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HEREDITARY PROPERTIES BETWEEN A RING AND ITS MAXIMAL SUBRINGS

СПАДКОВІ ВЛАСТИВОСТІ МІЖ КІЛЬЦЕМ ТА ЙОГО МАКСИМАЛЬНИМИ ПІДКІЛЬЦЯМИ

We study the existence of maximal subrings and hereditary properties between a ring and its maximal subrings. Some new techniques for establishing the existence of maximal subrings are given. It is shown that if R is an integral domain and S is a maximal subring of R, then the relation $\dim(R) = 1$ implies that $\dim(S) = 1$ and vice versa if and only if $(S : R) = 0$. Consequently, we show that if S is a maximal subring of a Dedekind domain R that is integrally closed in R, then S is a Dedekind domain if and only if S is Noetherian and $(S : R) = 0$. We also give some properties of maximal subrings of one-dimensional valuation domains and zero-dimensional rings. Some other hereditary properties such as semiprimarity, semisimplicity, and regularity are also studied.

Вивчається існування максимальних підкілець та спадкові властивості між кільцем та його максимальними підкільцями. Наведено деякі нові методи встановлення існування максимальних підкілець. Показано, що якщо R — інтегральна область, а S — її максимальне підкільце, то із співвідношення $\dim(R) = 1$ випливає, що $\dim(S) = 1$, і навпаки тоді і тільки тоді, коли $(S : R) = 0$. Як наслідок показано, що, якщо S є максимальним підкільцем дедекіндової області R, яка є інтегрально замкненою в R, то S є дедекіндовим підкільцем тоді і тільки тоді, коли S є нетеровим та $(S : R) = 0$. Наведено також деякі властивості максимальних підкілець одновимірних областей нормування та нульвимірних кілець. Також вивчено деякі інші спадкові властивості, такі як напівпримарність, напівпростота та регулярність.

1. Introduction. All rings in this article are commutative with $1 \neq 0$; all modules are unital. If S is a subring of a ring R, then $1_R \in S$. In this paper the characteristic of a ring R is denoted by $\mathrm{Char}(R)$. Let us call a ring with a maximal subring, a submaximal ring. Recently the existence of maximal subrings and also some hereditary properties between a ring and any of its maximal subrings are studied, see [1 – 5]. Let us, for the sake of the reader, first cite some old and new hereditary properties which are shared between a ring and each of its maximal subrings. M. L. Modica proved that, if D is a maximal subring of a ring R such that D is an integral domain and is integrally closed in R, then R is an integral domain too, see [12] (Theorem 10). The latter result is also an immediate consequence of [7] (Theorem 2.7). We remind the reader that all finite rings except $\mathbb{Z}_n$, where n is a natural number, are submaximal. It is also interesting to note that whenever S is a finite maximal subring of a ring R, then R must be finite, see [6] (Theorem 8), hence finiteness is a nice hereditary property shared between a ring R and any maximal subring of R. The latter interesting fact is also an easy consequence of [3] (the proof of Theorem 2.9) or [4] (Theorem 3.8). In [4] (Theorem 3.8) it is shown that if S is a maximal subring of a commutative ring R, then S is Artinian if and only if R is Artinian and is integral over S. Some other hereditary properties such as Noetherianity, zero-dimensionality and von Neumann regularity are also studied in [4]. In [2], it is also shown that if S is a maximal subring of an integral domain R, then S is a G-domain if and only if R is a G-domain. It is also proved that if S is a maximal subring of $\mathbb{R}$, then S is never a field and, similar to $\mathbb{R}$, the only ring endomorphism of S is the identity, see [2].

Now, let us sketch a brief outline of this paper. In Section 2, the existence of maximal subrings in fields (via G-domain), Artinian rings, infinite direct product (via ideals) and finally the existence of maximal subrings via R-modules are studied. In Section 3, we study hereditary properties between a ring and its maximal subrings and properties of maximal subrings in some special rings such as zero-dimensional rings and one-dimensional valuation domains. We first prove that under some conditions the Hilbert property can be shared between a ring and its maximal subrings. Next we prove that if R is an integral domain and S is a maximal subring of it, then $\dim(R) = 1$ implies $\dim(S) = 1$ and vice versa if and only if $(S : R) = 0$. Consequently, we show that if S is a maximal subring of a Dedekind domain R, which is integrally closed in R, then S is a Dedekind domain if and only if S is Noetherian and $(S : R) = 0$. We also show that if S is a maximal subring of a zero-dimensional ring R and S is integrally closed in R, then $\dim(S) = 1$ and we give the structures of $\mathrm{Max}(S)$ and $\mathrm{Spec}(S)$. The properties of maximal subrings of one-dimensional valuation domains are also studied. Some other hereditary properties such as semiprimarity, semisimplicity and regularity are also studied in Section 3.

Next, let us recall some standard definitions and notation for commutative ring theory which will be used throughout the paper, see [9]. An integral domain D is called G-domain if the quotient field of D is finitely generated as a ring over D. A prime ideal P of a ring R is called G-ideal if $\dfrac{R}{P}$ is a G-domain. A ring R is called Hilbert if every G-ideal of R is maximal. We also call a ring R, not necessarily Noetherian, semi-local (resp. local) if $\mathrm{Max}(R)$ is finite (resp. $|\mathrm{Max}(R)| = 1$). The set of minimal prime ideals and the prime ideals, of a ring R are denoted by $\mathrm{Min}(R)$ and $\mathrm{Spec}(R)$, respectively. As usual, let $U(R)$ denote the set of all units of a ring R. The Jacobson and the nil radical of a ring R are also denoted by $J(R)$ and $N(R)$, respectively. If D is an integral domain, then the quotient field and the integral closure of D are denoted by $Q(D)$ and D', respectively. More generally, if S is a subring of a ring R, then the integral closure of S in R is denoted by S'_R. Finally, let $S \subseteq R$ be two rings, then the conductor ideal of this extension is denoted by $(S : R)$, that is the largest ideal of R which is contained in S, i.e., $(S : R) = \{x \in R \mid Rx \subseteq S\}$.

The following two theorems, whose proofs could be found in either [8] or [12], are needed. Before presenting them, let us recall that whenever S is a maximal subring of a ring R, then one can easily see that either R is integral over S or S is integrally closed in R.

Theorem 1.1. *Let S be a maximal subring of a ring R. Then the following statements are true*:

(1) $(S : R) \in \mathrm{Spec}(S)$.

(2) $(S : R) \in \mathrm{Max}(S)$ *if and only if R is integral over S.*

(3) *If S is integrally closed in R, then* $(S : R) \in \mathrm{Spec}(R)$.

Let us before presenting the second theorem, recall that if S is a maximal subring of a ring R and X is a multiplicatively closed set in S, then one can easily see that either S_X, the ring of fractions of S with respect to X, is a maximal subring of R_X or $S_X = R_X$.

Theorem 1.2. *Let S be a maximal subring of a ring R. Then there exists a unique maximal ideal M of S such that S_M is a maximal subring of $R_{S\setminus M}$ and for any other prime ideal P of S we have $S_P = R_{S\setminus P}$. Moreover, $(S : R) \subseteq M$.*

The unique maximal ideal in the previous theorem is called the crucial maximal ideal of the extension $S \subseteq R$ and we denote it by $\mathrm{cru}_R(S)$.

2. Existence of maximal subrings and generalizations. In [3], it is proved that uncountable fields, nonabsolutely algebraic fields (i.e., fields which are not algebraic over their prime subfields), and fields with zero characteristic are submaximal. In what follows, for the sake of completeness, we give new proofs to some slight extensions of these results, which are basic in our study. First, we give the following theorem about the extension of G-domains. Let us recall that a ring R is submaximal if and only if there exists a proper subring S of R and $\alpha \in R\backslash S$ with $S[\alpha] = R$. In particular, R has a maximal subring containing S but not α, see [1] (Theorem 2.5).

Theorem 2.1. *Let R be a G-domain with the quotient field $F \neq R$. Then any algebraic field extension K of F is the quotient field of a nontrivial G-domain.*

Proof. First, note that by the preceding comment, we may assume that R is a maximal subring of F. Let $\bar{R}$ be the integral closure of R in K. By [1] (Theorem 3.4) R is a G-domain and is integrally closed in its quotient field F. We may assume that $K \neq F$. Since R is integrally closed in F, we infer that $\bar{R} \neq K$. Now we claim that there exists an element $u \in R$ with $K = \bar{R}[u^{-1}]$ and this completes the proof. To this end, we note that since R is a G-domain we have $F = R[u^{-1}]$, where u is a nonunit element of R. We show that we also have $K = \bar{R}[u^{-1}]$. Clearly, $\bar{R}[u^{-1}] \subseteq K$. Hence let $x \in K$ be any element and we are to show that $x \in \bar{R}[u^{-1}]$. Since K is algebraic over F we infer that $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ for some $a_0, a_1, \ldots, a_{n-1} \in F$. But $F = R[u^{-1}]$ implies that for any a_i, $i = 1, 2, \ldots, n-1$, there exists a nonnegative integer m_i such that $a_i = u^{-m_i} b_i$ for some $b_i \in R$. Now let m be a natural number such that $m \geq m_i$ for all i and multiply the above equation by u^{mn}. Hence we have the equation $(u^m x)^n + u^m a_{n-1}(u^m x)^{n-1} + \ldots + u^{m(n-1)} a_1 (u^m x) + u^{mn} a_0 = 0$. This implies that $u^m x$ is integral over R (note, since $a_i = u^{-m_i} b_i$, $b_i \in R$, the coefficient of the above equation are in R) and therefore $u^m x \in \bar{R}$ which implies that $x \in \bar{R}[u^{-1}]$, and we are done.

Theorem 2.1 is proved.

***Corollary* 2.1.** *Let F be a field with a nonfield maximal subring. Then any field extension of F is submaximal. In particular, every field with zero characteristic and any nonabsolutely algebraic field* (*such as uncountable fields*) *are submaximal.*

Proof. Let K/F be a field extension. If K is algebraic over F, then we are done by the previous theorem and the comment preceding it. Hence assume that K is not algebraic over F. Suppose that S is a transcendence basis for K over F. Then K is algebraic over $F(S)$ and hence we are done by the previous theorem and the fact that $F(S)$ has a maximal subring which is not a field. The final part is evident.

***Corollary* 2.2.** *Let F be a field with a nonfield maximal subring. Then any F-algebra is submaximal.*

Proof. Let R be an F-algebra and M be a maximal ideal of R, then R/M contains a copy of F. Thus R/M and therefore R are submaximal by Corollary 2.1.

In [4] (Theorem 3.8), it is proved that whenever S is a maximal subring of a ring R, then S is Artinian if and only if R is Artinian and integral over S. The following is now in order.

***Corollary* 2.3.** *Let R be an Artinian ring with a non-Artinian maximal subring. Then any R-algebra is submaximal.*

Proof. Let S be a non-Artinian maximal subring of R. Then we infer that R is not integral over S, by [4] (Theorem 3.8). Therefore $P = (S : R) \in \operatorname{Max}(R)$ but $P \notin \operatorname{Max}(S)$, by Theorem 1.1. Hence S/P is a nonfield maximal subring of the field R/P. Now, let T be any R-algebra, since P is a prime ideal of R, we infer that there exists a prime ideal Q of T, such that $Q \cap R \subseteq P$. But $\dim(R) = 0$ and $Q \cap R \in \operatorname{Spec}(R)$, hence $Q \cap R = P$. Thus T/Q contains a copy of R/P and therefore we are done by Corollary 2.2.

The next proposition is the converse of Theorem 1.2. Before presenting it, we need a well-known fact which follows. Let R be a ring and M be an R-module. If N is an R-submodule of M and for any prime ideal (or maximal ideal) P of R we have $N_P = M_P$, then $M = N$, see [9, p. 24] (Ex. 5).

Proposition 2.1. *Let S be a proper subring of a ring R. If there exists a unique maximal ideal M of S such that S_M is a maximal subring of $R_{S\setminus M}$; and for any maximal ideal $N \neq M$ of S, we have $S_N = R_{S\setminus N}$, then S is a maximal subring of R.*

Proof. Let L be a subring of R with $S \subseteq L \subseteq R$ and we are to show that $L = S$ or $L = R$. For each maximal ideal $N \neq M$ of S, Since $S_N = R_{S\setminus N}$, we infer that $S_N = L_{S\setminus N} = R_{S\setminus N}$. We also have $S_M \subseteq L_{S\setminus M} \subseteq R_{S\setminus M}$. Hence either $S_M = L_{S\setminus M}$, in which case $S = L$ (note, $S_P = L_P$ for all maximal ideals P of S, where $S \subseteq L$ are considered as S-modules), or $L_{S\setminus M} = R_{S\setminus M}$ which also implies that $S_P = L_P$ for all maximal ideals P of S, i.e., $L = R$ and we are done.

Proposition 2.1 is proved.

In [3] (Corollary 2.3), it is shown that whenever $R_1, \ldots, R_n$ are rings and $n \geq 2$, then the ring $R_1 \times \ldots \times R_n$ is submaximal if and only if either at least one of the R_i is submaximal or $R_i/M_i \cong$ $\cong R_j/M_j$ for two distinct i and j and maximal ideals M_k of R_k, $k = i, j$. Also, in [4] (Theorem 3.17), it is proved that if $\{R_i\}_{i\in I}$ is an infinite family of rings, then $\prod_{i\in I} R_i$ is submaximal. The following is in order.

Proposition 2.2. *Let R be a ring which is not submaximal and $\{M_i\}_{i\in I}$ be a family of maximal ideals in R. Then $T = \prod_{i\in I} R/M_i$ is submaximal if and only if $|I| = \infty$.*

Proof. If I is infinite then we are done by the preceding comments . Hence assume that T is submaximal but $|I| < \infty$. Thus $T = R/M_1 \times R/M_2 \times \ldots \times R/M_n$ is submaximal. Therefore, by the first part of the above comments, either there exists i, $1 \leq i \leq n$, such that R/M_i is submaximal, or there exist $i \neq j$, $1 \leq i, j \leq n$, such that $R/M_i \cong R/M_j$. The first case is not possible since R is not submaximal. We show that the second case is not possible either. To this end, if the second case holds, then we infer that $R/M_i \times R/M_j$ is submaximal, see [1]. Hence $R/(M_i \cap M_j)$ is submaximal which is absurd.

Proposition 2.2 is proved.

Next, we prove a generalization of [3] (Corollary 2.3). Before doing this generalization, we need some observations. In the proof of Theorem 3.17 in [4], it is shown that if $\{E_i\}_{i=1}^{\infty}$ is a family of infinite fields which are not submaximal and $\mathrm{Char}(E_i) = p$, where p is a fixed prime number, then the ring $R = \prod_{i=1}^{\infty} E_i$ is not integral over $\mathbb{Z}_p$. Moreover, it is proved that there exists a maximal ideal M of R such that R/M is not algebraic over $\mathbb{Z}_p$. See also [5] for a more general result. The following is now in order.

Theorem 2.2. *Let $\{R_i\}_{i\in I}$ be a family of rings. Then $R = \prod_{i\in I} R_i$ is submaximal if and only if at least one of the following conditions holds*:

(1) *There exists $i \in I$, such that R_i is submaximal.*

(2) *There exist $i \neq j$ in I, and maximal ideals M_k of R_k, $k = i, j$, such that $\dfrac{R_i}{M_i} \cong \dfrac{R_j}{M_j}$ $\Big($hence in this case R has distinct maximal ideals M and N such that $\dfrac{R}{M} \cong \dfrac{R}{N}\Big)$.*

(3) *There exists a maximal ideal M of R, such that $\dfrac{R}{M}$ is submaximal.*

Proof. If one of the above conditions holds then it is clear that R is submaximal (note, if item (2) holds, then $R_i \times R_j$ is submaximal, by the first part of the comment preceding Proposition 2.2, therefore R is submaximal too). Conversely, if R is submaximal, but conditions (1) and (2) do not hold. We show that item (3) holds. First note that by [3] (Corollary 2.3), I must be an infinite set. Now, For any $i \in I$, let M_i be a maximal ideal of R_i. Since for each $i \in I$, R_i is not submaximal, we infer that $\frac{R_i}{M_i}$ is not submaximal for each $i \in I$. Therefore by Corollary 2.1, $\operatorname{Char}\left(\frac{R_i}{M_i}\right) \neq 0$ for all $i \in I$. Also, note that for each $i, j \in I$, $i \neq j$, we have $\frac{R_i}{M_i} \not\cong \frac{R_j}{M_j}$. Now we have two cases.

1. The set $\left\{\operatorname{Char}\left(\frac{R_i}{M_i}\right) : i \in I\right\}$ is infinite. In this case let $\operatorname{Char}\left(\frac{R_{i_n}}{M_{i_n}}\right) = q_n$, where q_n, $n \in \mathbb{N}$, are distinct prime numbers and $i_n \in I$. Put $T = \prod_{n=1}^{\infty} \frac{R_{i_n}}{M_{i_n}}$. Thus T is a zero-dimensional ring with $\operatorname{Char}(T) = 0$. Hence there exists a prime (and therefore maximal) ideal N of T such that $T \cap \mathbb{Z} = 0$ (note $\mathbb{Z} \setminus \{0\}$ is a multiplicatively closed set in T). Thus $\frac{T}{N}$ is submaximal by Corollary 2.1.

2. The set $\left\{\operatorname{Char}\left(\frac{R_i}{M_i}\right) : i \in I\right\}$ is finite. Thus, in this case, there exists an infinite subset $\{i_n \colon n \in \mathbb{N}\}$ of I, such that $\operatorname{Char}\left(\frac{R_{i_n}}{M_{i_n}}\right) = p \neq 0$ for each $n \geq 1$. Also, we may assume that each $\frac{R_{i_n}}{M_{i_n}}$ is infinite field. (Note, the only finite fields with characteristic p which is not submaximal is $\mathbb{Z}_p$, up to isomorphism. Also note that item (2) is not hold.) Thus by the comments preceding Theorem 2.2, we infer that there exists a maximal ideal N in the ring $T = \prod_{n=1}^{\infty} \frac{R_{i_n}}{M_{i_n}}$ such that T/N is not algebraic over $\mathbb{Z}_p$ and therefore T/N is submaximal by Corollary 2.1.

Finally, let us put $J = I \setminus \{i_n \colon n \in \mathbb{N}\}$ and $M = N \times \prod_{j \in J} \frac{R_j}{M_j}$. Now it is clear that M is a maximal ideal of $S = \frac{\prod_{i \in I} R_i}{\prod_{i \in I} M_i}$ and $\frac{S}{M}$ is submaximal. This immediately implies that (3) holds and we are done.

Theorem 2.2 is proved.

We conclude this section with some observations about the existence of maximal subrings in a ring R, via R-modules. For a ring R let $\operatorname{Simp}(R)$ be the set of all simple R-modules, up to isomorphism. Now, the following is in order.

Proposition 2.3. *Let R be a ring which satisfies at least one of the following conditions*:

(1) *There exists a simple R-module M, such that either $|M| \geq 2^{\aleph_0}$ or M is a torsion free $\mathbb{Z}$-module.*

(2) $|\operatorname{Simp}(R)| > 2^{\aleph_0}$.

Then R is submaximal.

Proof. If (1) holds, then since M is a simple R-module, there exists a maximal ideal $\mathfrak{m}$ in R, such that $M \simeq R/\mathfrak{m}$ as an R-module. Hence either $|R/\mathfrak{m}| \geq 2^{\aleph_0}$ or $\operatorname{Char}(R/\mathfrak{m}) = 0$ and therefore by Corollary 2.1, $R/\mathfrak{m}$ and R are both submaximal. If (2) holds, then as we see in (1), for any simple R-module M, there exists a (unique) maximal ideal $\mathfrak{m}$ of R such that $M \simeq R/\mathfrak{m}$ as R-module. This implies that $|\operatorname{Simp}(R)| = |\operatorname{Max}(R)|$, hence $|\operatorname{Max}(R)| > 2^{\aleph_0}$ and therefore R is submaximal, by [2] (Proposition 2.6).

Proposition 2.3 is proved.

The next results are generalization of [3] (Proposition 2.4) and [4] (Theorem 2.9), respectively.

Proposition 2.4. *Let R be a ring. If there exists an uncountable Artinian R-module M, then R is submaximal.*

Proof. We first recall that Artinian modules over commutative rings are countably generated, see [10] (Corollary 4.2) for a generalization of this fact. Consequently, we may assume that $M = Rm_1+Rm_2+\ldots+Rm_k+\ldots$. Hence we infer that there exists $i \in \mathbb{N}$, such that $|Rm_i| \geq 2^{\aleph_0}$. Now note that $Rm_i \simeq R/\operatorname{ann}(m_i)$, which in turn implies that $R/\operatorname{ann}(m_i)$ is an uncountable Artinian ring. Thus, by [3] (Proposition 2.4), $R/\operatorname{ann}(m_i)$ and therefore R are both submaximal.

Proposition 2.4 is proved.

Proposition 2.5. *Let R be a ring. If there exists an R-module M, such that every cyclic R-submodule of M is Noetherian and $M = \sum_{i\in I} Rm_i$, where $|I| \leq 2^{\aleph_0} < |M|$ (in particular, if M is a Noetherian R-module with $|M| > 2^{\aleph_0}$), then R is submaximal.*

Proof. Since $|M| > 2^{\aleph_0}$, we infer that there exists $i \in I$ such that $|Rm_i| > 2^{\aleph_0}$. Now, we have $Rm_i \simeq R/\operatorname{ann}(m_i)$, as R-modules. Hence $R/\operatorname{ann}(m_i)$ is a Noetherian ring with $|R/\operatorname{ann}(m_i)| > 2^{\aleph_0}$. Thus we are done by [4] (Theorem 2.9).

Proposition 2.5 is proved.

Let R be a ring. By $\operatorname{Soc}(R)$ we mean the sum of all minimal ideals of R. Now the following corollary is immediate.

***Corollary* 2.4.** *Let R be a ring which is not submaximal and $\operatorname{Soc}(R) \neq (0)$. Then either $|\operatorname{Soc}(R)| \leq 2^{\aleph_0}$ or $\operatorname{Soc}(R)$ is not λ-generated where $\lambda \leq 2^{\aleph_0}$.*

3. Hereditary properties between a ring and its maximal subrings. We begin this section with the following interesting result about sharing the Hilbert property between a ring and its maximal subrings. We remind the reader that recently in [5] it is proved that every ring is either submaximal or it is a Hilbert ring. In particular, every Hilbert ring R with $|\operatorname{Spec}(R)| > 2^{2^{\aleph_0}}$ is submaximal. Now the following is in order.

Theorem 3.1. *Let S be a maximal subring of a ring R. Then the following are valid*:

(1) *If S is Hilbert, then R is Hilbert too.*

(2) *If R is Hilbert and is integral over S, then S is Hilbert too.*

Proof. (1) Assume that S is a Hilbert ring, and take $\alpha \in R \setminus S$. Thus $R = S[\alpha]$ and therefore R is Hilbert (note, if S is Hilbert ring, then the polynomial ring $S[x]$ and the epimorphic image of S are Hilbert, see [9]).

(2) We must show that every prime ideal P of S is an intersection of maximal ideals. Since R is integral over S, there exists a prime ideal Q of R such that $P = Q \cap R$. Since R is Hilbert, we infer that there exists a family $\{M_i\}_{i\in I}$ of maximal ideals of R such that $Q = \bigcap_{i\in I} M_i$. Inasmuch as R is integral over S, we infer that $M_i \cap S \in \operatorname{Max}(S)$, for each $i \in I$. Hence $P = Q \cap S = \bigcap_{i\in I}(M_i \cap S)$ and we are done.

Theorem 3.1 is proved.

Definition 3.1. *Let S be a subring of a ring R. We say that S is essential in R and denote it by $S \subseteq_e R$, if $S \cap I \neq 0$, for every nonzero ideal I of R. Also, we say that S is an essential S-submodule of R, if it intersects every nonzero S-submodule of R nontrivially and we denote it by $S \leq_e R$.*

It is manifest that in the above definition $S \leq_e R$ implies that $S \subseteq_e R$. The following is now in order.

Lemma 3.1. *Let S be a maximal subring of a ring R. Then either $S \subseteq_e R$ or $\operatorname{Soc}(R) \neq 0$ (i.e., R has a nonzero minimal ideal).*

Proof. Assume that S is not essential in R. So there exists a nonzero ideal I of R such that $S \cap I = 0$ and therefore $R = S \oplus I$. We show that I is a minimal ideal of R and we are done. Let A be a nonzero ideal of R such that $A \subseteq I$. Hence we have $S \cap A = 0$. Thus $R = S \oplus A$. But $I = A + (I \cap S) = A + 0$, i.e., $I = A$ and we are done.

***Corollary* 3.1.** *Let R be a reduced ring without nontrivial idempotents. Then any maximal subring S of R is essential in R. In particular, any maximal subring of an integral domain is essential.*

Proof. Let S be a maximal subring of R which is not essential. So we infer that $\mathrm{Soc}(R) \neq 0$ by the previous lemma. Hence assume that I is a nonzero minimal ideal in R. Since R is reduced, we infer that $I^2 = I$. Now, let $0 \neq x \in I$, thus we have $I = Rx = Rx^2$ and therefore $x = rx^2$ for some $r \in R$. Now by putting $e = rx$ we infer that e is a nontrivial idempotent, which is a contradiction. The last part is now evident.

***Remark* 3.1.** We recall that whenever S is a maximal subring of a ring R, then R is algebraic over S (note, for any $x \in R$, either $x^2 \in S$ or $x \in S[x^2]$). Now assume that R is an integral domain and S is a maximal subring of R which is integrally closed in R. Then one can easily see that R must be an overring of S, i.e., $S \subset R \subseteq Q(S)$. In particular, in the latter case $Q(R) = Q(S)$. Moreover, if R is integrally closed, and S is a maximal subring of R, which is integrally closed in R, then S is integrally closed too. To see this, let $E = Q(S) = Q(R)$, then we have $S \subseteq R \subseteq E$. Hence $S \subseteq S' \subseteq R' = R$, where S' and R' are the integral closures of S and R in E, respectively. Thus $S' = S$, since S is a maximal subring of R which is integrally closed in it.

It is clear that whenever S is a subring of a ring R, then for any prime ideal P of S, there exists a prime ideal Q of R such that $S \cap Q \subseteq P$ (note, we might have $0 = S \cap Q \subseteq P$). In what follows we show that if R is an integral domain and S is a maximal subring of R which is integrally closed in R, then $S \cap Q \neq 0$ whenever $P \neq 0, \mathrm{cru}_R(S)$.

Lemma 3.2. *Let R be an integral domain and S be a maximal subring of R which is integrally closed in R. Then for any $0 \neq P \in \mathrm{Spec}(S) \setminus \{\mathrm{cru}_R(S)\}$ there exists a nonzero prime ideal Q of R such that $0 \neq Q \cap S \subseteq P$.*

Proof. First note that, we have $S_P = R_{S \setminus P}$, by Theorem 1.2. Since S_P is a local integral domain which is not a field, we infer that $R_{S\setminus P}$ is also a local ring with a nonzero unique maximal ideal $Q_{S\setminus P}$. Hence $Q \neq 0$ and therefore $0 \neq Q \cap S \subseteq P$, by Corollary 3.1.

Lemma 3.2 is proved.

The following interesting result is now in order.

Proposition 3.1. *Let R be an integral domain and S be a maximal subring of R which is integrally closed in R. Then we have the following statements*:

(1) *If* $\dim(R) = 1$, *then* $\dim(S) = 1$ *if and only if* $(S : R) = 0$.

(2) *If* $\dim(S) = 1$, *then* $\dim(R) = 1$ *if and only if* $(S : R) = 0$.

Proof. (1) Assume that $\dim(S) = \dim(R) = 1$. We show that $(S : R) = 0$. We have $(S : R) \in \mathrm{Spec}(S) \setminus \mathrm{Max}(S)$, by Theorem 1.1, therefore $(S : R) = 0$. Conversely, let $0 \neq P \in \mathrm{Spec}(S) \setminus \{\mathrm{cru}_R(S)\}$, thus by the above lemma, there exists a nonzero prime ideal Q of R, such that $0 \neq Q \cap S \subseteq P$. But $Q \nsubseteq S$, since $(S : R) = 0$. Thus we infer that $Q \cap S \in \mathrm{Max}(S)$, (note, $S + Q = R$, $R/Q \cong S/(S \cap Q)$ and $Q \in \mathrm{Max}(R)$, since $\dim(R) = 1$). Thus $Q \cap S = P \in \mathrm{Max}(S)$ and we are done. The proof of part (2) is similar to the proof of part (1).

Proposition 3.1 is proved.

Now we are ready to present the following fact about sharing the Dedekind property between a ring and its maximal subrings, which is an immediate corollary of Remark 3.1, Proposition 3.1 and [9] (Theorem 96).

***Corollary* 3.2.** *Let R be an integral domain and S be a maximal subring of R which is integrally closed in R. Then the following statements hold*:

(1) *If S is a Dedekind domain, then R is a Dedekind domain and $(S : R) = 0$.*

(2) *If R is a Dedekind domain, then S is a Dedekind domain if and only if S is Noetherian and $(S : R) = 0$.*

***Remark* 3.2.** Let us record the simple needed fact which follows. If R is an integral domain and S is a maximal subring of R. Then the field $Q(R)$ is a simple (finite) algebraic extension of the field $Q(S)$ (note, we may have $Q(R) = Q(S)$, e.g., when S is integrally closed in R, see Remark 3.1), and when R is integral over S, then in fact for any $x \in R \setminus S$, we have $Q(R) = Q(S)[x]$, for $R = S[x]$ implies that $R \subseteq Q(S)[x]$, hence $Q(R) = Q(S)[x]$.

It is manifest that if R is an integral domain then $\mathrm{Soc}(R) = 0$. In what follows we give a kind of the converse of this fact. Assume that S is an integral domain which is a maximal subring of a ring R (note, one can easily show the interesting fact, essentially due to D. E. Dobbs, that any ring S is a maximal subring of a bigger ring R, see the introduction of [4]), then R is an integral domain if and only if $\mathrm{Soc}(R) = 0$. For the converse, note that by [7] (Theorem 2.7), since $\mathrm{Soc}(R) = 0$ we infer that R must be an integral domain.

Theorem 3.2. *Let S be a nonfield maximal subring of a ring R, such that S is a Dedekind domain. Then the following statements hold*:

(1) *If R is integral over S, then R is a Dedekind domain if and only if $R = S'_E$ where $E = Q(R)$.*

(2) *If R is an integral domain, then R/I is an Artinian ring for any nonzero ideal I of R.*

Proof. For item (1), if R is a Dedekind domain, then since $S \subset R \subseteq E$, we infer that $R \subseteq S'_E \subseteq \subseteq R'_E = R$. Thus $S'_E = R$. Conversely, if $R = S'_E$, then by [9] (Theorem 98) and the previous remark R is a Dedekind domain too. For item (2), assume that I is a nonzero ideal of R, if $I \subseteq S$, then S/I is an Artinian maximal subring of R/I (note, S is a Dedekind domain). Hence by [4] (Theorem 3.8) we conclude that R/I is an Artinian ring too. If $I \nsubseteq S$, then $S + I = R$ and therefore $R/I \cong S/(S \cap I)$, and we note that $S \cap I \neq 0$ by Lemma 3.1. Hence R/I is Artinian and we are done.

Theorem 3.2 is proved.

In the next two results we give some properties of maximal subrings in zero-dimensional rings and one-dimensional valuation domains. We recall that, if S is a maximal subring of a ring R, then S is zero-dimensional if and only if R is zero-dimensional and integral over S, see [4] and [5] for more similar properties. The following result, which concerns the integrally closed maximal subrings of a zero-dimensional ring, is now in order.

Proposition 3.2. *Let R be a zero-dimensional ring and S be a maximal subring of it which is integrally closed in R. Then the following statements hold*: (1) *$S/(S : R)$ is one-dimensional valuation domain with a unique nonzero prime ideal* $\mathrm{cru}_R(S)/P$. (2) $\mathrm{Max}(S) = \{\mathrm{cru}_R(S)\} \cup \{N \cap S \mid N \in \mathrm{Max}(R),\ N \nsubseteq S\}$. (3) $\mathrm{Spec}(S) = \{(S : R)\} \cup \mathrm{Max}(S)$. (4) $\dim(S) = 1$ *and in fact* $(S : R) \subset \mathrm{cru}_R(S)$ *is the only chain of length* 1 *in* S.

Proof. Assume that $P = (S : R)$ and $M = \mathrm{cru}_R(S)$. Hence by Theorem 1.1, we infer that $P \in \in \mathrm{Spec}(R) \cap \mathrm{Spec}(S)$ but $P \notin \mathrm{Max}(S)$; also by Theorem 1.2, we have $P \subset M$. Since $\dim(R) = 0$, we infer that P is a maximal ideal of R. Thus S/P is a maximal subring of the field R/P. Hence S/P is one-dimensional valuation domain with a unique nonzero prime ideal M/P. This proves (1). Now by Theorem 1.1, for any prime ideal $Q \in \mathrm{Spec}(S) \setminus \{M\}$, we have $S_Q = R_{S \setminus Q}$. But S_Q is a local ring, hence $R_{S\setminus Q}$ is local too. Thus let $Q'_{S\setminus Q}$ be the unique prime ideal of $R_{S\setminus Q}$. Hence $Q' \in \mathrm{Max}(R)$. Now we have the following two cases.

1. If $Q = P$, then $Q' = P$.

2. If $Q \neq P$, then we first show that $Q' \nsubseteq S$. To see this, let $Q' \subseteq S$, hence $Q' = P$. But $P = Q' \cap S = Q' \subseteq Q$, hence $P \subseteq Q$ and therefore by (1) we have $Q = M$, which is a contradiction (note, $S_M = S_Q$ is a maximal subring of $R_{S\setminus M}$, but $S_Q = R_{S\setminus Q}$). Thus we have $Q' \nsubseteq S$ and therefore $Q' \cap S \in \mathrm{Max}(S)$ (note $S + Q' = R$). Since $Q' \cap S \subseteq Q$, we infer that $Q' \cap S = Q$.

Hence, every prime ideal $Q \neq P$ of S is maximal. Thus (2), (3) and therefore (4) hold.

Proposition 3.2 is proved.

Proposition 3.3. *Let (V, Q) be a one-dimensional valuation domain. If R is a maximal subring of V, then either R is one dimensional or R is a local integral domain with* $\dim(R) = 2$.

Proof. If V is integral over R or R is integrally closed in V and $(R : V) = 0$, then $\dim(R) = 1$, by Proposition 3.1. Hence assume that R is integrally closed in V, $(R : V) \neq 0$ and $M = \mathrm{cru}_R(S)$. Hence by Theorems 1.1 and 1.2, we infer that $(R : V) = Q \subset M$ and therefore R/Q is a one-dimensional valuation domain with a unique nonzero prime ideal M/Q (note, R/Q is a nonfield maximal subring of the field V/Q). Now assume that $P \neq 0, M$ be any prime ideal in R. Hence by Theorem 1.2, we have $R_P = V_{R\setminus P}$. But R_P is not a field, hence we infer that $V_{R\setminus P}$ is also not a field. Since V is a maximal subring of $Q(V)$, see [9, p. 43] (Ex. 29), we infer that $V = V_{R\setminus P}$. Thus $R_P = V$ and therefore $ht(P) = 1$. This shows that the only chain of prime ideals of maximum length in R is $0 \subset Q \subset M$ and therefore $\dim(R) = 2$. Now it remains to be shown that R is local. Note that R_M is a maximal subring of V_M and also $R \subseteq R_M \subset V_M$ and $V \subseteq V_M \subseteq Q(V)$. Since V is a maximal subring of $Q(V)$ we infer that either $V_M = Q(V)$ which implies that R_M is a maximal subring of $Q(V)$ or $V = V_M$. Since $\dim(R_M) = 2$, we infer that R_M cannot be a maximal subring of $Q(V)$, see [9, p. 43] (Ex. 29). Therefore $V = V_M$ and R_M is a maximal subring of V which contains R, hence $R = R_M$ and we are done.

Proposition 3.3 is proved.

***Corollary* 3.3.** *Let R be a ring with* $\dim(R) = n < \infty$ *and S be a maximal subring of R. Then* $\dim(S) \leq n + 1$.

Proof. If R is integral over S then we are done. Hence assume that R is not integral over S and $M = \mathrm{cru}_R(S)$. Thus for any prime ideal $P \neq M$ of S we have $S_P = R_{S\setminus P}$, by Theorem 1.2. Therefore $ht(P) = \dim(R_{S\setminus P}) \leq \dim(R) = n$. This also shows that $ht(M) \leq n + 1$ and therefore $\dim(S) \leq n + 1$ and we are done.

Proposition 3.4. *Let S be a maximal subring of a ring R. Then the following statements hold*:

(1) *S is reduced if and only if either R is reduced or $R/N(R) \cong S$.*

(2) *If S is reduced and S is integrally closed in R, then R is reduced.*

Proof. (1) It is clear that if R is reduced or $R/N(R) \cong S$, then S is reduced too. Conversely, if S is reduced, then either R is reduced and we are done or $N(R) \neq 0$. Now since S is reduced we infer that $S \cap N(R) = 0$. Also, since S is maximal subring of R we have $S + N(R) = R$ and therefore $S \cong R/N(R)$.

(2) If S is integrally closed in R and $x \in N(R)$, then x is integral over S and therefore $x \in S$, i.e., $x = 0$ and we are done.

Proposition 3.4 is proved.

It is clear that if $S \subseteq R$ is an integral extension of rings, then $J(R) \cap S = J(S)$. Also we recall that if S is a maximal subring of a ring R, then S is Artinian if and only R is Artinian and it is integral over S, see [4] (Theorem 3.8). By a semisimple ring we mean an Artinian ring whose

Jacobson radical is zero. It is clear that a commutative ring is semisimple if and only if it is a finite direct product of fields.

Theorem 3.3. *Let S be a maximal subring of a ring R. Then the following statements hold*:

(1) *If S is a semisimple ring, then either R is semisimple or R is Artinian and $R/J(R) \cong S$ as rings.*

(2) *If R is a semisimple ring, then S is a semisimple ring if and only if S is zero-dimensional.*

Proof. (1) Since S is semisimple we infer that R is Artinian and integral over S by [4] (Theorem 3.8). Hence by the preceding comments, we conclude that $J(R) \cap S = J(S) = 0$. Now we have two cases, either $J(R) = 0$ and therefore R is semisimple, or $J(R) \neq 0$, thus $J(R) \nsubseteq S$ and therefore $S + J(R) = R$, hence we infer that $R/J(R) \cong S$ and we are done. (2) If S is semisimple then S is zero-dimensional and we are done. Conversely, assume that S is zero-dimensional. Since R is reduced we infer that S is reduced and therefore S is von Neumann regular (note, it is well-known that a zero-dimensional reduced ring is von Neumann regular). Also since S is zero dimensional we infer that $(S : R) \in \operatorname{Max}(S)$ and therefore R is integral over S, by Theorem 1.1. Hence again by the preceding comments, we conclude that S is Artinian too and since it is von Neumann regular, we infer that S is semisimple.

Theorem 3.3 is proved.

The ring R is called semiprimary if $J(R)$ is nilpotent and $R/J(R)$ is Artinian (semisimple). One can easily see that if R is a semiprimary ring, then R must be zero-dimensional. Next, we are interested in maximal subrings which are semiprimary.

Theorem 3.4. *Let S be a maximal subring of a ring R. Then the following statements hold*:

(1) *If R is a semiprimary ring, then S is a semiprimary ring if and only if R is integral over S.*

(2) *If S is a semiprimary ring, then R is semiprimary if and only if $J(R)$ is nilpotent.*

Proof. (1) If S is semiprimary, then S is zero-dimensional. Hence $(S : R) \in \operatorname{Max}(S)$ and therefore R is integral over S, by Theorem 1.1. Conversely, if R is integral over S, then $J(S) = J(R) \cap S$, by the comment preceding Theorem 3.3. Hence $J(S)$ is nilpotent. Now we have two cases.

(1) $J(R) \subseteq S$ and therefore $J(R) = J(S) = J$. Hence S/J is a maximal subring of R/J and R/J is integral over S/J. Thus S/J is Artinian by [4] (Theorem 3.8), and therefore S is semiprimary.

(2) $J(R) \nsubseteq S$, hence $S + J(R) = R$, since S is a maximal subring of R. The latter equality implies that $S/J(S) \cong R/J(R)$ and therefore $S/J(S)$ is Artinian. Consequently, S is semiprimary and we are done. Finally, the proof of (2) is similar to the part (1).

Theorem 3.4 is proved.

For the next result we need to recall some definitions, see [9]. A Noetherian local ring (R, M) is called a regular ring, if $\dim(R) = v \dim_K(M/M^2)$, where $K = R/M$. Now assume that R is a Noetherian ring, not necessary local, R is called regular if R_m is regular for every maximal ideal m of R. Now let R and S be rings, $f : S \longrightarrow R$ be a ring homomorphism and $pd_S(R) = n$ (the projective dimension of R as an S-module). Then for any right R-module M, we have $pd_S(M) \leq n + pd_R(M)$, see [11, p. 204] (Ex. 25).

Theorem 3.5. *Let S be a maximal subring of a regular ring R. If R is integral over S and $pd_S(R) < \infty$, then S is regular too.*

Proof. It is manifest that S is Noetherian, see [4]. Hence to show that S is regular it suffices to prove that for any maximal ideal M of S we have $pd_S(M) < \infty$, see [11] (Theorem 5.94). Considering the exact sequence $0 \to M \to S \to S/M \to 0$, we may show that $pd_S(S/M) < \infty$, for any maximal ideal M of S, see [11] (Theorem 5.20). Hence assume that M is a maximal ideal of S, since R is integral over S there exists a maximal ideal N of R such that $M = S \cap N$. Now we have two cases:

(1) If $N \subseteq S$, then $N = M$, hence by the previous comment we have $pd_S(N) \leq pd_S(R) + pd_R(N)$, thus $pd_S(N) < \infty$ (note, $pd_R(N) < \infty$, for R is regular).

(2) If $N \nsubseteq S$, then we consider two cases, either $S \cap N = 0$, in which case $R/N \cong S$ (note, $S + N = R$), hence S is field and we are done, or $S \cap N \neq 0$. In the latter case, we note that $R/N = (S+N)/N \simeq S/(S \cap N) = S/M$ as S-modules. But $pd_S(R/N)$ is finite, for by the above comment $pd_S(R/N) \leq pd_S(R) + pd_R(R/N) < \infty$. Hence $pd_S(S/M) < \infty$ and we are done.

Theorem 3.5 is proved.

We conclude this article with the next result.

Proposition 3.5. *Let S be a maximal subring of a ring R such that S is a von Neumann regular and R is self-injective. Then R_S is injective. Moreover at least one of the following holds*:

(1) *If $S \nsubseteq_e R$, then S is a self-injective ring.*

(2) *If $S \subseteq_e R$, then R is a von Neumann regular ring.*

Proof. Since S is von Neumann regular, we infer that R_S is flat, by [11] (Theorem 4.21). Hence R_S is injective by [11] (Corollary 3.6A). Now consider two cases.

(1) If $S \nsubseteq_e R$, then there exists a nonzero ideal I of R, such that $I \cap S = 0$. Hence $R = I \oplus S$, therefore S is injective as an S-module and we are done.

(2) If $S \subseteq_e R$, then $I \cap S \neq 0$, for each nonzero ideal I of R. But we have $J(R) \cap S = 0$ (note, S is a von Neumann regular subring of R), hence $J(R) = 0$. Thus R is reduced. Therefore $\mathcal{Z}(R) = 0$ (the singular ideal of R), by [11] (Lemma 7.8). To complete the proof, let us recall a general fact (it is even true in the noncommutative case and it is attributed to many people), namely, if R is self-injective then the Jacobson radical of R coincides with its singular ideal and $R/J(R)$ is a self-injective von Neumann regular ring, see the comment preceding [11] (Theorem 13.1), for some of the names of the above people and a generalization of this fact. Hence R is a von Neumann regular ring.

Proposition 3.5 is proved.

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