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# LOCALLY $\phi$-SYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

# ЛОКАЛЬНО $\phi$-СИМЕТРИЧНІ УЗАГАЛЬНЕНІ ФОРМИ ПРОСТОРУ САСАКЯНА

The object of the present paper is to find necessary and sufficient conditions for locally $\phi$-symmetric generalized Sasakian-space-forms to have constant scalar curvature, $\eta$-parallel Ricci tensor and cyclic parallel Ricci tensor. Illustrative examples are given.

Встановлено необхідні та достатні умови, при яких локально $\phi$-симетричні узагальнені форми простору Сасакяна мають сталу скалярну кривизну, $\eta$-паралельний тензор Річчі та циклічний паралельний тензор Річчі. Наведено приклади.

**1. Introduction.** The nature of a Riemannian manifold mostly depends on the curvature tensor $R$ of the manifold. It is well known that the sectional curvatures of a manifold determine curvature tensor completely. A Riemannian manifold with constant sectional curvature $c$ is known as real-space-form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

A Sasakian manifold with constant $\phi$-sectional curvature is a Sasakian-space-form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame P. Alegre, D. E. Blair and A. Carriazo introduced the notion of generalized Sasakian-space-forms in 2004 [1]. But, it is to be noted that generalized Sasakian-space-forms are not merely generalization of such space-forms. It also contains a large class of almost contact manifolds. For example, it is known that [2] any three-dimensional $(\alpha,\beta)$-trans Sasakian manifold with $\alpha,\beta$ depending on $\xi$ is a generalized Sasakian-space-form. However, we can find generalized Sasakian-space-forms with non-constant functions and arbitrary dimensions. In [1], the authors cited several examples of generalized Sasakian-space-forms in terms of warped product spaces. In this connection, it should be mentioned that in 1989 Z. Olszak [12] studied generalized complex-space-forms and proved its existence. A generalized Sasakian-space-form is defined as follows [1]:

Given an almost contact metric manifold $M(\phi,\xi,\eta,g)$, we say that $M$ is generalized Sasakian-space-form if there exist three functions $f_1$, $f_2$, $f_3$ on $M$ such that the curvature tensor $R$ is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\}+$$

$$+f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+$$

$$+f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},$$

for any vector fields $X$, $Y$, $Z$ on $M$. In such a case we denote the manifold as $M(f_1,f_2,f_3)$. Here we shall denote this manifold simply by $M$. In [1], the authors cited several examples of such manifolds.

If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form.

Generalized Sasakian-space-forms have been studied by several authors, viz., [1, 2, 9]. As a weaker notion of locally symmetric manifolds T. Takahashi [13] introduced and studied locally $\phi$-symmetric Sasakian manifolds. Locally $\phi$-symmetric manifolds have also been studied in the papers [5, 6]. Symmetry of a manifold primarily depends on curvature tensor and Ricci tensor of the manifold. In the paper [4], locally $\phi$-symmetric generalized Sasakian-space-forms have been studied and determined the condition for the manifold to be locally $\phi$-symmetric with the additional condition that the manifold is conformally flat. In the present paper, we study locally $\phi$-symmetric generalized Sasakian-space-forms and show that every locally $\phi$-symmetric generalized Sasakian-space-form is conformally flat. So, the present paper improves the result of the paper [4]. The present paper is organized as follows:

Section 2 of this paper contains some preliminary results. In Section 3, we study locally $\phi$-symmetric generalized Sasakian-space-forms, and prove that every generalized Sasakian-space-form which is locally $\phi$-symmetric is conformally flat. In this section, we also find the conditions for a locally $\phi$-symmetric generalized Sasakian-space-form to have constant scalar curvature, $\eta$-parallel Ricci tensor and cyclic parallel Ricci tensor. Interestingly, we show that in a locally $\phi$-symmetric generalized Sasakian-space-form all these properties hold if and only if $f_3$ is constant. The last section contains illustrative examples.

**2. Preliminaries.** This section contains some basic results and formulas which we will use in need for.

A $(2n+1)$-dimensional Riemannian manifold $(M, g)$ is called an almost contact metric manifold if the following results hold [3]:

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad g(X,\xi) = \eta(X). \tag{2.1}$$

Here $X$ is any vector field on the manifold, $\phi$ is a $(1,1)$ tensor, $\xi$ is a unit vector field, $\eta$ is an 1-form and $g$ is a Riemannian metric. This metric induces an inner product on the tangent space of the manifold. An almost contact metric manifold is called contact metric manifold if

$$d\eta(X,Y) = \Phi(X,Y) = g(X,\phi Y),$$

for any vector fields $X, Y$ on the manifold. $\Phi$ is called the fundamental two form of the manifold. An almost contact metric structure is said to be normal if the induced almost complex structure $J$ on the product manifold $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, where $X$ is tangent to $M$, $t$ is the coordinate of $\mathbb{R}$, and $f$ is a smooth function on $M \times \mathbb{R}$ [3]. A normal contact metric manifold is known as Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,$$

for any vector fields $X, Y$ on the manifold [3]. Here $\nabla$ is the Levi – Civita connection on the manifold. It is also called operator of covariant differentiation.

For a $(2n+1)$-dimensional generalized Sasakian-space-form we have [1]

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\}+$$

$$+f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+$$

$$+f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \tag{2.2}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.3}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{2.4}$$

Here $S$ is the Ricci tensor and $r$ is the scalar curvature of the space-form.

A generalized Sasakian-space-form of dimension greater than three is said to be conformally flat if its Weyl conformal curvature tensor vanishes. It is known that [9] a $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is conformally flat if and only if $f_2=0$.

**3. Locally $\phi$-symmetric generalized Sasakian space-forms.**

**Definition 3.1.** *A generalized Sasakian space form is said to be locally $\phi$-symmetric if*

$$\phi^2(\nabla_W R)(X,Y)Z = 0,$$

*for all vector fields $X,Y,Z$ orthogonal to $\xi$.*

This notion was introduced by T. Takahashi for Sasakian manifolds [13].

**Definition 3.2.** *The Ricci tensor $S$ of a generalized Sasakian-space-form is called $\eta$-parallel if it satisfies*

$$(\nabla_W S)(\phi X,\phi Y) = 0,$$

*for any vector fields $X,Y,W$.*

The notion of $\eta$-parallel Ricci tensor was introduced by M. Kon in the context of Sasakian geometry [11].

If $X,Y,Z$ are orthogonal to $\xi$, then (2.2) takes the form

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\}+$$

$$+f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}.$$

By covariant differentiation of $R(X,Y)Z$ with respect to $W$, we obtain from the above equation

$$(\nabla_W R)(X,Y)Z = \nabla_W R(X,Y)Z - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z - R(X,Y)\nabla_W Z =$$

$$= df_1(W)\{g(Y,Z)X - g(X,Z)Y\}+$$

$$+f_1\{\nabla_W g(Y,Z)X + g(Y,Z)\nabla_W X - \nabla_W g(X,Z)Y - g(X,Z)\nabla_W Y\}+$$

$$+df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+$$

$$+f_2\{\nabla_W g(X,\phi Z)\phi Y + g(X,\phi Z)\nabla_W(\phi Y)-$$

$$-\nabla_W g(Y,\phi Z)\phi X - g(Y,\phi Z)\nabla_W(\phi X)+$$

$$+2\nabla_W g(X,\phi Y)\phi Z + 2g(X,\phi Y)\nabla_W(\phi Z)\}-$$

$$-f_1\{g(Y,Z)\nabla_W X - g(\nabla_W X, Z)Y\}-$$

$$-f_2\{g(\nabla_W X,\phi Z)\phi Y - g(Y,\phi Z)\phi\nabla_W X + 2g(\nabla_W X,\phi Y)\phi Z\}-$$

$$-f_1\{g(\nabla_W Y, Z)X - g(X,Z)\nabla_W Y\}-$$

$$-f_2\{g(X,\phi Z)\phi\nabla_W Y - g(\nabla_W Y,\phi Z)\phi X + 2g(X,\phi\nabla_W Y)\phi Z\}-$$

$$-f_1\{g(Y,\nabla_W Z)X - g(X,\nabla_W Z)Y\}-$$

$$-f_2\{g(X,\phi\nabla_W Z)\phi Y - g(Y,\phi\nabla_W Z)\phi X + 2g(X,\phi Y)\phi\nabla_W Z\}. \tag{3.1}$$

Arranging the terms of the above equation, we have

$$(\nabla_W R)(X,Y)Z = df_1(W)\{g(Y,Z)X - g(X,Z)Y\}+$$

$$+df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+$$

$$+f_1\{\nabla_W g(Y,Z)X + g(Y,Z)\nabla_W X - \nabla_W g(X,Z)Y - g(X,Z)\nabla_W Y-$$

$$-g(Y,Z)\nabla_W X + g(\nabla_W X, Z)Y - g(\nabla_W Y, Z)X + g(X,Z)\nabla_W Y-$$

$$-g(Y,\nabla_W Z)X + g(X,\nabla_W Z)Y\}+$$

$$+f_2\{\nabla_W g(X,\phi Z)\phi Y + g(X,\phi Z)\nabla_W(\phi Y)-$$

$$-\nabla_W g(Y,\phi Z)\phi X - g(Y,\phi Z)\nabla_W(\phi X)+$$

$$+2\nabla_W g(X,\phi Y)\phi Z + 2g(X,\phi Y)\nabla_W(\phi Z)-$$

$$-g(\nabla_W X,\phi Z)\phi Y + g(Y,\phi Z)\phi\nabla_W X - 2g(\nabla_W X,\phi Y)\phi Z-$$

$$-g(X,\phi Z)\phi\nabla_W Y + g(\nabla_W Y,\phi Z)\phi X - 2g(X,\phi\nabla_W Y)\phi Z-$$

$$-g(X,\phi\nabla_W Z)\phi Y + g(Y,\phi\nabla_W Z)\phi X - 2g(X,\phi Y)\phi\nabla_W Z\}.$$

After canceling some terms in the coefficient of $f_1$ in the above equation, using the result $(\nabla_W\phi)X = \nabla_W(\phi X) - \phi\nabla_W X$ and arranging the terms, we get from the above equation

$$(\nabla_W R)(X,Y)Z = df_1(W)\{g(Y,Z)X - g(X,Z)Y\}+$$

$$+df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+$$

$$+f_1\{(\nabla_W g(Y,Z) - g(\nabla_W Y, Z) - g(Y,\nabla_W Z))X-$$

$$-(\nabla_W g(X,Z) - g(\nabla_W X, Z) - g(X, \nabla_W Z))Y\}+$$

$$+f_2\{(\nabla_W g(X,\phi Z) - g(\nabla_W X, \phi Z) - g(X, \nabla_W(\phi Z)))\phi Y+$$

$$+g(X,(\nabla_W \phi)Z)\phi Y - (\nabla_W g(Y,\phi Z) - g(\nabla_W Y, \phi Z)-$$

$$-g(Y, \nabla_W(\phi Z)))\phi X - g(Y,(\nabla_W\phi)Z)\phi X+$$

$$+2(\nabla_W g(X,\phi Y) - g(\nabla_W X, \phi Y)-$$

$$-g(X,\nabla_W(\phi Y)))\phi Z + 2g(X,(\nabla_W\phi)Y)\phi Z+$$

$$+g(X,\phi Z)(\nabla_W\phi)Y - g(Y,\phi Z)(\nabla_W\phi)X + 2g(X,\phi Y)(\nabla_W\phi)Z\}.$$

The operator $\nabla$ of the covariant differentiation is called metric connection if $(\nabla_W g)(X,Y) = 0$, i.e., $\nabla_W g(X,Y) - g(\nabla_W X, Y) - g(X, \nabla_W Y) = 0$. Here we take $\nabla$ as metric connection. Then, we also have $(\nabla_W g)(X, \phi Y) = 0$. Thus, the above equation gives

$$(\nabla_W R)(X,Y)Z = df_1(W)\{g(Y,Z)X - g(X,Z)Y\}+$$

$$+df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+$$

$$+f_2\{g(X,\phi Z)(\nabla_W\phi)Y - g(Y,\phi Z)(\nabla_W\phi)X+$$

$$+2g(X,\phi Y)(\nabla_W\phi)Z + g(X,(\nabla_W\phi)Z)\phi Y-$$

$$-g(Y,(\nabla_W\phi)Z)\phi X + 2g(X,(\nabla_W\phi)Y)\phi Z\}. \tag{3.2}$$

Applying $\phi^2$ on both sides of (3.2) and using (2.1), we get

$$\phi^2(\nabla_W R)(X,Y)Z = df_1(W)\{g(X,Z)Y - g(Y,Z)X\}+$$

$$+df_2(W)\{g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z - g(X,\phi Z)\phi Y\}+$$

$$+f_2\{g(X,\phi Z)\phi^2((\nabla_W\phi)Y) - g(Y,\phi Z)\phi^2((\nabla_W\phi)X)+$$

$$+2g(X,\phi Y)\phi^2((\nabla_W\phi)Z) - g(X,(\nabla_W\phi)Z)\phi Y+$$

$$+g(Y,(\nabla_W\phi)Z)\phi X - 2g(X,(\nabla_W\phi)Y)\phi Z\}. \tag{3.3}$$

Suppose that the manifold is locally $\phi$-symmetric. Then (3.3) yields

$$df_1(W)\{g(X,Z)Y - g(Y,Z)X\}+$$

$$+df_2(W)\{g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z - g(X,\phi Z)\phi Y\}+$$

$$+f_2\{g(X,\phi Z)\phi^2((\nabla_W\phi)Y) - g(Y,\phi Z)\phi^2((\nabla_W\phi)X)+$$

$$+2g(X,\phi Y)\phi^2((\nabla_W\phi)Z) - g(X,(\nabla_W\phi)Z)\phi Y + g(Y,(\nabla_W\phi)Z)\phi X-$$

$$-2g(X,(\nabla_W\phi)Y)\phi Z\} = 0. \tag{3.4}$$

Taking the inner product $g$ in both sides of the above equation with $W$ we have

$$df_1(W)\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}+$$

$$+df_2(W)\{g(Y,\phi Z)g(\phi X,W) - 2g(X,\phi Y)g(\phi Z,W) - g(X,\phi Z)g(\phi Y,W)\}+$$

$$+f_2\{g(X,\phi Z)g(\phi^2((\nabla_W\phi)Y),W) - g(Y,\phi Z)g(\phi^2((\nabla_W\phi)X),W)+$$

$$+2g(X,\phi Y)g(\phi^2((\nabla_W\phi)Z),W) - g(X,(\nabla_W\phi)Z)g(\phi Y,W) + g(Y,(\nabla_W\phi)Z)g(\phi X,W)-$$

$$-2g(X,(\nabla_W\phi)Y)g(\phi Z,W)\} = 0. \tag{3.5}$$

In (3.5) putting $X = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, $i = 1,2,3,\ldots,2n+1$, we get

$$2ndf_1(W)g(Y,Z) + 3df_2(W)g(Y,Z) - f_2\{g(\phi Z,\phi^2(\nabla_{e_i}\phi)Y)-$$

$$-\sum_i g(Y,\phi Z)g(\phi^2(\nabla_{e_i}\phi)e_i,e_i) + 2g(\phi Y,\phi^2(\nabla_{e_i}\phi)Z)-$$

$$-g((\nabla_W\phi)Z,\phi Y) - 2g((\nabla_W\phi)Y,\phi Z)\} = 0. \tag{3.6}$$

Putting $Z = \phi Y$, we have from the above equation

$$f_2\{g(\phi^2 Y,\phi^2(\nabla_{e_i}\phi Y)) - \sum_i g(Y,\phi^2 Y)g(\phi^2(\nabla_{e_i}\phi)e_i,e_i)+$$

$$+2g(\phi Y,(\nabla_{e_i}\phi)\phi Y) - g((\nabla_W\phi)\phi Y,\phi Y) - 2g((\nabla_W\phi)Y,\phi^2 Y)\} = 0. \tag{3.7}$$

The above equation is true for any arbitrary $Y$ orthogonal to $\xi$. We observe from (3.7) that for $Y \neq \xi$

$$g(\phi^2 Y,\phi^2(\nabla_{e_i}\phi Y)) - \sum_i g(Y,\phi^2 Y)g(\phi^2(\nabla_{e_i}\phi)e_i,e_i)+$$

$$+2g(\phi Y,(\nabla_{e_i}\phi)\phi Y) - g((\nabla_W\phi)\phi Y,\phi Y) - 2g((\nabla_W\phi)Y,\phi^2 Y) \neq 0.$$

Hence, in view of (3.7) we must have

$$f_2 = 0. \tag{3.8}$$

It is known that [9] a generalized Sasakian-space-form is conformally flat if and only if $f_2 = 0$. Thus, we have the following theorem.

**Theorem 3.1.** *A locally $\phi$-symmetric generalized Sasakian-space-form is conformally flat.*

The above theorem gives a new result regarding the relation between locally $\phi$-symmetric generalized Sasakian-space-forms and conformally flat generalized Sasakian-space-forms.

By virtue of (3.8), (3.5) takes the form

$$df_1(W) = 0.$$

The above equation yields $f_1$ is a constant. Hence, for a locally $\phi$-symmetric generalized Sasakian-space-form $f_2 = 0$ and $f_1$ is constant. Therefore, from (2.4), it follows that

$$r = 2n(2n+1)f_1 - 4nf_3.$$

The above quation yields

$$dr(W) = -4ndf_3(W). \tag{3.9}$$

In view of the above equation we obtain the following theorem.

**Theorem 3.2.** *The scalar curvature of a locally $\phi$-symmetric generalized Sasakian-space-form is constant if and only if $f_3$ is constant.*

From (2.3) we have

$$(\nabla_W S)(\phi X, \phi Y) = d(2nf_1 + 3f_2 - f_3)(W)g(\phi X, \phi Y), \tag{3.10}$$

where $X, Y$ are orthogonal to $\xi$. If the manifold is locally $\phi$-symmetric, then the above equation takes the form

$$(\nabla_W S)(\phi X, \phi Y) = -d(f_3)(W)g(X, Y).$$

The above equation leads us to state the following theorem.

**Theorem 3.3.** *A locally $\phi$-symmetric generalized Sasakian-space-form has $\eta$-parallel Ricci tensor if and only if $f_3$ is constant.*

A. Gray [8] introduced two classes of Riemannian manifolds determined by the covariant derivative of the Ricci tensor. The first one is the class $\mathcal{A}$ consisting of all Riemannian manifolds whose Ricci tensor $S$ is a Codazzi tensor, that is,

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z).$$

The second one is the class $\mathcal{B}$ consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, that is,

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0.$$

It is known that [10] the Ricci tensor of Cartan hypersurface is cyclic parallel. Now, we like to find under what condition a locally $\phi$-symmetric generalized Sasakian space-form has cyclic parallel Ricci tensor. In view of (2.3), and for $X, Y, Z$ orthogonal to $\xi$, we get

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = d(2nf_1 + 3f_2 - f_3)(X)g(Y, Z) +$$

$$+ d(2nf_1 + 3f_2 - f_3)(Y)g(X, Z) + d(2nf_1 + 3f_2 - f_3)(Z)g(X, Y). \tag{3.11}$$

For a locally $\phi$-symmetric generalized Sasakian-space-form $f_2 = 0$ and $f_1$ is consant. Hence, the above equation yields

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) =$$

$$= -d(f_3)(X)g(Y, Z) - d(f_3)(Y)g(X, Z) - d(f_3)(Z)g(X, Y). \tag{3.12}$$

The above equation enables us to state the following theorem.

**Theorem 3.4.** *A locally $\phi$-symmetric generalized Sasakian-space-form has cyclic parallel Ricci tensor if and only if $f_3$ is constant.*

By virtue of Theorems 3.2, 3.3, 3.4, we obtain the following corollary.

***Corollary* 3.1.** *For a locally $\phi$-symmetric generalized Sasakian-space-form the following conditions are equivalent*:

(i) *the manifold has constant scalar curvature,*

(ii) *the manifold has $\eta$-parallel Ricci tensor,*

(iii) *the manifold has cyclic parallel Ricci tensor.*

The above corollary gives a new result.

***Remark* 3.1.** The notion of quarter-symmetric metric connection was introduced by $S$. Golab [7]. The torsion tensor of the quarter-symmetric metric connection is given by

$$T(X,Y) = \eta(Y)X - \eta(X)Y.$$

If $X$, $Y$ are orthogonal to $\xi$, then the torsion tensor vanishes and the quarter-symmetric metric connection reduces to Levi-Civita connection. Therefore, all the results of the present paper are of the same form with respect to quarter-symmetric metric connection and Levi-Civita connection.

**4. Examples.** Let us now give an example of a generalized Sasakian-space-form which is locally $\phi$-symmetric.

***Example* 4.1.** In [1], it is shown that $\mathbb{R} \times_f \mathbb{C}^m$ is a generalized Sasakian-space-form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$, $t \in \mathbb{R}$ and $f'$ denotes derivative of $f$ with respect to $t$. If we choose $m = 4$, and $f(t) = e^t$, then $M$ is a 5-dimensional conformally flat generalized Sasakian-space-form, because $f_2 = 0$. We also see that $f_3 = 0$, which is a constant. Therefore, by the results obtained in the present paper $M$ is locally $\phi$-symmetric and has constant scalar curvature, $\eta$-parallel Ricci tensor and cyclic parallel Ricci tensor.

***Example* 4.2.** Let $N(a,b)$ be a generalized complex space-form of dimension 4, then by [1], $M = \mathbb{R} \times_f N$, endowed with the almost contact metric structure $(\phi, \xi, \eta, g_f)$ is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ of dimension 5 with

$$f_1 = \frac{a - f'^2}{f^2}, \quad f_2 = \frac{b}{f^2}, \quad f_3 = \frac{a - f'^2}{f^2} + \frac{f''}{f}$$

where $f$ is a function of $t \in \mathbb{R}$ and $f'$ denotes differentiation of $f$ with respect to $t$. Let us choose $f$ and $a$ as constants and $b = 0$. Then $f_2 = 0$ and $f_3$ is a constant. Therefore, by theorems obtained in the present paper $M$ locally $\phi$-symmetric and has constant scalar curvature, $\eta$-parallel Ricci tensor and cyclic parallel Ricci tensor.

***Example* 4.3.** For a Sasakian-space-form of dimension greater than three and of constant $\phi$-sectional curvature 1, $f_1 = 1$, $f_2 = f_3 = 0$. Therefore, by theorems obtained in the present paper $M$ is locally $\phi$-symmetric and has constant scalar curvature, $\eta$-parallel Ricci tensor and cyclic parallel Ricci tensor.

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