformed by a D-homothetic deformation of the structure tensors [11]. If $H > -3$, then choosing a constant $\theta = \dfrac{H+3}{4}$, we get a K-contact Riemannian manifold $M\left(\phi, \dfrac{1}{\theta}\xi, \theta\eta, \theta g + (\theta^2 - \theta)\eta \otimes \eta\right)$ of constant ϕ-sectional curvature [11]. Hence Tanno posed a natural question that does there exist contact metric manifolds of constant ϕ-sectional curvature which are not Sasakian [11]. Since the normal almost contact metric manifold contains both the Sasakian and non-Sasakian structures, the existance of a non-Sasakian manifold of both constant and non-constant ϕ-sectional curvature is ensured in our paper, which gives rise to the answer of the question of Tanno [11] as affirmative.

In a Sasakian manifold, the Ricci operator Q commutes with the structure tensor ϕ, that is, $Q\phi = \phi Q$. But in $(2n+1)$-dimensional normal almost contact metric manifold $Q\phi \neq \phi Q$, in general.

The present paper is organized as follows: After preliminaries in Section 3, we prove some important lemmas. In Section 4, we study the properties of the expression $Q\phi - \phi Q$ in $(2n+1)$-dimensional normal almost contact metric manifolds and prove that $Q\phi = \phi Q$ in these manifolds, provided α, β are constants. Beside this, in this section we also prove that the expression $Q\phi - \phi Q$ of these manifolds is invariant under a D-homothetic deformation, provided α is constant. Section 5 deals with the study of $(2n+1)$-dimensional η-Einstein normal almost contact metric manifolds and prove that these manifolds are invariant under a D-homothetic deformation, provided $\alpha = 0$. Section 6 is devoted to study ϕ-sectional curvature tensor in a $(2n+1)$-dimensional normal almost contact metric manifold and we show that there exists a $(2n+1)$-dimensional normal almost contact metric manifold (non-Sasakian) with non-zero and non-constant ϕ-sectional curvature. Section 7 deals with locally ϕ-symmetric three dimensional normal almost contact metric manifold and we prove this manifold is also invariant under a D-homothetic deformation, provided $\alpha = \mathrm{constant}$. Finally in Section 8, we set an example of a three dimensional normal almost contact metric manifold which verifies some theorems of Section 6.

2. Preliminaries. For a normal almost contact metric structure (ϕ, ξ, η, g) on M, we have [8]

$$(\nabla_X \phi)(Y) = g(\phi \nabla_X \xi, Y) - \eta(Y) \phi \nabla_X \xi, \tag{2.1}$$

$$\nabla_X \xi = \alpha[X - \eta(X)\xi] - \beta \phi X, \tag{2.2}$$

where $2\alpha = \operatorname{div} \xi$ and $2\beta = \operatorname{tr}(\phi \nabla \xi)$, $\operatorname{div} \xi$ is the divergent of ξ defined by $\operatorname{div} \xi = \operatorname{trace}\{X \longrightarrow \longrightarrow \nabla_X \xi\}$ and $\operatorname{tr}(\phi \nabla \xi) = \operatorname{trace}\{X \longrightarrow \phi \nabla_X \xi\}$. Using (2.2) in (2.1), we get

$$(\nabla_X \phi)(Y) = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X] + \beta[g(X, Y)\xi - \eta(Y)X]. \tag{2.3}$$

Also in this manifold the following relation holds:

$$R(X,Y)\xi = [Y\alpha + (\alpha^2 - \beta^2)\eta(Y)]\phi^2 X - [X\alpha + (\alpha^2 - \beta^2)\eta(X)]\phi^2 Y +$$

$$+[Y\beta + 2\alpha\beta\eta(Y)]\phi X - [X\beta + 2\alpha\beta\eta(X)]\phi Y, \tag{2.4}$$

$$S(X,\xi) = -X\alpha - (\phi X)\beta - [\xi\alpha + 2(\alpha^2 - \beta^2)]\eta(X), \tag{2.5}$$

$$\xi\beta + 2\alpha\beta = 0, \tag{2.6}$$

where R denotes the curvature tensor and S is the Ricci tensor.

$$(\nabla_X \eta)(Y) = \alpha g(\phi X, \phi Y) - \beta g(\phi X, Y). \tag{2.7}$$

On the other hand, the curvature tensor in a three dimensional Riemannian manifold always satisfies

$$R(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY -$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \tag{2.8}$$

where r is the scalar curvature of the manifold.

By (2.4), (2.5) and (2.8) we can derive

$$S(Y,Z) = \left(\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2\right) g(\phi Y, \phi Z) -$$

$$-\eta(Y)(Z\alpha + (\phi Z)\beta) - \eta(Z)(Y\alpha + (\phi Y)\beta) - 2(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \tag{2.9}$$

From (2.6) it follows that if $\alpha, \beta = \text{constant}$, then the manifold is either β-Sasakian or α-Kenmotsu [6] or cosymplectic [1]. Also we have a 3-dimensional normal almost contact metric manifold is quasi-Sasakian if and only if $\alpha = 0$ [8].

3. Some lemmas. In this section we shall state and prove some lemmas which will be needed to prove the main results.

Lemma 3.1. *In a normal almost contact metric manifold M the following relation holds*:

$$g(R(X,Y)\phi Z,W) + g(R(X,Y)Z,\phi W) = (X\alpha)[g(\phi Y,Z)\eta(W)-$$

$$-g(\phi Y,W)\eta(Z)] + (X\beta)[g(Y,Z)\eta(W)-$$

$$-g(Y,W)\eta(Z)] + (Y\alpha)[g(\phi X,W)\eta(Z)-$$

$$-g(\phi X,Z)\eta(W)] + (Y\beta)[g(X,W)\eta(Z) - g(X,Z)\eta(W)]+$$

$$+(\alpha^2 - \beta^2)[g(\phi X,W)g(Y,Z) + g(\phi Y,Z)g(X,W)-$$

$$-g(\phi Y,W)g(X,Z) - g(\phi X,Z)g(Y,W)] + 2\alpha\beta[g(\phi Y,W)g(\phi X,Z)-$$

$$-g(\phi X,W)g(\phi Y,Z) + g(X,W)g(Y,Z) - g(Y,W)g(X,Z)]. \tag{3.1}$$

Proof. Differentiating (1.7) covariantly with respect to X and using (2.3) and (2.7) we obtain

$$(\nabla_X\Phi)(Y,Z) = \alpha[g(\phi X,Z)\eta(Y) - g(\phi X,Y)\eta(Z)]+$$

$$+\beta[g(X,Z)\eta(Y) - g(X,Y)\eta(Z)]. \tag{3.2}$$

Again differentiating (3.2) covariantly and using (2.2), (2.3) and (2.7) yields

$$(\nabla_X\nabla_Y\Phi)(Z,W) = (X\alpha)[g(\phi Y,W)\eta(Z)-$$

$$-g(\phi Y,Z)\eta(W)] + (X\beta)[g(Y,W)\eta(Z)-$$

$$-g(Y,Z)\eta(W)] + \alpha^2[g(\phi Y,W)g(\phi X,\phi Z)-$$

$$-g(\phi Y,Z)g(\phi X,\phi W) - g(\phi X,W)\eta(Y)\eta(Z)+$$

$$+g(\phi X,Z)\eta(Y)\eta(W)] + \beta^2[g(\phi X,W)g(Y,Z)-$$

$$-g(\phi X,Z)g(Y,W)] + \alpha\beta[g(\phi X,W)g(\phi Y,Z)-$$

$$-g(\phi X,Z)g(\phi Y,W) + g(Y,W)g(\phi X,\phi Z)-$$

$$-g(Y,Z)g(\phi X,\phi W) + g(X,Z)\eta(Y)\eta(W)-$$

$$-g(X,W)\eta(Y)\eta(Z)] + \alpha[g(\phi\nabla_X Y,W)\eta(Z)-$$

$$-g(\phi\nabla_X Y, Z)\eta(W)] + \beta[g(\nabla_X Y, W)\eta(Z) - g(\nabla_X Y, Z)\eta(W)]. \tag{3.3}$$

Using (3.2) and (3.3) we obtain

$$(\nabla_X \nabla_Y \Phi)(Z, W) - (\nabla_Y \nabla_X \Phi)(Z, W) - (\nabla_{[X,Y]}\Phi)(Z, W) =$$

$$= (X\alpha)[g(\phi Y, W)\eta(Z) - g(\phi Y, Z)\eta(W)]+$$

$$+(X\beta[g(Y, W)\eta(Z) - g(Y, Z)\eta(W)]-$$

$$-(Y\alpha)[g(\phi X, W)\eta(Z) - g(\phi X, Z)\eta(W)]-$$

$$-(Y\beta)[g(X, W)\eta(Z) - g(X, Z)\eta(W)]+$$

$$+(\alpha^2 - \beta^2)[g(\phi Y, W)g(X, Z) - g(\phi X, W)g(Y, Z)-$$

$$-g(X, W)g(\phi Y, Z) + g(Y, W)g(\phi X, Z)] + 2\alpha\beta[g(\phi X, W)g(\phi Y, Z)-$$

$$-g(\phi X, Z)g(\phi Y, W) + g(X, Z)g(Y, W) - g(X, W)g(Y, Z)]. \tag{3.4}$$

Then using (3.4) and by Ricci identity we easily obtain (3.1).

Lemma 3.2. *Let $M(\phi, \xi, \eta, g)$ be a normal almost contact metric manifold of dimension $(2n+$ $+\,1)$. Then for any X, Y, Z and W on M, the following relation holds*:

$$g(R(X, Y)\phi Z, \phi W) = g(R(X, Y)Z, W) + (X\alpha)[g(Y, Z)\eta(W)-$$

$$-g(Y, W)\eta(Z)] - (X\beta)[g(\phi Y, Z)\eta(W)-$$

$$-g(\phi Y, W)\eta(Z)] + (Y\alpha)[g(X, W)\eta(Z)-$$

$$-g(X, Z)\eta(W)] + (Y\beta)[g(\phi X, Z)\eta(W)-$$

$$-g(\phi X, W)\eta(Z)] + (\alpha^2 - \beta^2)[g(X, W)g(Y, Z)-$$

$$-g(X, Z)g(Y, W) + g(\phi X, Z)g(\phi Y, W)-$$

$$-g(\phi X, W)g(\phi Y, Z)] + 2\alpha\beta[g(Y, W)g(\phi X, Z)-$$

$$-g(X, W)g(\phi Y, Z) + g(X, Z)g(\phi Y, W) - g(Y, Z)g(\phi X, W)]. \tag{3.5}$$

Proof. Replacing W by ϕW in (3.1) and using (1.1), (1.6) and (2.4) we easily obtain (3.5).

Lemma 3.3. *Let $M(\phi, \xi, \eta, g)$ be a normal almost contact metric manifold of dimension $(2n+$ $+\,1)$. Then for any X, Y, Z and W on M, the following relation holds*:

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X, Y)Z, W) + (\alpha^2 - \beta^2)[g(Y, Z)\eta(X)\eta(W)-$$

$$-g(X, Z)\eta(Y)\eta(W) + g(X, W)\eta(Y)\eta(Z) - g(Y, W)\eta(X)\eta(Z)]+$$

$$+2\alpha\beta[2g(\phi X,W)g(Y,Z)-2g(\phi Y,W)g(X,Z)+$$

$$+2g(\phi Y,Z)g(X,W)-2g(\phi X,Z)g(Y,W)+$$

$$+g(\phi Y,W)\eta(X)\eta(Z)-g(\phi X,W)\eta(Y)\eta(Z)+$$

$$+g(\phi X,Z)\eta(Y)\eta(W)-g(\phi Y,Z)\eta(X)\eta(W)]+$$

$$+(Z\alpha)[g(X,W)\eta(Y)-g(Y,W)\eta(X)]-$$

$$-(Z\beta)[g(\phi Y,W)\eta(X)-g(\phi X,W)\eta(Y)]+$$

$$+(W\alpha)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]+$$

$$+(W\beta)[g(\phi Y,Z)\eta(X)-g(\phi X,Z)\eta(Y)]+$$

$$+(\phi X\alpha)[g(\phi Y,Z)\eta(W)-g(\phi Y,W)\eta(Z)]-$$

$$-(\phi X\beta)[g(Y,W)\eta(Z)-g(Y,Z)\eta(W)]+$$

$$+(\phi Y\alpha)[g(\phi X,W)\eta(Z)-g(\phi X,Z)\eta(W)]+$$

$$+(\phi Y\beta)[g(X,W)\eta(Z)-g(X,Z)\eta(W)]. \tag{3.6}$$

Proof. Putting ϕX and ϕY instead of X and Y respectively in (3.5) and using (1.1), (1.6) and (3.5) we easily obtain (3.6).

Proposition 3.1. *In a $(2n+1)$-dimensional η-Einstein normal almost contact metric manifold $M(\phi,\xi,\eta,g)$, the Ricci tensor is expressed as*

$$S(X,Y)=\left[\frac{r}{2n}+\xi\alpha+(\alpha^2-\beta^2)\right]g(X,Y)-$$

$$-\left[\frac{r}{2n}+(2n+1)\xi\alpha+(2n+1)(\alpha^2-\beta^2)\right]\eta(X)\eta(Y). \tag{3.7}$$

Proof. From (1.8) we have by contraction

$$r=(2n+1)\lambda+\mu, \tag{3.8}$$

where r is the scalar curvature of the manifold. Again putting $X=\xi$ in (2.5), we obtain

$$\lambda+\mu=-2n\xi\alpha-2n(\alpha^2-\beta^2). \tag{3.9}$$

Solving above two equations we get

$$\lambda=\frac{r}{2n}+\xi\alpha+(\alpha^2-\beta^2), \tag{3.10}$$

and

$$\mu = -\frac{r}{2n} - (2n+1)\xi\alpha - (2n+1)(\alpha^2 - \beta^2). \tag{3.11}$$

Putting the values of λ and μ in (1.8) we get (3.7).

Proposition 3.1 is proved.

4. Properties of the expression $Q\phi - \phi Q$. In this section we investigate the properties of the expression $Q\phi - \phi Q$ in a $(2n+1)$-dimensional normal almost contact metric manifold M.

Let $\{e_i, \phi e_i, \xi\}$, $i = 1, 2, \ldots, n$, be a local ϕ-basis at any point of the manifold. Then putting $Y = Z = e_i$ in (3.6) and taking summation over $i = 1$ to n, we obtain by virtue of $\eta(e_i) = 0$,

$$-\sum_{i=1}^{n} \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{n} R(X, e_i)e_i + n(\alpha^2 - \beta^2)\eta(X)\xi +$$

$$+[(n-1)\operatorname{grad}\alpha - (\phi \operatorname{grad}\beta)]\eta(X) +$$

$$+4(n-2)\alpha\beta(\phi X) + (X\alpha)\xi + (n-1)(\phi X\beta)\xi. \tag{4.1}$$

Again putting $Y = Z = \phi e_i$ in (3.6) and taking summation over $i = 1$ to n then using (1.1) and $\eta(e_i) = 0$, we obtain

$$-\sum_{i=1}^{n} \phi R(\phi X, e_i)e_i = \sum_{i=1}^{n} R(X, \phi e_i)\phi e_i +$$

$$+n(\alpha^2 - \beta^2)\eta(X)\xi + [(n-1)\operatorname{grad}\alpha - (\phi \operatorname{grad}\beta)]\eta(X) +$$

$$+4(n-2)\alpha\beta(\phi X) + (X\alpha)\xi + (n-1)(\phi X\beta)\xi. \tag{4.2}$$

Adding (4.1) and (4.2) and using the definition of Ricci operator, we obtain

$$-\phi Q(\phi X) + \phi R(\phi X, \xi)\xi = QX - R(X, \xi)\xi +$$

$$+2n(\alpha^2 - \beta^2)\eta(X)\xi + 8(n-2)\alpha\beta(\phi X) +$$

$$+2[(n-1)\operatorname{grad}\alpha - \phi(\operatorname{grad}\beta)]\eta(X) + 2(X\alpha)\xi + 2(n-1)(\phi X\beta)\xi. \tag{4.3}$$

From (2.4) by virtue of (2.6), it follows that

$$R(\phi X, \xi)\xi = -[\xi\alpha + (\alpha^2 - \beta^2)](\phi X). \tag{4.4}$$

In view of (2.4), (2.6) and (4.4), the relation (4.3) takes the form

$$-\phi Q(\phi X) = QX + 2n(\alpha^2 - \beta^2)\eta(X)\xi + 8(n-2)\alpha\beta(\phi X) +$$

$$+2[(n-1)\operatorname{grad}\alpha - \phi(\operatorname{grad}\beta)]\eta(X) + 2(X\alpha)\xi + 2(n-1)(\phi X\beta)\xi. \tag{4.5}$$

Operating ϕ on both sides of (4.5) and using (1.1) we get

$$Q\phi X - \phi QX = S(\phi X, \xi)\xi + 8(n-2)\alpha\beta(\phi^2 X) +$$

$$+2[(n-1)\phi(\operatorname{grad}\alpha)-\phi^2(\operatorname{grad}\beta)]\eta(X). \tag{4.6}$$

From (2.5) we have

$$S(\phi X,\xi)=-(\phi X)\alpha-(\phi^2X)\beta. \tag{4.7}$$

By virtue of (4.7) and (2.6), (4.6) reduces to

$$[Q\phi-\phi Q]X=(X\beta)\xi-(n-2)(4\xi\beta)X-(\phi X\alpha)\xi+$$

$$+(4n-7)(\xi\beta)\eta(X)\xi+2[(n-1)\phi(\operatorname{grad}\alpha)-\phi^2(\operatorname{grad}\beta)]\eta(X). \tag{4.8}$$

Hence we state the following theorem.

Theorem 4.1. *In a $(2n+1)$-dimensional normal almost contact metric manifold $Q\phi=\phi Q$, provided α, β are constants.*

By virtue of (2.7), the relation (1.10) reduces to

$$W(X,Y)=(1-a)[\eta(Y)\phi X+\eta(X)\phi Y]+\left(1-\frac{1}{a}\right)\alpha[g(X,Y)-\eta(X)\eta(Y)]\xi. \tag{4.9}$$

In view of (2.2), (2.3) and (2.7), the relation (4.9) yields

$$(\nabla_XW)(Y,Z)=(1-a)[\alpha\{g(\phi X,Y)\eta(Z)\xi+$$

$$+g(\phi X,Z)\eta(Y)\xi+g(X,Z)\phi Y+g(X,Y)\phi Z-$$

$$-\eta(X)\eta(Y)\phi Z-\eta(X)\eta(Z)\phi Y-2\eta(Y)\eta(Z)\phi X\}+\beta\{g(X,Y)\eta(Z)\xi+$$

$$+g(X,Z)\eta(Y)\xi-g(\phi X,Z)\phi Y-g(\phi X,Y)\phi Z-2\eta(Y)\eta(Z)X\}]+$$

$$+\frac{a-1}{a}(X\alpha)[g(Y,Z)-\eta(Y)\eta(Z)]\xi-\frac{a-1}{a}\alpha[\alpha\{g(X,Y)\eta(Z)\xi+$$

$$+g(X,Z)\eta(Y)\xi+g(Y,Z)\eta(X)\xi-g(Y,Z)X+\eta(Y)\eta(Z)X-$$

$$-3\eta(X)\eta(Y)\eta(Z)\xi\}+\beta\{g(Y,Z)\phi X-g(\phi X,Z)\eta(Y)\xi-$$

$$-g(\phi X,Y)\eta(Z)\xi-\eta(Y)\eta(Z)\phi X\}]. \tag{4.10}$$

Using (4.9) and (4.10) into (1.11), we obtain by virtue of (2.4) and (2.7) that

$$\bar R(X,Y)Z=R(X,Y)Z+(1-a)[\alpha\{g(\phi X,Z)\eta(Y)\xi-$$

$$-g(\phi Y,Z)\eta(X)\xi+2g(\phi X,Y)\eta(Z)\xi+g(X,Z)\phi Y-g(Y,Z)\phi X+$$

$$+\eta(X)\eta(Z)\phi Y-\eta(Y)\eta(Z)\phi X\}+\beta\{g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi-$$

$$-2g(\phi X,Y)\phi Z-g(\phi X,Z)\phi Y+g(\phi Y,Z)\phi X-2\eta(Y)\eta(Z)X+$$

$$+2\eta(X)\eta(Z)Y\}] + \frac{a-1}{a}(X\alpha)[g(Y,Z)-$$

$$-\eta(Y)\eta(Z)]\xi - \frac{a-1}{a}(Y\alpha)[g(X,Z) - \eta(X)\eta(Z)]\xi + \frac{a-1}{a}\alpha[\alpha\{g(Y,Z)X-$$

$$-g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}+$$

$$+\beta\{g(X,Z)\phi Y - g(Y,Z)\phi X + 2g(\phi X,Y)\eta(Z)\xi + g(\phi X,Z)\eta(Y)\xi-$$

$$-g(\phi Y,Z)\eta(X)\xi + \eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y\}]+$$

$$+(1-a)^2[\eta(X)\eta(Z)\phi^2 Y - \eta(Y)\eta(Z)\phi^2 X]-$$

$$-\frac{(1-a)^2}{a}[\alpha\{g(\phi Z,X)\eta(Y)\xi - 2g(\phi X,Y)\eta(Z)\xi+$$

$$+g(Y,Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X + g(\phi Y,Z)\eta(X)\xi\}]. \tag{4.11}$$

Putting $Y = Z = \xi$ in (4.11) and using (1.1) we obtain

$$\bar{R}(X,\xi)\xi = R(X,\xi)\xi + 2(1-a)[\beta(\phi^2 X) - \alpha(\phi X)] - (1-a)^2\phi^2 X. \tag{4.12}$$

Let $\{e_i, \phi e_i, \xi\}$, $i = 1, 2, \ldots, n$, be a local ϕ-basis at any point of the manifold. Then putting $Y = Z = e_i$ in (4.11) and taking summation over $i = 1$ to n we obtain by virtue of $\eta(e_i) = 0$,

$$\sum_{i=1}^{n} \bar{R}(X,e_i)e_i = \sum_{i=1}^{n} R(X,e_i)e_i-$$

$$-(1-a)[\alpha(n-1)(\phi X) + \beta\{n\eta(X)\xi - 3X\}] + \frac{a-1}{a}(n-1)(X\alpha)\xi+$$

$$+\frac{a-1}{a}\alpha^2(n-1)X - \frac{a-1}{a}\alpha\beta(n-1)\phi X - \frac{(1-a)^2}{a}\alpha(n-1)\phi X. \tag{4.13}$$

Again, putting $Y = Z = \phi e_i$ in (4.11) and taking summation over $i = 1$ to n then using (1.1) and $\eta(e_i) = 0$, we obtain

$$\sum_{i=1}^{n} \bar{R}(X,\phi e_i)\phi e_i = \sum_{i=1}^{n} R(X,\phi e_i)\phi e_i-$$

$$-(1-a)[\alpha(n-1)(\phi X) + \beta\{n\eta(X)\xi - 3X\}] + \frac{a-1}{a}(n-1)(X\alpha)\xi+$$

$$+\frac{a-1}{a}\alpha^2(n-1)X - \frac{a-1}{a}\alpha\beta(n-1)\phi X - \frac{(1-a)^2}{a}\alpha(n-1)\phi X. \tag{4.14}$$

Adding (4.13) and (4.14) and using the definition of Ricci operator we have

$$\bar{Q}X - \bar{R}(X,\xi)\xi = QX - R(X,\xi)\xi - 2(1-a)[\alpha\{(n-1)\phi X\}+$$

$$+\beta\{n\eta(X)\xi - 3X\}] + \frac{2(a-1)}{a}(n-1)(X\alpha)\xi + \frac{2(a-1)}{a}\alpha^2(n-1)X -$$

$$-\frac{2(a-1)}{a}\alpha\beta(n-1)\phi X - \frac{2(1-a)^2}{a}\alpha(n-1)\phi X. \tag{4.15}$$

In view of (4.12) we get from (4.15)

$$\bar{S}(X,Y) = S(X,Y) - 2(1-a)[\alpha n g(\phi X, Y) -$$

$$-\beta\{g(\phi^2 X, Y) + n\eta(X)\eta(Y) - 3g(X,Y)\}] +$$

$$+\frac{2(a-1)}{a}(n-1)[(X\alpha)\eta(Y) + \alpha^2 g(X,Y) -$$

$$-\alpha\beta g(\phi X, Y) - (a-1)\alpha g(\phi X, Y)], \tag{4.16}$$

which implies that

$$\bar{Q}X = QX - 2(1-a)[\alpha n\phi X - \beta\{\phi^2 X + n\eta(X)\xi - 3X\}] +$$

$$+\frac{2(a-1)}{a}(n-1)[(X\alpha)\xi + \alpha^2 X - \alpha\beta(\phi X) - (a-1)\alpha(\phi X)]. \tag{4.17}$$

Operating $\bar{\phi} = \phi$ on both sides of (4.17) from the left we have

$$\bar{\phi}\bar{Q}X = \phi QX - 2(1-a)[\alpha n(\phi^2 X) + 4\beta(\phi X)] +$$

$$+\frac{2(a-1)}{a}(n-1)[\alpha^2(\phi X) - \alpha\beta(\phi^2 X) - (a-1)\alpha(\phi^2 X)]. \tag{4.18}$$

Again, putting $\bar{\phi}X = \phi X$ in (4.17) we have

$$\bar{Q}\bar{\phi}X = Q\phi X - 2(1-a)[\alpha n(\phi^2 X) + 4\beta(\phi X)] +$$

$$+\frac{2(a-1)}{a}(n-1)[(\phi X\alpha)\xi + \alpha^2(\phi X) - \alpha\beta(\phi^2 X) - (a-1)\alpha(\phi^2 X)]. \tag{4.19}$$

Subtracting (4.18) and (4.19) we get

$$(\bar{\phi}\bar{Q} - \bar{Q}\bar{\phi})X = (\phi Q - Q\phi)X - \frac{2(a-1)}{a}(n-1)(\phi X\alpha)\xi. \tag{4.20}$$

Therefore we can state the following theorem.

Theorem 4.2. *Under a D-homothetic deformation, the expression $Q\phi - \phi Q$ of a $(2n+1)$-dimensional normal almost contact metric manifold is invariant, provided α is constant.*

In view of (4.20) we state the following corollary.

***Corollary* 4.1.** *Under a D-homothetic deformation, the expression $Q\phi - \phi Q$ of a 3-dimensional normal almost contact metric manifold is invariant.*

5. η-Einstein normal almost contact metric manifolds. Let $M(\phi,\xi,\eta,g)$ be a $(2n+1)$-dimensional η-Einstein normal almost contact metric manifold which reduces to $M(\bar\phi,\bar\xi,\bar\eta,\bar g)$ under a D-homothetic deformation. Then from (4.16) it follows by virtue of (3.7) that

$$\bar S(X,Y)=\bar\lambda\bar g(X,Y)+\bar\mu\bar\eta(X)\bar\eta(Y)+\frac{2(a-1)}{a^2}(n-1)(X\alpha)\bar\eta(Y)-$$

$$-\left[\frac{2(1-a)}{a}\alpha n+\frac{2(a-1)}{a^2}\alpha\beta(n-1)+\frac{2(a-1)^2}{a^2}(n-1)\alpha\right]\bar g(\bar\phi X,Y), \tag{5.1}$$

where $\bar\lambda$, $\bar\mu$ are smooth functions given by

$$\bar\lambda=\frac{1}{a}\left[\frac{r}{2n}+\xi\alpha+(\alpha^2-\beta^2)\right]-8\frac{(1-a)}{a}\beta+\frac{2(a-1)}{a^2}(n-1)\alpha^2 \tag{5.2}$$

and

$$\bar\mu=-\frac{a-1}{a}\left[\frac{r}{2n}+\xi\alpha+(\alpha^2-\beta^2)\right]-\frac{1}{a^2}\left\{\frac{r}{2n}+(2n+1)(\xi\alpha+\alpha^2-\beta^2)\right\}+$$

$$+2\beta(n+1)\frac{1-a}{a^2}-8\beta\frac{(a-1)^2}{a}-2\alpha^2(n-1)\frac{(a-1)^2}{a^2}. \tag{5.3}$$

In view of the relation (5.1) we state the following theorem.

Theorem 5.1. *Under a D-homothetic deformation, a $(2n+1)$-dimensional η-Einstein normal almost contact metric manifold is invariant, provided $\alpha=0$.*

6. ϕ-Sectional curvature of normal almost contact metric manifolds. In this section we consider the ϕ-sectional curvature on a $(2n+1)$-dimensional normal almost contact metric manifold.

From (4.11) it can be easily seen that

$$\bar K(X,\phi X)-K(X,\phi X)=\frac{a-1}{a}[3a\beta-\alpha^2] \tag{6.1}$$

and hence we state the following theorem.

Theorem 6.1. *Under a D-homothetic deformation, the ϕ-sectional curvature of a $(2n+1)$-dimensional normal almost contact metric manifold is invariant.*

If a $(2n+1)$-dimensional normal almost contact metric manifold $M(\bar\phi,\bar\xi,\bar\eta,\bar g)$ satisfies $R(X,Y)\xi=0$ for all X,Y (for example the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure with $R(X,Y)\xi=0$), then it can be easily seen that $K(X,\phi X)=0$ and hence from (6.1) it follows that

$$\bar K(X,\phi X)=\frac{a-1}{a^2}[3a\beta-\alpha^2]\neq 0$$

for $a\neq 1$ and $\alpha^2\neq 3a\beta$, where X is a unit vector field orthogonal to ξ and $K(X,\phi X)$ is the ϕ-sectional curvature. This implies that the ϕ-sectional curvature $\bar K(X,\phi X)$ is non-vanishing and non-constant for $a\neq 1$ and $\alpha^2\neq 3a\beta$. Therefore, we state the following theorem.

Theorem 6.2. *There exists $(2n+1)$-dimensional normal almost contact metric manifold* (*non-Sasakian*) *with non-zero and non-constant ϕ-sectional curvature.*

7. Locally ϕ-Ricci symmetric three dimensional normal almost contact metric manifolds. In this section we study locally ϕ-Ricci symmetry on a three dimensional normal almost contact metric manifold.

Differentiating (4.17) covariantly with respect to W and using (2.3) we obtain

$$(\nabla_W \bar{Q})(X) = (\nabla_W Q)(X) - 2(1-a)(W\alpha)\phi X -$$

$$-2(1-a)\alpha[\alpha\{g(\phi W, X)\xi - \eta(X)\phi W\} + \beta\{g(W,X)\xi - \eta(X)W\}] -$$

$$-(1-a)^2(\nabla_W \eta)(X)\xi - (1-a)^2\eta(X)\nabla_W\xi. \tag{7.1}$$

Operating ϕ^2 on both sides of (7.1) and taking X as an orthonormal vector to ξ we obtain

$$\bar{\phi}^2(\nabla_W \bar{Q})(X) = \phi^2(\nabla_W Q)(X) + 2(1-a)(W\alpha)(\phi X). \tag{7.2}$$

In view of the relation (7.2) we state the following theorem.

Theorem 7.1. *Under a D-homothetic deformation a locally ϕ-Ricci symmetry on a three dimensional normal almost contact metric manifold is invariant, provided* $\alpha = \mathrm{constant}$.

8. Example. We consider the three dimensional manifold $M = \{(x,y,z) \in R^3, z \neq 0\}$, where (x,y,z) are standard coordinate of R^3. The vector fields

$$e_1 = z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be a Riemannian metric defined by

$$g(e_1,e_3) = g(e_1,e_2) = g(e_2,e_3) = 0,$$

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using the identity of ϕ and g, we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi–Civita connection with respect to the metric g. Then we have

$$[e_1,e_3] = ye_2 - z^2e_3, \quad [e_1,e_3] = -\frac{1}{z}e_1 \quad \text{and} \quad [e_2,e_3] = -\frac{1}{z}e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) -$$

$$-g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \tag{8.1}$$

which is known as Koszul's formula. Using (8.1) we can easily calculate the following:

$$\nabla_{e_1} e_3 = -\frac{1}{z}e_1 + \frac{z^2}{2}e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2}z^2 e_3, \quad \nabla_{e_1} e_1 = \frac{1}{z}e_3,$$

$$\nabla_{e_2} e_3 = -\frac{1}{z}e_2 - \frac{1}{2}z^2 e_1, \quad \nabla_{e_2} e_2 = ye_1 + \frac{1}{z}e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2}z^2 e_3 - ye_2, \tag{8.2}$$

$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = -\frac{1}{2}z^2 e_1, \quad \nabla_{e_3} e_1 = \frac{1}{2}z^2 e_2.$$

From (8.2) it can be easily seen that (ϕ, ξ, η, g) is a normal almost contact metric manifold with $\alpha = -\frac{1}{z} \neq 0$ and $\beta = -\frac{1}{2}z^2 \neq 0$.

It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{8.3}$$

With the help of (8.3) and using (8.2) we can easily calculate

$$R(e_1,e_2)e_1 = \left(\frac{3z^4}{4} + \frac{1}{z^2} + y^2\right)e_2 + (yz^2)e_3, \quad R(e_2,e_1)e_2 = \left(\frac{3z^4}{4} + \frac{1}{z^2} + y^2\right)e_1 + \frac{y}{z}e_3,$$

$$R(e_1,e_3)e_3 = \left(\frac{z^4}{4} - \frac{2}{z^2}\right)e_1, \quad R(e_2,e_3)e_3 = \left(\frac{z^4}{4} - \frac{2}{z^2}\right)e_2,$$

$$R(e_3,e_1)e_1 = \left(\frac{z^4}{4} - \frac{2}{z^2}\right)e_3 - (yz^2)e_2, \quad R(e_3,e_2)e_2 = \left(\frac{z^4}{4} - \frac{2}{z^2}\right)e_3 - \frac{y}{z}e_1.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1,e_1) = g(R(e_1,e_2)e_2,e_1) + g(R(e_1,e_3)e_3,e_1) = -\frac{z^4}{2} - \frac{3}{z^2} - y^2.$$

Similarly we have

$$S(e_2,e_2) = -\frac{z^4}{2} - \frac{3}{z^2} - y^2 \quad \text{and} \quad S(e_3,e_3) = \frac{z^4}{2} - \frac{4}{z^2}.$$

Therefore

$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) = -\frac{z^4}{2} - \frac{10}{z^2} - 2y^2.$$

Now using (2.9) in (2.8) we get

$$g(R(X,Y)Z,W) = \left[\frac{r}{2} + \xi\alpha + (\alpha^2 - \beta^2)\right]\left[g(\phi Y, \phi Z)g(X,W) - \right.$$

$$-g(\phi X,\phi Z)g(Y,W)+g(\phi X,\phi W)g(Y,Z)-g(\phi Y,\phi W)g(X,Z)]-$$

$$-\{X\alpha+(\phi X)\beta\}[g(Y,Z)\eta(W)-g(Y,W)\eta(Z)-$$

$$-\{Y\alpha+(\phi Y)\beta\}[g(X,W)\eta(Z)-g(X,Z)\eta(W)]-$$

$$-\{W\alpha+(\phi W)\beta\}[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]-$$

$$-2(\alpha^2-\beta^2)[g(X,W)\eta(Y)\eta(Z)-g(Y,W)\eta(X)\eta(Z)+$$

$$+g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)]-$$

$$-\frac{r}{2}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)].$$

In view of the above relation we get

$$K(e_1,\phi e_1)=K(e_2,\phi e_2)=2(\beta^2-\alpha^2)-2(\xi\alpha)-\frac{r}{2}.$$

Now, in this example we have

$$K(e_1,\phi e_1)=g(R(e_1,\phi e_1)e_1,\phi e_1)=g(R(e_1,e_2)e_1,e_2)=$$

$$=\frac{3z^4}{4}+\frac{1}{z^2}+y^2=2(\beta^2-\alpha^2)-2(\xi\alpha)-\frac{r}{2}.$$

Similarly we have

$$K(e_2,\phi e_2)=\frac{3z^4}{4}+\frac{1}{z^2}+y^2=2(\beta^2-\alpha^2)-2(\xi\alpha)-\frac{r}{2}.$$

Again from (4.11) it can be easily shown that

$$\bar{K}(e_1,\phi e_1)=\frac{3z^4}{4}+\frac{1}{z^2}+y^2+\frac{a-1}{a}(3\alpha\beta-\alpha^2)=$$

$$=K(e_1,\phi e_1)+\frac{a-1}{a}\left(-\frac{3az^2}{2}-\left(-\frac{1}{z}\right)^2\right),$$

which implies that

$$\bar{K}(e_1,\phi e_1)-K(e_1,\phi e_1)=\frac{a-1}{a}(3a\beta-\alpha^2).$$

Similarly, we have

$$\bar{K}(e_2,\phi e_2)-K(e_2,\phi e_2)=\frac{a-1}{a}(3a\beta-\alpha^2).$$

Therefore such a normal almost contact metric manifold satisfies the relation (6.1) and hence Theorem 6.1 is verified.

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