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FIXED-POINT RESULTS ON COMPLETE G-METRIC SPACES FOR MAPPINGS SATISFYING AN IMPLICIT RELATION OF A NEW TYPE

РЕЗУЛЬТАТИ ПРО НЕРУХОМУ ТОЧКУ НА ПОВНИХ G-МЕТРИЧНИХ ПРОСТОРАХ ДЛЯ ВІДОБРАЖЕНЬ, ЩО ЗАДОВОЛЬНЯЮТЬ НЕЯВНЕ СПІВВІДНОШЕННЯ НОВОГО ТИПУ

We prove some general fixed-point theorems in complete G-metric space that generalize some recent results.

Доведено загальні теореми про нерухому точку у повних G-метричних просторах, що узагальнюють деякі результати, отримані нещодавно.

1. Introduction. In [3, 4] Dhage introduced a new class of generalized metric space, named D-metric space. Mustafa and Sims [7, 8] proved that most of the claims concerning the fundamental topological structures on D-metric spaces are incorrect and introduced appropriate notion of generalized metric space, named G-metric space. In fact, Mustafa, Sims and other authors [2, 9 – 11] studied many fixed-point results for self mappings in G-metric spaces under certain conditions.

Quite recently [12], Mustafa et al. obtained new results for mappings in G-metric spaces.

In [13, 14], Popa initiated the study of fixed points in metric spaces for mappings satisfying an implicit relation.

Let T be a self mapping of a metric space (X, d). We denote by $\mathrm{Fix}\,(T)$ the set of all fixed points of T. T is said to satisfy property (P) if $\mathrm{Fix}\,(T) = \mathrm{Fix}\,(T^n)$ for each $n \in \mathbb{N}$. An interesting fact about mappings satisfying property (P) is that they have not nontrivial periodic points. Papers dealing with property (P) are, between others, [2, 13 – 15].

The purpose of this paper is to prove a general fixed-point theorem in complete G-metric space which generalize the results from [1, 10 – 12] for mappings satisfying a new form of implicit relation.

In the last part of this paper is proved a general theorem for mappings in G-metric space satisfying property (P), which generalize some results from [1].

2. Preliminaries.

Definition 2.1 [8]. *Let X be a nonempty set and $G\colon X^3 \to \mathbb{R}_+$ be a function satisfying the following properties*:

(G_1) $G(x, y, z) = 0$ *if* $x = y = z$;

(G_2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$;

(G_3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$;

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (*symmetry in all three variables*);

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ (*rectangle inequality*).

Then the function G is called a G-metric and the pair (X, G) is called a G-metric space.

Note that if $G(x, y, z) = 0$ then $x = y = z$ [8].

Lemma 2.1 [8]. $G(x, y, y) \leq 2G(x, x, y)$ *for all* $x, y \in X$.

Definition 2.2 [8]. *Let (X, G) be a metric space. A sequence (x_n) in X is said to be*:

a) *G-convergent to $x \in X$ if for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k$*;

b) *G-Cauchy if for* $\varepsilon > 0$, *there exists* $k \in \mathbb{N}$ *such that for all* $n, m, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$ *that is* $G(x_n, x_m, x_p) \to 0$ *as* $m, n, p \to \infty$.

A G-metric space is said to be G-complete if every G-Cauchy sequence in X *is G-convergent.*

Lemma 2.2 [8]. *Let* (X, G) *be a G-metric space. Then, the following properties are equivalent*:

1) (x_n) *is G-convergent to* x;

2) $G(x, x_n, x_n) \to 0$ *as* $n \to \infty$;

3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.

Lemma 2.3 [8]. *Let* (X, G) *be a G-metric space. Then the following properties are equivalent*:

1) *The sequence* (x_n) *is G-Cauchy.*

2) *For every* $\varepsilon > 0$, *there exists* $k \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$ *for* $n, m > k$.

Definition 2.3 [8]. *Let* (X, G) *and* (X', G') *be two G-metric spaces and* $f\colon (X, G) \to (X', G')$. *Then,* f *is said to be G-continuous at* $x \in X$ *if for* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x, y \in X$ *and* $G(a, x, y) < \delta$, *then* $G'(fa, fx, fy) < \varepsilon$. f *is G-continuous if it is G-continuous at each* $a \in X$.

Lemma 2.4 [8]. *Let* (X, G) *and* (X', G') *be two G-metric spaces. Then, a function* $f\colon (X, G) \to$ $\to (X', G')$ *is G-continuous at a point* $x \in X$ *if and only if* f *is sequentially continuous, that is, whenever* (x_n) *is G-convergent to* x *we have that* $f(x_n)$ *is G-convergent to* fx.

Lemma 2.5 [8]. *Let* (X, G) *be a G-metric space. Then, the function* $G(x, y, z)$ *is continuous in all three of its variables.*

Quite recently, the following theorem is proved in [12].

Theorem 2.1. *Let* (X, G) *be a complete G-metric space and* $T\colon X \to X$ *be a mapping which satisfies the following condition, for all* $x, y \in X$

$$G(Tx, Ty, Ty) \leq \max\{aG(x, y, y), b[G(x, Tx, Tx) + 2G(y, Ty, Ty)],$$

$$b[G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)]\}, \tag{2.1}$$

where $a \in [0, 1)$ *and* $b \in \left[0, \dfrac{1}{3}\right)$. *Then* T *has a unique fixed point.*

The purpose of this paper is to prove a general fixed point theorem in G-metric space for mappings satisfying a new type of implicit relation which generalize Theorem 2.1 and other results from [1, 2, 10 – 12].

3. Implicit relations.

Definition 3.1. *Let* $\mathfrak{F}_u$ *be the set of all continuous functions* $F(t_1, \ldots, t_6)\colon \mathbb{R}^6_+ \to \mathbb{R}$ *such that*

(F_1) F *is nonincreasing in variables* t_5 *and* t_6;

(F_2) *there exists* $h \in [0, 1)$ *such that for each* $u, v \geq 0$ *and* $F(u, v, v, u, u + v, 0) \leq 0$, *then* $u \leq hv$;

(F_3) $F(t, t, 0, 0, t, 2t) > 0 \quad \forall t > 0$.

Example **3.1.** $F(t_1, \ldots, t_6) = t_1 - \max\{at_2, b(t_3 + 2t_4), b(t_4 + t_5 + t_6)\}$, where $a \in [0, 1)$ and $b \in \left[0, \dfrac{1}{3}\right)$.

(F_1) Obviously.

(F_2) Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, 0) = u - \max\{av, b(v + 2u)\} \leq 0$. If $u > v$, then $u[1 - \max\{a, 3b\}] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $h =$ $= \max\{a, 3b\} < 1$.

(F_3) $F(t, t, 0, 0, t, 2t) = t(1 - \max\{a, 3b\}) > 0 \quad \forall t > 0$.

***Example* 3.2.** $F(t_1, \ldots, t_6) = t_1 - at_2 - b(t_3 + 2t_4) - c(t_5 + t_6)$, where $a, b, c \geq 0$, $a + 3b + 2c < 1$ and $a + 3c < 1$.

(F_1) Obviously.

(F_2) Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, 0) = u - av - b(v + 2u) - c(u + v) \leq 0$. Then $u \leq hv$, where $h = \dfrac{a + b + c}{1 - 2b - c} < 1$.

(F_3) $F(t, t, 0, 0, t, 2t) = t[1 - (a + 3c)] > 0 \quad \forall t > 0$.

***Example* 3.3.** $F(t_1, \ldots, t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_5, t_6\}$, where $a, b, c \geq 0$, $a + b + 2c < 1$.

(F_1) Obviously.

(F_2) Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, 0) = u - av - b\max\{u, v\} - c(u + v) \leq 0$. If $u > v$, then $u[1 - (a + b + 2c)] \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $h = \dfrac{a + b + c}{1 - c} < 1$.

(F_3) $F(t, t, 0, 0, t, 2t) = t[1 - (a + 2c)] > 0 \quad \forall t > 0$.

***Example* 3.4.** $F(t_1, \ldots, t_6) = t_1 - k\max\{t_2, t_3, \ldots, t_6\}$, where $k \in \left[0, \dfrac{1}{2}\right)$.

(F_1) Obviously.

(F_2) Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, 0) = u - k(u + v) \leq 0$ which implies $u \leq hv$, where $h = \dfrac{k}{k - 1} < 1$.

(F_3) $F(t, t, 0, 0, t, 2t) = t(1 - 2k) > 0 \quad \forall t > 0$.

***Example* 3.5.** $F(t_1, \ldots, t_6) = t_1 - at_2 - bt_3 - c\max\{t_4 + t_5, 2t_6\}$, where $a, b, c \geq 0$, $a + b + 3c < < 1$, $a + 4c < 1$.

(F_1) Obviously.

(F_2) Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, 0) = u - av - bv - c(2u + v) \leq 0$. Then $u \leq hv$, where $h = \dfrac{a + b + c}{1 - 2c} < 1$.

(F_3) $F(t, t, 0, 0, t, 2t) = t[1 - (a + 4c)] > 0 \quad \forall t > 0$.

***Example* 3.6.** $F(t_1, \ldots, t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \dfrac{2t_4 + t_6}{3}, \dfrac{2t_4 + t_3}{3}, \dfrac{t_5 + t_6}{3}\right\} \leq 0$, where $k \in [0, 1)$.

(F_1) Obviously.

(F_2) Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, 0) = u - k\max\left\{u, v, \dfrac{2u}{3}, \dfrac{2u + v}{3}, \dfrac{u + v}{3} \leq 0\right\}$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $h = k < 1$.

(F_3) $F(t, t, 0, 0, t, 2t) = t(1 - k) > 0\ \forall t > 0$.

***Example* 3.7.** $F(t_1, \ldots, t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\right\}$, where $k \in \left[0, \dfrac{2}{3}\right)$.

(F_1) Obviously.

(F_2) Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, 0) = u - k\max\left\{u, v, \dfrac{u + v}{2}\right\} \leq 0$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $h = k < 1$.

(F_3) $F(t, t, 0, 0, t, 2t) = t - k\max\left\{t, \dfrac{3t}{2}\right\} = t\left[1 - \dfrac{3k}{2}\right] > 0 \quad \forall t > 0$.

***Example* 3.8.** $F(t_1,\ldots,t_6)=t_1^2-t_1(at_2+bt_3+ct_4)-dt_5t_6$, where $a,b,c\geq 0$, $a+b+c<1$, $a+2d<1$.

(F_1) Obviously.

(F_2) Let $u,v\geq 0$ be and $F(u,v,v,u,u+v,0)=u^2-u(av+bv+cu)\leq 0$. If $u>0$, then $u-av-bv-cu\leq 0$ which implies $u\leq hv$, where $h=\dfrac{a+b}{1-c}<1$. If $u=0$, then $u\leq hv$.

(F_3) $F(t,t,0,0,t,2t)=t^2[1-(a+2d)]>0\quad\forall t>0$.

***Example* 3.9.** $F(t_1,\ldots,t_6)=t_1-k\max\left\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}$, where $k\in\left[0,\dfrac{2}{3}\right)$.

(F_1) Obviously.

(F_2) Let $u,v\geq 0$ be and $F(u,v,v,u,u+v,0)=u-k\max\left\{v,\dfrac{u+v}{2}\right\}\leq 0$. If $u>0$, then $u(1-k)\leq 0$, a contradiction. Hence $u\leq v$ which implies $u\leq hv$, where $h=k<1$.

(F_3) $F(t,t,0,0,t,2t)=t\left[1-\dfrac{3k}{2}\right]>0\quad\forall t>0$.

***Example* 3.10.** $F(t_1,\ldots,t_6)=t_1-k\max\left\{t_2,\sqrt{t_3t_4},\sqrt{t_5t_6}\right\}$, where $k\in\left[0,\dfrac{2}{3}\right)$.

(F_1) Obviously.

(F_2) Let $u,v\geq 0$ be and $F(u,v,v,u,u+v,0)=u-k\max\{v,\sqrt{uv}\}\leq 0$. If $u>v$, then $u(1-k)\leq 0$, a contradiction. Hence, $u\leq v$ which implies $u\leq hv$, where $0\leq h=k<1$.

(F_3) $F(t,t,0,0,t,2t)=t(1-\sqrt{2}k)>0\quad\forall t>0$.

4. Main results.

Theorem 4.1. *Let (X,G) be a G-metric space and $T\colon(X,G)\to(X,G)$ be a mapping such that*

$$F(G(Tx,Ty,Ty),G(x,y,y),G(x,Tx,Tx),G(y,Ty,Ty),G(x,Ty,Ty),G(y,Tx,Tx))\leq 0 \tag{4.1}$$

for all $x,y\in X$, where F satisfies property (F_3). Then T has at most a fixed point.

Proof. Suppose that T has two distinct fixed points u and v. Then by (4.1) we have successively

$$F(G(Tu,Tv,Tv),G(u,v,v),G(u,Tu,Tu),G(v,Tv,Tv),G(u,Tv,Tv),G(v,Tu,Tu))\leq 0,$$

$$F(G(u,v,v),G(u,v,v),0,0,G(u,v,v),G(v,u,u))\leq 0.$$

By Lemma 2.1 $G(v,u,u)\leq 2G(u,v,v)$. Since F is nonincreasing in variable t_6 we obtain

$$F(G(u,v,v),G(u,v,v),0,0,G(u,v,v),2G(u,v,v))\leq 0,$$

a contradiction of (F_3). Hence $u=v$.

Theorem 4.1 is proved.

Theorem 4.2. *Let (X,G) be a complete G-metric space and $T\colon(X,G)\to(X,G)$ satisfying inequality* (4.1) *for all $x,y\in X$, where $F\in\mathfrak{F}_u$. Then T has a unique fixed point.*

Proof. Let $x_0\in X$ be an arbitrary point in X. We define $x_n=Tx_{n-1}$, $n=1,2,\ldots$. Then by (4.1) we have successively

$$F(G(Tx_{n-1},Tx_n,Tx_n),G(x_{n-1},x_n,x_n),G(x_{n-1},Tx_{n-1},Tx_{n-1}),$$

$$G(x_n, Tx_n, Tx_n), G(x_{n-1}, Tx_n, Tx_n), G(x_n, Tx_{n-1}, Tx_{n-1})) \leq 0,$$

$$F(G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n),$$

$$G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), 0) \leq 0.$$

By (G_5), $G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})$. Since F is nonincreasing in variable t_5 we obtain

$$F(G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n),$$

$$G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}, 0) \leq 0$$

which implies by (F_2) that

$$G(x_n, x_{n+1}, x_{n+1}) \leq hG(x_{n-1}, x_n, x_n).$$

Then

$$G(x_n, x_{n+1}, x_{n+1}) \leq hG(x_{n-1}, x_n, x_n) \leq \ldots \leq h^n G(x_0, x_1, x_1).$$

Moreover, for all $m, n \in \mathbb{N}$, $m > n$, we have repeated use the rectangle inequality

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{m-1}, x_m, x_m) \leq$$

$$\leq (h^n + h^{n+1} + \ldots + h^{m-1}) G(x_0, x_1, x_1) \leq \frac{h^n}{1-h} G(x_0, x_1, x_1),$$

which implies $\lim_{n,m \to \infty} G(x_n, x_m, x_m) = 0$. Hence, (x_n) is a G-Cauchy sequence. Since (X, G) is G-complete, there exists $u \in X$ such that $\lim_{n \to \infty} x_n = u$.

We prove that $u = Tu$. By (F_1) we have successively

$$F(G(Tx_{n-1}, Tu, Tu), G(x_{n-1}, u, u), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}),$$

$$G(u, Tu, Tu), G(x_{n-1}, Tu, Tu), G(u, Tx_{n-1}, Tx_{n-1})) \leq 0,$$

$$F(G(x_n, Tu, Tu), G(x_{n-1}, u, u), G(x_{n-1}, x_n, x_n),$$

$$G(u, Tu, Tu), G(x_{n-1}, Tu, Tu), G(u, x_n, x_n)) \leq 0.$$

By continuity of F and G, letting n tend to infinity, we obtain

$$F(G(u, Tu, Tu), 0, 0, G(u, Tu, Tu), G(u, Tu, Tu), 0) \leq 0.$$

By (F_2) we obtain $G(u, Tu, Tu) = 0$, hence $u = Tu$ and u is a fixed point of T. By Theorem 4.1 u is the unique fixed point of T.

Theorem 4.2 is proved.

***Corollary* 4.1.** *Theorem* 2.1.

Proof. The proof follows from Theorem 4.2 and Example 3.1.

***Corollary* 4.2** (Theorem 2.2 [11]). *Let (X,G) be a G-complete metric space and $T\colon (X,G)\to$ $\to (X,G)$ be a mapping satisfying the following condition*:

$$G(Tx,Ty,Tz)\leq \alpha G(x,y,z)+\beta[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)], \qquad (4.2)$$

for all $x,y,z\in X$ and $0\leq \alpha+3\beta<1$. Then T has a unique fixed point.

Proof. By (4.2) for $z=y$ we obtain

$$G(Tx,Ty,Ty)\leq \alpha G(x,y,y)+\beta[G(x,Tx,Tx)+2G(y,Ty,Ty)],$$

for all $x,y\in X$. By Theorem 4.2 and Example 3.2 for $\alpha=a$, $\beta=b$ and $c=0$ it follows that T has a unique fixed point.

***Corollary* 4.3** (Theorem 2.3 [11]). *Let (X,G) be a G-complete metric space and $T\colon (X,G)\to$ $\to (X,G)$ be a mapping satisfying the condition*

$$G(Tx,Ty,Tz)\leq \alpha G(x,y,z)+\beta\max\{G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\}, \qquad (4.3)$$

for all $x,y,z\in X$ and $0\leq \alpha+\beta<1$. Then T has a unique fixed point.

Proof. By (4.3) for $z=y$ we obtain

$$G(Tx,Ty,Ty)\leq \alpha G(x,y,y)+\beta\max\{G(x,Tx,Tx),G(y,Ty,Ty)\},$$

for all $x,y\in X$. By Theorem 4.2 and Example 3.3 for $\alpha=a$, $\beta=b$ and $c=0$ it follows that T has a unique fixed point.

***Corollary* 4.4** (Theorem 2.1 [10]). *Let (X,G) be a G-complete metric space and $T\colon (X,G)\to$ $\to (X,G)$ be a mapping satisfying the condition*

$$\begin{aligned}G(Tx,Ty,Tz)\leq k\max\{&G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),\\ &G(y,Tz,Tz),G(x,Ty,Ty),G(y,Tz,Tz),G(z,Tx,Tx)\},\end{aligned} \qquad (4.4)$$

for all $x,y,z\in X$, where $k\in\left[0,\frac{1}{2}\right)$. Then T has a unique fixed point.

Proof. By (4.4) for $z=y$ we obtain

$$G(Tx,Ty,Ty)\leq k\max\{G(x,y,y),G(x,Tx,Tx),G(y,Ty,Ty),G(x,Ty,Ty),G(y,Tx,Tx)\}.$$

By Theorem 4.2 and Example 3.4, T has a unique fixed point.

***Corollary* 4.5.** *Let (X,G) be a G-complete metric space and $T\colon (X,G)\to (X,G)$ be a mapping which satisfy the following inequality for all $x,y\in X$,*

$$G(Tx,Ty,Ty)\leq k\max\{G(y,Ty,Ty)+G(x,Ty,Ty),2G(y,Tx,Tx)\}, \qquad (4.5)$$

where $k\in\left[0,\frac{1}{3}\right)$. Then T has a unique fixed point.

Proof. By Theorem 4.2 and Example 3.5 for $a=b=0$ and $c=k$, T has a unique fixed point.

***Remark* 4.1.** In Theorem 2.8 [10], $k\in\left[0,\frac{1}{2}\right)$.

***Corollary* 4.6.** *Let (X,G) be a G-metric space and $T\colon (X,G)\to(X,G)$ be a mapping satisfying the following inequality for all $x,y,z\in X$,*

$$G(Tx,Ty,Tz)\le h\max\Big\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz),$$

$$\frac{G(y,Tx,Tx)+G(y,Ty,Ty)+G(y,Tz,Tz)}{3},\frac{G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)}{3}\Big\}, \tag{4.6}$$

where $k\in[0,1)$. Then T has a unique fixed point.

Proof. If $y=z$, by (4.6) we obtain that

$$G(Tx,Ty,Ty)\le h\max\Big\{G(x,y,y),G(x,Tx,Tx),G(y,Ty,Ty),$$

$$\frac{G(y,Tx,Tx)+2G(y,Ty,Ty)}{3},\frac{G(x,Tx,Tx)+2G(y,Ty,Ty)}{3}\Big\}\le$$

$$\le h\max\Big\{G(x,y,y),G(x,Tx,Tx),G(y,Ty,Ty),\frac{G(y,Tx,Tx)+2G(y,Ty,Ty)}{3},$$

$$\frac{G(x,Tx,Tx)+2G(y,Ty,Ty)}{3},\frac{G(x,Ty,Ty)+G(y,Tx,Tx)}{3}\Big\},$$

for all $x,y\in X$.

By Theorem 4.2 and Example 3.6, T has a unique fixed point.

***Remark* 4.2.** Corollary 4.6 is a generalization of Theorem 2.6 [1], where $k\in\left[0,\dfrac{1}{2}\right)$.

***Remark* 4.3.** By Theorem 4.2 and Examples 3.7 – 3.10 we obtain new results.

5. Property (P) in G-metric spaces.

Theorem 5.1. *Under the conditions of Theorem* 4.2, *T has property (P).*

Proof. By Theorem 4.2, T has a fixed point. Therefore, $\mathrm{Fix}\,(T^n)\neq\varnothing$ for each $n\in\mathbb{N}$. Fix $n>1$ and assume that $p\in\mathrm{Fix}\,(T^n)$. We prove that $p\in\mathrm{Fix}\,(T)$. Using (4.1) we have

$$F(G(T^np,T^{n+1}p,T^{n+1}p),G(T^{n-1}p,T^np,T^np),G(T^{n-1}p,T^np,T^np),G(T^np,T^{n+1}p,T^{n+1}p),$$

$$G(T^{n-1}p,T^{n+1}p,T^{n+1}p),G(T^np,T^np,T^np))\le 0.$$

By rectangle inequality

$$G(T^{n-1}p,T^{n+1}p,T^{n+1}p)\le G(T^{n-1}p,T^np,T^np)+G(T^np,T^{n+1}p,T^{n+1}p).$$

By (F_1) we obtain

$$F(G(T^np,T^{n+1}p,T^{n+1}p),G(T^{n-1}p,T^np,T^np),G(T^{n-1}p,T^np,T^np),G(T^np,T^{n+1}p,T^{n+1}p),$$

$$G(T^{n-1}p,T^np,T^np)+G(T^np,T^{n+1}p,T^{n+1}p),0)\le 0.$$

By (F_2) we obtain

$$G(T^n p, T^{n+1} p, T^{n+1} p) \le hG(T^{n-1} p, T^n p, T^n p) \le \ldots \le h^n G(p, Tp, Tp).$$

Since $p \in T^n p$, then

$$G(p, Tp, Tp) = G(T^n p, T^{n+1} p, T^{n+1} p).$$

Therefore

$$G(p, Tp, Tp) \le h^n G(p, Tp, Tp)$$

which implies $G(p, Tp, Tp) = 0$, i.e., $p = Tp$ and T has property (P).

Theorem 5.1 is proved.

***Corollary* 5.1.** *In the condition of Corollary* 4.6, T *has property* (P).

***Remark* 5.1.** Corollary 5.1 is a generalization of the results from Theorem 2.6 [1].

***Corollary* 5.2.** *In the condition of Corollary* 4.4 *with* $k \in \left[0, \frac{1}{2}\right)$, *instead* $k \in [0, 1)$, T *has property* (P).

***Remark* 5.2.** We obtain other new results from Examples 3.1 – 3.10.

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