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AN EXTENDED TAUBERIAN THEOREM FOR THE WEIGHTED MEAN METHOD OF SUMMABILITY

УЗАГАЛЬНЕНА ТАУБЕРОВА ТЕОРЕМА ДЛЯ МЕТОДУ ЗВАЖЕНОГО СЕРЕДНЬОГО ДЛЯ ЗНАХОДЖЕННЯ СУМ

We prove a new Tauberian-like theorem that establishes the slow oscillation of a real sequence $u = (u_n)$ on the basis of the weighted mean summability of its generator sequence $(V_{n,p}^{(0)}(\Delta u))$ and some conditions.

Доведено нову теорему тауберового типу, яка встановлює повільні коливання дійсної послідовності $u = (u_n)$ на основі збіжності її генеруючої послідовності $(V_{n,p}^{(0)}(\Delta u))$ у зважених середніх та певних умов.

1. Introduction. Let $u = (u_n)$ be a sequence of real numbers. Assume that $p = (p_n)$ is a sequence of nonnegative numbers with $p_0 > 0$ such that

$$P_n = \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty.$$

The nth weighted mean of (u_n) is defined by

$$\sigma_{n,p}^{(1)}(u) = \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k.$$

A sequence (u_n) is said to be summable by the weighted mean method determined by the sequence p, in short; $(\overline{N}, p)$ summable to a finite number s if

$$\lim_{n\to\infty} \sigma_{n,p}^{(1)}(u) = s. \tag{1}$$

If the limit

$$\lim_{n\to\infty} u_n = s \tag{2}$$

exists, then (1) also exists. However, the converse is not always true. Notice that (1) may imply (2) under a certain condition which is called a Tauberian condition. Any theorem which states that convergence of sequences follows from $(\overline{N}, p)$ summability method and some Tauberian condition is said to be a Tauberian theorem.

If $p_n = 1$ for all nonnegative n, then $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

The difference between u_n and its nth weighted mean $\sigma_{n,p}^{(1)}(u)$ which is called the weighted Kronecker identity is given by the identity

$$u_n - \sigma_{n,p}^{(1)}(u) = V_{n,p}^{(0)}(\Delta u), \tag{3}$$

where $V_{n,p}^{(0)}(\Delta u) := \dfrac{1}{P_n} \sum_{k=1}^{n} P_{k-1} \Delta u_k$.

Since $\dfrac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(1)}(u)=V_{n,p}^{(0)}(\Delta u)$, the Kronecker identity can be written as

$$u_n=V_{n,p}^{(0)}(\Delta u)+\sum_{k=1}^{n}\frac{p_k}{P_{k-1}}V_{k,p}^{(0)}(\Delta u). \tag{4}$$

Because of the identity (4), the sequence $\left(V_{n,p}^{(0)}(\Delta u)\right)$ is called the generator sequence of (u_n).

For each integer $m\geq 0$, we define $\sigma_{n,p}^{(m)}(u)$ and $V_{n,p}^{(m)}(\Delta u)$ by

$$\sigma_{n,p}^{(m)}(u)=\begin{cases}\dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n}p_k\sigma_{k,p}^{(m-1)}(u), & m\geq 1,\\ u_n, & m=0,\end{cases}$$

and

$$V_{n,p}^{(m)}(\Delta u)=\begin{cases}\dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n}p_kV_{k,p}^{(m-1)}(\Delta u), & m\geq 1,\\ V_{n,p}^{(0)}(\Delta u), & m=0,\end{cases}$$

respectively.

The weighted classical control modulo of (u_n) is denoted by $\omega_{n,p}^{(0)}(u)=\dfrac{P_{n-1}}{p_n}\Delta u_n$ and the weighted general control modulo of integer order $m\geq 1$ of (u_n) is defined in [1] by $\omega_{n,p}^{(m)}(u)=$ $=\omega_{n,p}^{(m-1)}(u)-\sigma_{n,p}^{(1)}(\omega^{m-1}(u))$.

If $p_n=1$ for all nonnegative n, then the weighted classical and general control modulo reduce to the classical and general control modulo, respectively. The classical and general control modulo have been used as Tauberian conditions for various summability methods [2–5].

For a sequence $u=(u_n)$, we define

$$\left(\frac{P_{n-1}}{p_n}\Delta\right)_m u_n=\left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1}\left(\frac{P_{n-1}}{p_n}\Delta u_n\right)=\frac{P_{n-1}}{p_n}\Delta\left(\left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1}u_n\right),$$

where $\left(\dfrac{P_{n-1}}{p_n}\Delta\right)_0 u_n=u_n$, and $\left(\dfrac{P_{n-1}}{p_n}\Delta\right)_1 u_n=\dfrac{P_{n-1}}{p_n}\Delta u_n$.

Note that by the definition of the weighted general control modulo,

$$\omega_{n,p}^{(1)}(u)=\omega_{n,p}^{(0)}(u)-\sigma_{n,p}^{(1)}(\omega^{(0)}(u))=\frac{P_{n-1}}{p_n}\Delta u_n-V_{n,p}^{(0)}(\Delta u)=\frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(0)}(\Delta u)$$

and

$$\omega_{n,p}^{(2)}(u)=\omega_{n,p}^{(1)}(u)-\sigma_{n,p}^{(1)}(\omega^{(1)}(u))=$$

$$=\frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(0)}(\Delta u)-\frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(1)}(\Delta u)=\left(\frac{P_{n-1}}{p_n}\Delta\right)_2 V_{n,p}^{(1)}(\Delta u).$$

A sequence (u_n) is said to be slowly oscillating [6] if

$$\lim_{\lambda\to 1^+}\limsup_{n\to\infty}\max_{n+1\leq k\leq[\lambda n]}|u_k-u_n|=0. \tag{5}$$

Denote by $\mathcal{S}$ the class of slowly oscillating sequences.

The weighted de la Vallée Poussin means of (u_n) are defined by

$$\tau^{>}_{n,[\lambda n],p}(u)=\frac{1}{P_{[\lambda n]}-P_n}\sum_{k=n+1}^{[\lambda n]}p_k u_k$$

for $\lambda>1$ and sufficiently large n, and

$$\tau^{<}_{n,[\lambda n],p}(u)=\frac{1}{P_n-P_{[\lambda n]}}\sum_{k=[\lambda n]+1}^{n}p_k u_k$$

for $0<\lambda<1$ and sufficiently large n.

For the definitions of O and o we refer to [7, p. 149].

A number of authors such as Hardy [7], Tietz [8], Tietz and Zeller [9], and Móricz and Rhoades [10] obtained Tauberian theorems for $(\overline{N},p)$ summability method. Tietz [8], Tietz and Zeller [9] established Tauberian conditions controlling the oscillatory behavior of sequences for $(\overline{N},p)$ summability method. Móricz and Rhoades [10] obtained necessary and sufficient conditions for $(\overline{N},p)$ summable (u_n) to be convergent. Hardy [7] proved that $\omega^{(0)}_{n,p}(u)=O(1)$ is a Tauberian condition for the weighted mean summability method. Recently, Çanak and Totur [1] have shown that under some certain conditions imposed on the sequence $p=(p_n)$ the condition

$$\omega^{(1)}_{n,p}(u)\geq -C \tag{6}$$

for some positive constant C is a Tauberian condition for $(\overline{N},p)$ summability method.

Instead of recovering convergence of (u_n) out of the existence of (1) and additional condition imposed on the sequences (u_n) and $p=(p_n)$, we can obtain more general information on (u_n) by replacing $(\overline{N},p)$ summability of (u_n) by $(\overline{N},p)$ summability of its generator sequence $\big(V^{(0)}_{n,p}(\Delta u)\big)$.

In this paper, we shall prove the following extended Tauberian theorem.

Theorem 1. *Let*

$$\frac{P_{n-1}}{p_n}=O(n). \tag{7}$$

For a real sequence $u=(u_n)$ let there exist a nonnegative sequence $M=(M_n)$ with $\left(\sum_{k=1}^{n}\frac{M_k}{k}\right)\in$ $\in\mathcal{S}$ such that

$$\omega^{(2)}_{n,p}(u)\geq -M_n, \tag{8}$$

$$\limsup_{n\to\infty}\left(\frac{P_{[\lambda n]}-P_n}{P_n}\right)\limsup_{n\to\infty}\tau^{>}_{n,[\lambda n],p}(M)=o(1),\quad \lambda\to 1^{+}, \tag{9}$$

and

$$\limsup_{n\to\infty}\left(\frac{P_n-P_{[\lambda n]}}{P_{[\lambda n]}}\right)\limsup_{n\to\infty}\tau^{<}_{n,[\lambda n],p}(M)=o(1),\quad \lambda\to 1^{-}. \tag{10}$$

If $\big(V^{(0)}_{n,p}(\Delta u)\big)$ is $(\overline{N},p)$ summable to s, then $u=(u_n)$ is slowly oscillating.

If we take $p_n=1$ for all n, we have Theorem 2.1 in [11].

2. Auxiliary results. For the proof of Theorem 1 we shall need the following lemmas.

Lemma 1 [1]. *Let $v = (v_n)$ be a sequence of real numbers.*

(i) *For $\lambda > 1$ and sufficiently large n,*

$$v_n - \sigma^{(1)}_{n,p}(v) = \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(\sigma^{(1)}_{[\lambda n],p}(v) - \sigma^{(1)}_{n,p}(v)\right) - \frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k(v_k - v_n),$$

where $[\lambda n]$ denotes the integer part of λn.

(ii) *For $0 < \lambda < 1$ and sufficiently large n,*

$$v_n - \sigma^{(1)}_{n,p}(v) = \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}}\left(\sigma^{(1)}_{n,p}(v) - \sigma^{(1)}_{[\lambda n],p}(v)\right) + \frac{1}{P_n - P_{[\lambda n]}}\sum_{k=[\lambda n]+1}^{n} p_k(v_n - v_k).$$

The next lemma represents the difference between the weighted de la Vallée Poussin means and the weighted means of sequence (v_n).

Lemma 2. *Let $v = (v_n)$ be a sequence of real numbers.*

(i) *For $\lambda > 1$ and sufficiently large n,*

$$\tau^{>}_{n,[\lambda n],p}(v) - \sigma^{(1)}_{n,p}(v) = \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(\sigma^{(1)}_{[\lambda n],p}(v) - \sigma^{(1)}_{n,p}(v)\right).$$

(ii) *For $0 < \lambda < 1$ and sufficiently large n,*

$$\tau^{<}_{n,[\lambda n],p}(v) - \sigma^{(1)}_{n,p}(v) = \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}}\left(\sigma^{(1)}_{n,p}(v) - \sigma^{(1)}_{[\lambda n],p}(v)\right).$$

Proof. (i) From the definition of the weighted de la Vallée Poussin means of (v_n) we have, for $\lambda > 1$,

$$\tau^{>}_{n,[\lambda n],p}(v) = \frac{1}{P_{[\lambda n]} - P_n}\sum_{j=n+1}^{[\lambda n]} p_j v_j = \frac{1}{P_{[\lambda n]} - P_n}\left(\sum_{j=0}^{[\lambda n]} p_j v_j - \sum_{j=0}^{n} p_j v_j\right) =$$

$$= \frac{1}{P_{[\lambda n]} - P_n}\left(P_{[\lambda n]}\sigma^{(1)}_{[\lambda n],p}(v) - P_n\sigma^{(1)}_{n,p}(v)\right) =$$

$$= \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\sigma^{(1)}_{[\lambda n],p}(v) - \frac{P_n}{P_{[\lambda n]} - P_n}\sigma^{(1)}_{n,p}(v) =$$

$$= \sigma^{(1)}_{n,p}(v) + \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(\sigma^{(1)}_{[\lambda n],p}(v) - \sigma^{(1)}_{n,p}(v)\right),$$

which proves Lemma 2 (i).

(ii) From the definition of the weighted de la Vallée Poussin means of (v_n) we have, for $0 < \lambda < 1$,

$$\tau^{<}_{n,[\lambda n],p}(v) = \frac{1}{P_n - P_{[\lambda n]}}\sum_{j=[\lambda n]+1}^{n} p_j v_j = \frac{1}{P_n - P_{[\lambda n]}}\left(\sum_{j=0}^{n} p_j v_j - \sum_{j=0}^{[\lambda n]} p_j v_j\right) =$$

$$= \frac{1}{P_n - P_{[\lambda n]}} \Big(P_n \sigma^{(1)}_{n,p}(v) - P_{[\lambda n]} \sigma^{(1)}_{[\lambda n],p}(v) \Big) =$$

$$= \frac{P_n}{P_n - P_{[\lambda n]}} \sigma^{(1)}_{n,p}(v) - \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}} \sigma^{(1)}_{[\lambda n],p}(v) =$$

$$= \sigma^{(1)}_{n,p}(v) + \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}} \Big(\sigma^{(1)}_{n,p}(v) - \sigma^{(1)}_{[\lambda n],p}(v) \Big),$$

which proves Lemma 2 (ii).

The next lemma states that if (v_n) is $(\overline{N}, p)$ summable to s, then the sequence of the weighted de la Vallée Poussin means of (v_n) converges to s.

Lemma 3. *If (v_n) is $(\overline{N}, p)$ summable to s, then*

(i) $\lim_{n\to\infty} \tau^{>}_{n,[\lambda n],p}(v) = s$,

(ii) $\lim_{n\to\infty} \tau^{<}_{n,[\lambda n],p}(v) = s$.

Proof. (i) Since

$$\lim_{n\to\infty} \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n} \Big(\sigma^{(1)}_{[\lambda n],p}(v) - \sigma^{(1)}_{n,p}(v) \Big) = 0,$$

we have

$$\lim_{n\to\infty} \tau^{>}_{n,[\lambda n],p}(v) = \lim_{n\to\infty} \sigma^{(1)}_{n,p}(v) = s.$$

Proof of (ii) is similar.

Lemma 4. *For a real sequence $v = (v_n)$ let there exist a nonnegative sequence $M = (M_n)$ such that*

$$\frac{P_{n-1}}{p_n} \Delta v_n \geq -M_n.$$

Then

(i) $-\Big(\tau^{>}_{n,[\lambda n],p}(v) - v_n \Big) \leq \dfrac{P_{[\lambda n]} - P_n}{P_n} \tau^{>}_{n,[\lambda n],p}(M)$,

(ii) $v_n - \tau^{<}_{n,[\lambda n],p}(v) \geq -\dfrac{P_n - P_{[\lambda n]}}{P_{[\lambda n]}} \tau^{<}_{n,[\lambda n],p}(M)$.

Proof. (i) Since $\Delta v_n \geq -\dfrac{p_n}{P_{n-1}} M_n$, we have

$$- \sum_{j=n+1}^{k} \Delta v_j = -(v_k - v_n) \leq \sum_{j=n+1}^{k} \frac{p_j}{P_{j-1}} M_j. \tag{11}$$

Multiplying the inequality (11) by p_k, summing the resulting inequality from $k = n+1$ to $[\lambda n]$ and then dividing it by $P_{[\lambda n]} - P_n$, we get

$$-(\tau^{>}_{n,[\lambda n],p}(v) - v_n) = -\frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} (p_k v_k - p_k v_n) \leq$$

$$\leq \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k \sum_{j=n+1}^{k} \frac{p_j}{P_{j-1}} M_j \leq$$

$$\leq \frac{P_{[\lambda n]} - P_n}{P_n} \left(\frac{1}{P_{[\lambda n]} - P_n} \sum_{j=n+1}^{[\lambda n]} p_j M_j \right) =$$

$$= \frac{P_{[\lambda n]} - P_n}{P_n} \tau^{>}_{n,[\lambda n],p}(M).$$

(ii) Since $\Delta v_n \geq -\frac{p_n}{P_{n-1}} M_n$, we have

$$\sum_{j=k+1}^{n} \Delta v_j = v_n - v_k \geq -\sum_{j=k+1}^{n} \frac{p_j}{P_{j-1}} M_j. \tag{12}$$

Multiplying the inequality (12) by p_k, summing the resulting inequality from $k = [\lambda n] + 1$ to n and then dividing it by $P_n - P_{[\lambda n]}$, we get

$$v_n - \tau^{<}_{n,[\lambda n],p}(v) = \frac{1}{P_n - P_{[\lambda n]}} \sum_{k=[\lambda n]+1}^{n} (p_k v_n - p_k v_k) \geq$$

$$\geq -\frac{1}{P_n - P_{[\lambda n]}} \sum_{k=[\lambda n]+1}^{n} p_k \sum_{j=k+1}^{n} \frac{p_j}{P_{j-1}} M_j =$$

$$= -\frac{P_n - P_{[\lambda n]}}{P_{[\lambda n]}} \left(\frac{1}{P_n - P_{[\lambda n]}} \sum_{j=[\lambda n]+1}^{n} p_j M_j \right) = -\frac{P_n - P_{[\lambda n]}}{P_{[\lambda n]}} \tau^{<}_{n,[\lambda n],p}(M).$$

3. Proof of Theorem 1. Since $(V^{(0)}_{n,p}(\Delta u))$ is $(\overline{N}, p)$ summable to s, $(V^{(1)}_{n,p}(\Delta u))$ is convergent to s, and then $\left(V^{(2)}_{n,p}(\Delta u)\right)$ is convergent to s. Applying the identity (3) to $(V^{(1)}_{n,p}(\Delta u))$, we obtain that

$$V^{(1)}_{n,p}(\Delta u) - V^{(2)}_{n,p}(\Delta u) = V^{(0)}_{n,p}(\Delta V^{(1)}(\Delta u)) = \frac{P_{n-1}}{p_n} \Delta V^{(2)}_{n,p}(\Delta u).$$

Therefore, we have $\frac{P_{n-1}}{p_n} \Delta V^{(2)}_{n,p}(\Delta u) = o(1)$. Under the assumptions of Theorem 1 we now prove that $\frac{P_{n-1}}{p_n} \Delta V^{(1)}_{n,p}(\Delta u) = o(1)$. Let $v_n := \frac{P_{n-1}}{p_n} \Delta V^{(1)}_{n,p}(\Delta u)$. Then $\sigma^{(1)}_{n,p}(v) = \frac{P_{n-1}}{p_n} \Delta V^{(2)}_{n,p}(\Delta u)$. Applying Lemma 1 (i) to v_n, and using Lemma 2 (i) and Lemma 4 (i), we have

$$v_n - \sigma^{(1)}_{n,p}(v) \leq \tau^{>}_{n,[\lambda n],p}(v) - \sigma^{(1)}_{n,p}(v) + \frac{P_{[\lambda n]} - P_n}{P_n} \tau^{>}_{n,[\lambda n],p}(M). \tag{13}$$

Taking $\limsup$ of both sides of (13), we obtain

$$\limsup_{n\to\infty} \left(v_n - \sigma^{(1)}_{n,p}(v) \right) \leq \limsup_{n\to\infty} \left(\tau^{>}_{n,[\lambda n],p}(v) - \sigma^{(1)}_{n,p}(v) \right) + \limsup_{n\to\infty} \left(\frac{P_{[\lambda n]} - P_n}{P_n} \tau^{>}_{n,[\lambda n],p}(v) \right). \tag{14}$$

Noticing that the first term on the right-hand side of (14) vanishes by Lemma 3 (i), we deduce that

$$\limsup_{n\to\infty} (v_n - \sigma^{(1)}_{n,p}(v)) \leq \limsup_{n\to\infty} \left(\frac{P_{[\lambda n]} - P_n}{P_n} \tau^{>}_{n,[\lambda n],p}(M) \right) \leq$$

$$\leq \limsup_{n\to\infty}\left(\frac{P_{[\lambda n]}-P_n}{P_n}\right)\limsup_{n\to\infty}\tau^{>}_{n,[\lambda n],p}(M).$$

Hence, it follows by (9) that

$$\limsup_{n\to\infty}(v_n-\sigma^{(1)}_{n,p}(v))\leq 0. \tag{15}$$

Similarly, applying Lemma 1 (ii) to v_n, and using Lemma 2 (ii) and Lemma 4 (ii), we have

$$v_n-\sigma^{(1)}_{n,p}(v)\geq \tau^{<}_{n,[\lambda n],p}(v)-\sigma^{(1)}_{n,p}(v)-\frac{P_n-P_{[\lambda n]}}{P_{[\lambda n]}}\tau^{<}_{n,[\lambda n],p}(M). \tag{16}$$

Taking the $\liminf$ of both sides of (16), we obtain

$$\liminf_{n\to\infty}\left(v_n-\sigma^{(1)}_{n,p}(v)\right)\geq \liminf_{n\to\infty}\left(\tau^{<}_{n,[\lambda n],p}(v)-\sigma^{(1)}_{n,p}(v)\right)+\liminf_{n\to\infty}\left(-\frac{P_n-P_{[\lambda n]}}{P_{[\lambda n]}}\tau^{<}_{n,[\lambda n],p}(M)\right). \tag{17}$$

Noticing that the first term on the right-hand side of (17) vanishes by Lemma 3 (ii), we deduce that

$$\liminf_{n\to\infty}(v_n-\sigma^{(1)}_{n,p}(v))\geq \liminf_{n\to\infty}\left(-\frac{P_n-P_{[\lambda n]}}{P_{[\lambda n]}}\tau^{<}_{n,[\lambda n],p}(M)\right)\geq$$

$$\geq -\limsup_{n\to\infty}\left(\frac{P_n-P_{[\lambda n]}}{P_{[\lambda n]}}\right)\limsup_{n\to\infty}\tau^{<}_{n,[\lambda n],p}(M).$$

Hence, it follows by (10) that

$$\liminf_{n\to\infty}(v_n-\sigma^{(1)}_{n,p}(v))\geq 0. \tag{18}$$

Combining (15) and (18) yields

$$v_n=\frac{P_{n-1}}{p_n}\Delta V^{(1)}_{n,p}(\Delta u)=o(1). \tag{19}$$

Applying the identity (3) to $(V^{(0)}_{n,p}(\Delta u))$, we obtain that

$$V^{(0)}_{n,p}(\Delta u)-V^{(1)}_{n,p}(\Delta u)=V^{(0)}_{n,p}(\Delta V^{(0)}(\Delta u))=\frac{P_{n-1}}{p_n}\Delta V^{(1)}_{n,p}(\Delta u).$$

Since $(V^{(0)}_{n,p}(\Delta u))$ is $(\overline{N},p)$ summable to s, it follows from (19) that $(V^{(0)}_{n,p}(\Delta u))$ converges to s. From the representation $u_n=V^{(0)}_{n,p}(\Delta u)+\sum_{k=1}^{n}\dfrac{p_kV^{(0)}_k(\Delta u)}{P_{k-1}}$ and the condition (7), it follows that (u_n) is slowly oscillating.

As a corollary we have the following classical Tauberian theorem for $(\overline{N},p)$ summability method.

***Corollary* 1.** *Let (p_n) satisfy the condition* (7). *For a real sequence $u=(u_n)$ let there exist a nonnegative sequence $M=(M_n)$ with $\left(\sum_{k=1}^{n}\dfrac{M_k}{k}\right)\in\mathcal{S}$ such that* (8), (9) *and* (10) *are satisfied. If (u_n) is $(\overline{N},p)$ summable to s, then (u_n) is convergent to s.*

Proof. Assume that (u_n) is $(\overline{N}, p)$ summable to s. It follows by (3) that $(V_{n,p}^{(0)}(\Delta u))$ is $(\overline{N}, p)$ summable to 0. By Theorem 1 (u_n) is slowly oscillating. Finally, (u_n) is convergent to s by Theorem 6 in [1].

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