UDC 517.9

T. Donchev (Univ. Al. I. Cuza, Iaşi, Romania),
A. Nosheen (Abdus Salam School Math. Sci., Pakistan)

FUZZY FUNCTIONAL DIFFERENTIAL EQUATIONS UNDER DISSIPATIVE-TYPE CONDITIONS*

НЕЧІТКІ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ З УМОВАМИ ДИСИПАТИВНОГО ТИПУ

Fuzzy functional differential equations with continuous right-hand side are studied. Existence and uniqueness of a solution under dissipative-type conditions are proved. The continuous dependence of the solution on the initial conditions is shown. The existence of a solution on an infinite interval and its stability are also considered.

Вивчаються нечіткі функціонально-диференціальні рівняння з неперервною правою частиною. Доведено існування та єдиність розв'язку за умов дисипативного типу. Встановлено неперервну залежність розв'язку від початкових умов. Також розглянуто питання про існування розв'язку на нескінченному інтервалі та його стійкість.

1. Introduction. The real life models depend not only on the current state of the subject but also on its prehistory. There are many papers devoted to delayed differential equations. We refer the readers to [6, 14] where many examples of functional differential equations, representing models in population dynamics, mathematical biology and medicine are provided. Often the models depend on uncertainty which requires the usage of a special mathematical approach as stochastic models, multivalued differential equations or fuzzy set models. The fuzzy differential equations had been studied first by Kandel and Byatt in [4]. Starting with the work of Kaleva [5], the theory of fuzzy differential equations was rapidly developed. We refer the books [8, 12] where the fuzzy differential equations are comprehensively studied (see also [9, 11, 15]).

There are relatively small number of papers devoted to fuzzy functional differential equations as [1], where the authors prove the existence of solutions using fixed point arguments. In the very interesting paper [10], the author proves the existence and uniqueness of solution when the right-hand side satisfied the locally Lipschitz condition (with respect to Kamke function), that paper is also provided with significant examples. In [15] the authors prove existence and uniqueness of the solution of fuzzy (ordinary) differential equations using Lyapunov-like functions.

We extend the results of the last two papers to fuzzy functional differential equations satisfying Razumikhin dissipative-type conditions.

In the next section we give the needed preliminaries from the theory of fuzzy sets. In Section 3, the system description and a needed preliminary lemma is proved. In Section 4, we prove the local and global existence as well as uniqueness of the solution under dissipative conditions. Afterward continuous dependence on the initial conditions and stability of the solution are shown. In the last section the existence and uniqueness results with the help of Lyapunov-like function are proved.

2. Preliminaries. The space of fuzzy numbers is $\mathbb{E} = \{x\colon \mathbb{R}^n \to [0,1];\ x \text{ satisfies 1)– 4)}\}$.

1) x is normal, i.e., there exists $y_0 \in \mathbb{R}^n$ such that $x(y_0) = 1$,

* This research is supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0154. The second author is partially supported by Higher Education Commission of Pakistan.

2) x is fuzzy convex, i.e., $x(\lambda y+(1-\lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

3) x is upper semicontinuous, i.e., for any $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta(y_0, \varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $|y - y_0| < \delta,\ y \in \mathbb{R}^n$,

4) the closure of the set $\{y \in \mathbb{R}^n;\ x(y) > 0\}$ is compact.

The set $[x]^\alpha = \{y \in \mathbb{R}^n;\ x(y) \geq \alpha\}$ is called α-level set of x.

It follows from 1) – 4) that the α-level sets $[x]^\alpha$ are in $C_c(\mathbb{R}^n)$ for all $\alpha \in (0,1]$, where $C_c(\mathbb{R}^n)$ denotes the compact convex subsets of $\mathbb{R}^n$. The fuzzy zero is defined as

$$\hat{0}(y) = \begin{cases} 0 & \text{if} \quad y \neq 0, \\ 1 & \text{if} \quad y = 0. \end{cases}$$

The metric in $\mathbb{E}$ is defined by $D(x,y) = \sup_{\alpha \in (0,1]} D_H([x]^\alpha, [y]^\alpha)$, where $D_H(\cdot,\cdot)$ means the Hausdorff distance in $C_c(\mathbb{R}^n)$.

Some properties of $D(x,y)$ are as follows:

(1) $D(x+z, y+z) = D(x,y)$ and $D(x,y) = D(y,x)$,

(2) $D(\lambda x, \lambda y) = \lambda D(x,y)$,

(3) $D(x,y) \leq D(x,z) + D(z,y)$, for all $x, y, z \in \mathbb{E}$ and $\lambda \in \mathbb{R}$.

We recall some properties of differentiability and integrability for fuzzy functions from [13]. Let $T > 0$, further in the paper $I = [0,T]$.

The map $f\colon I \to \mathbb{E}$ is said to be differentiable at $\hat{t} \in I = [0,T]$ iff for small $h > 0$ the differences $f(\hat{t}+h) - f(\hat{t}), f(\hat{t}) - f(\hat{t}-h)$ (in sense of Hukuhara) exist and there exists a fuzzy number $A = \dot{f}(\hat{t})$ such that

$$\lim_{h \to 0^+} \frac{f(\hat{t}+h) - f(\hat{t})}{h} = A = \lim_{h \to 0^+} \frac{f(\hat{t}) - f(\hat{t}-h)}{h}.$$

At the end points of I we consider only the one sided (left or right) derivative. It is easy to check that

$$D(f(\hat{t}+h), f(\hat{t}) + h\dot{f}(\hat{t})) = o(h) \text{ where } \lim_{h \to 0^+} \frac{o(h)}{h} = 0. \tag{1}$$

The fuzzy valued function $F\colon I \to \mathbb{E}$ is said to be (strongly) measurable if for each $\alpha \in [0,1]$, the set-valued function $F^\alpha\colon I \to C_c(\mathbb{R}^n)$ is measurable (see, e.g., [2]).

The integral of F over I, denoted by $\displaystyle\int_I F(t)dt$ is defined level-wise as

$$\left[\int_I F(t)dt\right]^\alpha = \int_I F^\alpha(t)dt = \left\{\int_I f(t)dt,\ f(t) \in F^\alpha(t) \text{ is integrable}\right\}$$

for all $0 \leq \alpha \leq 1$.

The function $F\colon \mathbb{R} \to \mathbb{E}$ ($G : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$) is said to be continuous if it is continuous w.r.t. the metric $D(\cdot,\cdot)$. The set of continuous functions from $[a,b]$ into $\mathbb{E}$ is denoted by $C([a,b], \mathbb{E})$.

Proposition 1 [5]. *If $F\colon I \to \mathbb{E}$ is continuous, then it is integrable over I. Moreover, in this case, the function $G(t) = \displaystyle\int_0^t F(s)ds$ is differentiable and $\dot{G}(t) = F(t)$.*

Let $m\colon [-\tau, \infty) \to \mathbb{R}^+$ be continuous. Denote $\dot{m}^+(t) = \overline{\lim}_{h \to 0^+} \dfrac{m(t+h) - m(t)}{h}$ the upper right Dini derivative.

We need the following known results:

Lemma 1 (cf. Lemma 6.1.1 of [7]). *Let* $g\colon I \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $m\colon [-\tau, T] \to \mathbb{R}$ *be two continuous functions and* $\dot{m}^+(t) \le g(t, m(t))$ *for every* $t \in [0,T] \backslash G$ (*where* G *is a countable set*) *with* $m(t) = \max_{s\in[-\tau,0]} m(t+s)$. *If* $m(0) \le r_0$, *then* $m(t) \le r(t)$, *where* $r(\cdot)$ *is the maximal solution of* $\dot{r}(t) = g(t, r),\ r(0) = r_0$ (*provided that* $r(\cdot)$ *exists on* I).

Lemma 2 [7]. *Let the assumptions of Lemma* 1 *hold. If* $[0, t_1] \subset [0, \infty)$, *then there exists an* $\varepsilon_0 > 0$ *such that* $0 < \varepsilon < \varepsilon_0$ *and the maximum solution* $r(t, u_0, \varepsilon)$ *of*

$$\dot{u} = g(t, u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon$$

exists on $[0, t_1]$ *and* $\lim_{\varepsilon\to 0} r(t, t_0, u_0, \varepsilon) = r(t, t_0, u_0)$ *uniformly on* $[0, t_1]$.

3. System description. In the paper we consider the following fuzzy functional differential equation:

$$\dot{x}(t) = f(t, x_t), \quad x_0 = \xi, \quad t \in I, \tag{2}$$

here $f\colon I \times \mathbb{E}_1 \to \mathbb{E}$, $\xi \in \mathbb{E}_1 := C([-\tau, 0], \mathbb{E})$ and $x_t \in \mathbb{E}_1$ is defined by $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$. For $\beta, \gamma \in \mathbb{E}_1$ we denote $D_C(\beta, \gamma) = \max_{s\in[-\tau,0]} D(\beta(s), \gamma(s))$. If $\gamma \equiv \hat{0}$ then we will write $D_C(\beta, \hat{0})$.

We refer the reader to [3] for a general theory of functional differential equations.

Notice that the system (2) is very general. It includes for example:

1. Variable times delay system $\dot{x} = f(t, x(t-h(t)))$, where $h(\cdot)$ is continuous with $0 \le h(t) \le \tau$.

2. Distributed time delay $\dot{x}(t) = f\left(t, \int_{-\tau}^{0} k(s, x(s))ds\right)$ as in the well know predator–prey model of Lotka – Wolterra:

$$\dot{x}(t) = x(t)\left[r_1 - \gamma_1 y(t) - \int\limits_{-\tau}^{0} F_1(s) y(t+s) ds\right],$$

$$\dot{y}(t) = y(t)\left[\gamma_2 x(t) - r_2 + \int\limits_{-\tau}^{0} F_2(s) x(t+s) ds\right],$$

where $x(\cdot)$, $y(\cdot)$ represent the population densities of prey and predator at time $t \ge 0$, $r_1 > 0$ is the intrinsic growth rate of the prey, $r_2 > 0$ is the death rate of the predators: $\gamma_1 > 0$ and $\gamma_2 > 0$ are the interaction constants.

The book [14] contains many examples of delayed differential equation from natural sciences. Some examples of functional differential equations in fuzzy set settings are contained in [10].

The continuous function $g\colon I \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be Kamke function if $g(t, 0) \equiv 0$, $g(t, \cdot)$ is monotone nondecreasing and $u \equiv 0$ is the only solution of the scalar differential equation

$$\dot{u} = g(t, u),\ u(0) = 0. \tag{3}$$

We will use the following hypotheses:

(H_1) $f\colon I \times \mathbb{E}_1 \to \mathbb{E}$ is continuous.

***Remark* 1.** (H_1) implies that there exist positive numbers a, b, M such that $D(f(t, \varphi), \hat{0}) \le M$ for every $(t, \varphi) \in [0, a] \times S_b(\xi)$. Furthermore, one can take $a \le \dfrac{b}{M+1}$. Here $S_b(\xi) = \{\varphi \in \mathbb{E}_1 \colon D_C(\varphi, \xi) \le b\}$.

(H_2) There exists a Kamke function $g\colon I \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{h\to 0^+} \frac{[D(\beta(0) + hf(t,\beta), \gamma(0) + hf(t,\gamma)) - D(\beta(0),\gamma(0))]}{h} \le g(t, D(\beta(0),\gamma(0))),$$

whenever $(\beta,\gamma) \in \Omega$. Here Ω is given by $\Omega = \{\beta,\gamma \in E_1\colon D_C(\beta,\gamma) = D(\beta(0),\gamma(0))\}$.

Lemma 3 [5]. *Under* (H_1) *the AC function* $x(\cdot)$ *is a solution of* (2) *if and only if* $x_0 = \xi$ *and*

$$x(t) = \xi(0) + \int_0^t f(s, x_s)\, ds. \tag{4}$$

Our next result is concerned with the construction of approximate solution to the problem (2).

Lemma 4. *Let* (H_1) *hold and let* $a, b, M > 0$ *be as in Remark* 1. *Then for every positive integer* n *there exists a continuous function* $x^n\colon [-\tau, a] \to \mathbb{E}$, *which is differentiable for every* $t \in [0,a]\setminus\mathcal{N}_n$ (*here* $\mathcal{N}_n \subset I$ *is countable set*) *and satisfies the following conditions*:

(i) $D(\dot{x}^n(t), f(t, x_t^n)) \le \dfrac{1}{n}$,

(ii) $D(\dot{x}^n(t), \hat{0}) \le M$,

(iii) $x_t^n \in S_b(\xi)$ (*recall that* $x_t^n \in \mathbb{E}_1$).

Proof. If $x(\cdot)$ satisfies (ii) then it is Lipschitz with a constant M. Indeed for $0 \le s < t \le a$, one has that $x(t) = x(s) + \displaystyle\int_s^t \dot{x}(\tau)d\tau$. Hence

$$D\left(x(t), x(s)\right) = \int_s^t D\left(\dot{x}(\tau), \hat{0}\right) d\tau \le M(t-s).$$

Suppose first that the needed $x^n(\cdot)$ exists on $[0,s]$, where $0 \le s \le a$. If $s = a$, then the prove is complete. Otherwise, for $t > s$ we define

$$x^n(t) = x_s^n(0) + (t-s)f(s, x_s^n).$$

The continuity of $f(\cdot,\cdot)$ implies that there exists $\tilde{t} > s$ such that

$$D\left(\dot{x}^n(t), f(t, x_t^n)\right) = D\left(f(s, x_s^n), f(t, x_t^n)\right) \le \frac{1}{n} \quad \text{for all} \quad t \in [s, \tilde{t}],$$

i.e., (i) is satisfied on $[0,\tilde{t}]$. Since $D\left(f(s, x_s^n), \hat{0}\right) \le M$, we have that (ii) and (iii) also hold on $[0,\tilde{t}]$.

Let $0 < \tilde{t} < a$ and let $x^n(\cdot)$ be defined on $[0,\tilde{t})$. Since $x^n(\cdot)$ is M-Lipschitz, then $\lim_{t\to\tilde{t}-0} x^n(t) = x^n(\tilde{t})$ exists, i.e., $x^n(\cdot)$ is defined on $[0,\tilde{t}]$. By standard application of Zorn's lemma, one can conclude that the needed solution $x^n(\cdot)$ exists on $[0,a]$.

4. Existence and uniqueness of solution. In this section we prove the main results in the paper. The first theorem is devoted to the local existence and uniqueness of solution for (2).

Theorem 1. *Let the hypothesis* (H_1) *and* (H_2) *hold. Then for each* $\xi \in \mathbb{E}_1$ *there exists unique solution* $x(\cdot)$ *of the initial value problem* (2) *defined on* $[0,a]$.

Proof. It follows from Lemma 4 that the approximate solution $x^n(\cdot)$ exists on $[0, a]$ and $D(f(t, x_t^n), \hat{0}) \leq M$.

Let $x^n(\cdot)$ and $x^k(\cdot)$ be two approximate solutions. Denote $m(t) = D(x^n(t), x^k(t))$. By virtue of the triangle inequality and (1) we have

$$m(t+h) - m(t) = D(x^n(t+h), x^k(t+h)) - D(x^n(t), x^k(t)) \leq$$

$$\leq D(x^n(t) + h\dot{x}^n(t), x^k(t) + h\dot{x}^k(t)) + o(h) - D(x^n(t), x^k(t)).$$

Thus

$$\lim_{h\to 0^+} \frac{m(t+h) - m(t)}{h} = \lim_{h\to 0^+} \frac{D(x^n(t) + h\dot{x}^n(t), x^k(t) + h\dot{x}^k(t)) - D(x^n(t), x^k(t))}{h} \leq$$

$$\leq \lim_{h\to 0^+} \frac{D(x^n(t) + hf(t, x_t^n), x^k(t) + hf(t, x_t^k)) - D(x^n(t), x^k(t))}{h} + \frac{1}{n} + \frac{1}{k}.$$

By (H_2) we have

$$\dot{m}(t)^+ \leq g(t, m(t)) + \frac{1}{n} + \frac{1}{k}$$

for every $t \in [0, a]$ with $m(t) = \max_{s\in[t-\tau,t]} m(s)$. It follows from Lemma 1 that $D(x^n(t), x^k(t)) \leq \leq r_{n,k}(t)$, where $r_{n,k}(t)$ is the maximal solution of

$$\dot{r}(t) = g(t, r(t)) + \frac{1}{n} + \frac{1}{k}, \quad r(0) = 0.$$

Since $g(\cdot, \cdot)$ is a Kamke function, one has that $x^n(\cdot)$ is a Cauchy sequence in $C([0, a], \mathbb{E})$ and hence $x^n(\cdot) \to x(\cdot)$ uniformly on $[0, a]$. Therefore $x_t^n \to x_t$ in $\mathbb{E}_1$ uniformly on $t \in [0, a]$. Furthermore, $f(t, \cdot)$ is continuous and hence $f(t, x_t^n) \to f(t, x_t)$ uniformly on $[0, a]$. Therefore

$$\int_0^t f(s, x_s^n)ds \to \int_0^t f(s, x_s)ds.$$

Consequently $x(t) = \xi(0) + \int_0^t f(s, x_s)ds$ and hence it is a local solution of (2) thanks to Lemma 3.

Now we prove that the solution is unique.

Let $x(t)$ and $y(t)$ be two solutions, we define $m(t) = D(x(t), y(t))$ and $m_t = D_C(x_t, y_t)$. It is easy to see that $m_t = m(t)$ if and only if $(x_t, y_t) \in \Omega$. In this case triangle inequality and (1) implies

$$\lim_{h\to 0^+} \frac{m(t+h) - m(t)}{h} = \lim_{h\to 0^+} \frac{[D(x(t+h), y(t+h)) - D(x(t), y(t))]}{h} \leq$$

$$\leq \lim_{h\to 0^+} \frac{[D(x(t) + hf(t, x_t), y(t) + hf(t, y_t)) - D(x(t), y(t)) + o(h)]}{h}.$$

Since $\lim_{h\to 0^+} h^{-1}.o(h) = 0$, one has that

$$\lim_{h\to 0^+} \frac{m(t+h) - m(t)}{h} = \lim_{h\to 0^+} \frac{[D(x(t) + hf(t, x_t), y(t) + hf(t, y_t)) - D(x(t), y(t))]}{h}.$$

Using (H_2) we derive

$$\lim_{h\to 0^+}\frac{m(t+h)-m(t)}{h}\le g(t,D(x(t),y(t))).$$

Thus we have $\dot m^+(t)\le g(t,m(t))$ for any t for which $m(t)=m_t$. From Lemma 1 and the last inequality it follows that $D(x(t),y(t))\le r(t)$, where $r(\cdot)$ is the maximal solution of $\dot r(t) = g(t,r(t))$, $r(0)=D_C(x_0,y_0)$. If $x_0=y_0$, then $r(t)=0$, i.e., $x(t)\equiv y(t)$ on $[0,a]$.

We are ready to study global existence of solution.

We need the following additional hypothesis:

(H_3) There exists a continuous function $w\colon I\times R^+\to R^+$ such that

(1) The maximal solution of $\dot\mu(t)=w(t,\mu(t))$, $\mu(0)=D_C(\xi,\hat 0)$ exists on I;

(2) for any $t\in I$ with $D_C(x_t,\hat 0)=D(x(t),\hat 0)$, we have

$$\lim_{h\to 0^+}\frac{[D(x(t)+hf(t,x_t),\hat 0)-D(x(t),\hat 0)]}{h}\le w(t,D(x(t),\hat 0));$$

(H_4) $f(\cdot,\cdot)$ is bounded on the bounded sets, i.e., for every $\vartheta>0$ there exists M such that $D(f(t,x(t)),\hat 0)\le M$ if $x\in\mathbb{E}_1$, $t>0$ and $D(x,\hat 0)+t\le\vartheta$.

Theorem 2. *Under* (H_1) – (H_4) *for every* $\xi\in\mathbb{E}_1$ *there exists unique solution of* (2) *defined on* $[0,T]$.

Proof. By Theorem 1 we know there exists $a>0$ such that fuzzy functional differential equation (2) admits unique solution on $[0,a]$. Suppose that the maximal interval of existence for $x(\cdot)$ (unique solution of (2)) is $[0,S)$, where $S<T$. It follows from (H_3) that $\dot D^+(x(t),\hat 0)\le w(t,D(x(t),\hat 0))$ for $t\in I$ with $D(x(t),\hat 0)=D_C(x_t,\hat 0)$. Using Lemma 1 we have $D(x(t),\hat 0)\le\mu(t)$ where $\mu(t)$ is the maximal solution of

$$\dot\mu(t)=w(t,\mu(t)),\ \mu(0)=D_C(\xi,\hat 0).$$

Since $\mu(\cdot)$ exists on the whole interval $[0,T]$, one has that there exist a constant $M>0$ such that $\mu(t)\le M-T$, for all $t\in[0,T]$.

Furthermore $f(\cdot,\cdot)$ is bounded on the bounded sets and hence $x(\cdot)$ is ϑ-Lipschitz on $[0,S)$ and hence $x(S)=\lim_{t\to S-0}x(t)$ exists. Consider the equation (2) on the interval $[S,T]$. The initial condition x_S is well defined, because $S>0$.

It follows from Theorem 1 that there exists $\delta>0$ such that (2) has a solution on $[S,S+\delta]$, where $S<S+\delta\le T$, a contradiction. Hence $x(t)$ exists on the interval $[0,T]$, and Zorn's lemma implies the existence of unique solution $x(\cdot)$ (as it is shown in Theorem 1) on $[0,T]$.

The following corollary shows the continuous dependence of the solution of (2) on the initial condition.

***Corollary* 1.** *Let* $f\colon I\times\mathbb{E}_1\to\mathbb{E}$ *satisfy* (H_1) – (H_3) *and let* $x(\cdot,\varphi)$, $y(\cdot,\psi)$ *be solutions of* (2) *with different initial conditions* $x_0=\varphi$ *and* $y_0=\psi$ *where* $\varphi,\psi\in\mathbb{E}_1$. *Then* $D(x(t),y(t))\le r(t)$ *where* $r(t)$ *is the maximal solution of*

$$\dot r(t)=g(t,r(t)),\quad r(0)=D(\varphi,\psi).$$

Proof. Denote $m(t)=D(x(t,\varphi),y(t,\psi))$. Evidently $m(0)=D(\varphi,\psi)$. Using the same arguments as in the proof of Theorem 2 we can show that $\dot m^+(t)\le g(t,m(t))$. Since $g(\cdot,\cdot)$ is a Kamke function, one has that the map $\varphi\to x(\cdot,\varphi)$ from $\mathbb{E}_1$ into $C(I,\mathbb{E})$ is continuous.

Now we will study (2) on $[0,\infty)=\mathbb{R}^+$. We replace I in (H_1) and (H_2) by $[0,\infty)$, i.e., we set $T=\infty$.

The hypothesis (H_3) in that case has the form:

(H_3') For every $S>0$ there exists a continuous function $w\colon [0,S]\times\mathbb{R}^+\to\mathbb{R}^+$such that

(1) the maximal solution of $\dot\mu(t)=w(t,\mu(t))$, $\mu(0)=D_C(\xi,\hat0)$ exists on $[0,S]$;

(2) for any $t\in I$ with $D_C(x_t,\hat0)=D(x(t),\hat0)$, we have

$$\lim_{h\to0^+}\frac{[D(x(t)+hf(t,x_t),\hat0)-D(x(t),\hat0)]}{h}\le w(t,D(x(t),\hat0)).$$

Theorem 3. *Let* (H_1), (H_2) (*with I replaced by* $[0,\infty)$, (H_3') *and* (H_4) *hold. Then the differential equation* (2) *admits unique solution $x(\cdot)$ defined on* $[0,\infty)$.

Proof. Let $S>0$ be given. From Theorem 2 we know that the initial problem (2) has a solution on $[0,S]$. Since $S>0$ is arbitrary, by virtue of Zorn lemma the solution $x(\cdot)$ exists on $[0,\infty)$.

Definition 1. *The solution $x(\cdot)$ (defined on $[0,\infty)$) of* (2) *is said to be stable if for every $\varepsilon>0$ there exists δ such that $D(y(t),x(t))<\varepsilon$ for every $t>0$ when $D_C(y_0,x_0)<\delta$. It is called asymptotically stable if it is stable and there exists $\nu>0$ such that* $\lim_{t\to\infty}D(x(t),y(t))=0$ *when* $D_C(y_0,x_0)<\nu$.

The following theorem is then valid.

Theorem 4. *Under the assumptions of Theorem* 3, *the unique solution $x(\cdot)$ of* (2) *is* (*asymptotically*) *stable if the zero solution of $\dot r(t)=w(t,r(t))$ is* (*asymptotically*) *stable.*

Proof. Due to (H_2)

$$\lim_{h\to0^+}\frac{D(x_t(0)+hf(t,x_t),y_t(0)+hf(t,y_t))-D(x_t(0),y_t(0))}{h}\le g(t,D(x_t(0),y_t(0)))$$

when $D(x_t(0),y_t(0))=D_C(x_t,y_t)$. Consequently $D(x(t),y(t))\le r(t)$ for every $t>0$, where $r(\cdot)$ is the maximal solution of $\dot r(t)=g(t,r(t))$ with initial condition $r(0)=D(x_0,y_0)$.

5. Lyapunov-like function condition. In this section we relax the dissipative condition. Our results extend the existence results given in [10] and [15]. We assume that $f(\cdot,\cdot)$ satisfies (H_3). Let M be the constant from the proof of Theorem 2 and denote $\mathbb{B}_M=\{x\in\mathbb{E}: D(x,\hat0)\le M+1\}$.

Definition 2. *The continuous map $W\colon (x(0)+\mathbb{B}_M)\times(x(0)+\mathbb{B}_M)\to\mathbb{R}^+$ is said to be a Lyapunov-like function for (2) if it satisfies the following*:

1) $W(x,x)=0$, $W(x,y)>0$ *for $x\ne y$ and* $\lim_{n\to\infty}W(x^n,y^n)=0$ *implies* $\lim_{n\to\infty}D(x^n,y^n)=0$;

2) *there exists a constant $L>0$ such that*

$$|W(x^1,y^1)-W(x^2,y^2)|\le L\left(D(x^1,x^2)+D(y^1,y^2)\right);$$

3) *there exists a Kamke function $v(\cdot,\cdot)$ such that*

$$\lim_{h\to0^+}\frac{W(\beta(0)+hf(t,\beta),\gamma(0)+hf(t,\gamma))-W(\beta(0),\gamma(0))}{h}\le v(t,W(\beta(0),\gamma(0)))$$

for any $\beta,\gamma\in\mathbb{E}_1$ with $W(\beta(0),\gamma(0))=\max_{s\in[-\tau,0]}W(\beta(s),\gamma(s))$.

Here we also require the following hypothesis to obtain desired result:

(H_5) There exists a Lyapunov-like function $W(\cdot,\cdot)$ for the system (2).

Theorem 5. *Under the assumptions* (H_1), (H_3), (H_4) *and* (H_5) *the fuzzy functional differential equation* (2) *has unique solution defined on* I.

Proof. Let a, b, M be as in Remark 1. From Lemma 4 we know that there exists a sequence of approximate solutions $\{x^n(\cdot)\}_{n=1}^{\infty}$ such that $x_0^n = \xi$ and $D(\dot{x}^n(t), f(t, x_t^n)) \leq \dfrac{1}{2^n}$. Denote $m(t) = W(x^k(t), x^n(t))$. Notice that $x^n(t) = x_t^n(0)$. Then

$$m(t+h) - m(t) = W(x^k(t+h), x^n(t+h)) - W(x^k(t), x^n(t)) \leq$$

$$\leq W(x^k(t) + hf(t, x_t^k), x^n(t) + hf(t, x_t^n)) - W(x^k(t), x^n(t)) +$$

$$+ L\left[D(x^k(t) + hf(t, x_t^k), x^k(t+h)) + D(x^n(t) + hf(t, x_t^n), x^n(t+h))\right] \leq$$

$$\leq W(x^k(t) + hf(t, x_t^k), x^n(t) + hf(t, x_t^n)) -$$

$$- W(x^k(t), x^n(t)) + L.o(h) + L\left(\frac{1}{2^k} + \frac{1}{2^n}\right).$$

Consequently

$$\lim_{h \to 0^+} \frac{m(t+h) - m(t)}{h} \leq v(t, m(t)) + L\left(\frac{1}{2^k} + \frac{1}{2^n}\right),$$

i.e.,

$$m(0) = 0 \text{ and } \dot{m}^+(t) \leq v(t, m(t)) + L\left(\frac{1}{2^k} + \frac{1}{2^n}\right),$$

for every $t \in I$ with $m(t) = \max_{s \in [-\tau, 0]} m(s)$. It follows from Lemma 1 that $m(t) \leq r(t)$, where $r(\cdot)$ is the maximal solution of

$$\dot{r}(t) = v(t, r(t)) + L\left(\frac{1}{2^k} + \frac{1}{2^n}\right), \quad r(0) = 0.$$

Taking into account Definition 2 we get that the sequence $\{x^n(\cdot)\}_{n=1}^{\infty}$ is a Cauchy sequence in $C(I, \mathbb{E})$. One can show as in the proof of Theorem 1 that the function $x(t) = \lim_{n \to \infty} x^n(t)$ is the unique solution of (2).

The proof of the following corollary is omitted, because it follows from the proofs in the previous section with obvious modifications.

Corollary 2. *Let* (H_1), (H_3), (H_4), *and* (H_5) *hold and let* $x(\cdot.\varphi)$, $y(\cdot, \phi)$ *be solutions of* (2) *with different initial conditions* $x_0 = \varphi$ *and* $y_0 = \psi$ *where* $\varphi, \psi \in \mathbb{E}_1$. *Then* $W(x(t), y(t)) \leq r(t)$ *where* $r(t)$ *is the maximal solution of*

$$\dot{r}(t) = g(t, r(t)), \; r(0) = \max_{s \in [-\tau, 0]} W(\varphi(s), \psi(s)).$$

Replacing (H_3) by (H_3') one can prove the same result on $[0, \infty)$.

1. *Arara A., Benchohra M.* Fuzzy solutions for neutral functional differential equations with nonlocal conditions // Georg. Math. J. – 2004. – **11**. – P. 299 – 312.
2. *Castaing C., Valadier M.* Convex analysis and measurable multifunctions. – New York: Springer, 1977.
3. *Hale J.* Theory of functional differential equations. – Berlin: Springer, 1977.

4. *Kandel A., Byatt N.* Fuzzy differential equations // Proc. Int. Conf. Cybern. and Soc. – 1978. – P. 1213 – 1216.
5. *Kaleva O.* Fuzzy differential equations // Fuzzy Sets and Systems. – 1987. – **24**. – P. 301 – 317.
6. *Kuang Y.* Delay differential equations with applications in population dynamics. – Boston: Acad. Press, 1993.
7. *Lakshmikantham V., Leela S.* Differential and integral inequalities. – New York: Acad. Press, 1969. – Vols I, II.
8. *Lakshmikantham V., Mohapatra R.* Theory of fuzzy differential equations and inclusions. – London: Taylor - Francis, 2003.
9. *Lupulescu V.* Initial value problem for fuzzy differential equations under dissipative conditions // Inform. Sci. – 2008. – **178**. – P. 4523 – 4533.
10. *Lupulescu V.* On a class of fuzzy functional differential equations // Fuzzy sets and Systems. – 2009. – **160**. – P. 1547 – 1562.
11. *Park J. Y., Han H. K.* Existence and uniqueness theorem for a solution of fuzzy differential equations // Int. J. Math. and Math. Sci. – 1999. – **22**. – P. 271 – 279.
12. *Plotnikov A., Skripnik N.* Differential equations with "clear"and fuzzy multivalued right-hand side // Asymptotic Methods. – Odessa: Astroprint, 2009 (in Russian).
13. *Puri M., Ralescu D.* Fuzzy random variables // J. Math. Anal. and Appl. – 1986. – **114**. – P. 409 – 422.
14. *Smith H.* An introduction to delay differential equations with applications to the life sciences. – New York: Springer, 2010.
15. *Song S., Wu C., Xue X.* Existence and uniqueness of Cauchy problems for fuzzy differential equations under fuzzy dissipative conditions // Comput. Math. and Appl. – 2006. – **51**. – P. 1483 – 1492.

Received 02.05.12