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ON EQUIVALENT CONE METRIC SPACES*

ПРО ЕКВІВАЛЕНТНІ КОНІЧНІ МЕТРИЧНІ ПРОСТОРИ

We explore the necessary and sufficient conditions for the two cone metrics to be topologically equivalent.

Досліджено необхідні та достатні умови для топологічної еквівалентності двох конічних метрик.

1. Introduction. The concept of cone metric space was first introduced by Huang and Zhang [7] in 2007. They also obtained some fixed point theorems for mappings satisfying certain contractive conditions. Afterwards, many authors generalized fixed point theorems from metric spaces to cone metric spaces (see 1, 3, 8, 9, 10, 12). In 2010, Du [4] obtained an ordinary metric corresponding to a cone metric using the following nonlinear scalarization function: Let E be a Banach space and P be a cone in E. The nonlinear scalarization function $\xi_e\colon E \to \mathbb{R}$ is defined as follows:

$$\xi_e(y) = \inf\{r \in \mathbb{R}\colon y \in re - P\} \text{ for all } y \in E.$$

If (X, D) is a cone metric space, Du [4] showed that $\rho_D := \xi_e \circ D$ is an ordinary metric on X. Abdeljawad [2] proved that for every complete cone metric space there exists a correspondent complete usual metric space such that the spaces are topologically equivalent.

In this paper, we introduce the concept of equivalent cone metrics on the same cone. We present the relations between the notions of convergence and equivalence in cone metric spaces. We obtain the necessary and sufficient conditions for two cone metrics to be equivalent. We also present an alternative definition for the equivalence of cone metrics, which is called the Lipschitz equivalence. Finally, we compare these two definitions.

2. Preliminaries. Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a *cone* in E if it satisfies the following conditions:

(i) P is closed, nonempty and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$ [7].

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, $\operatorname{int} P$ denotes the interior of P [7].

In the sequel, one also has to note that by using the properties of the cone and the definition of the interior that $\operatorname{int} P + P \subseteq \operatorname{int} P$ [11].

Let X be a nonempty set. Suppose the mapping $D\colon X \times X \to E$ satisfies

(d_1) $0 < D(x, y)$ for all $x, y \in X$, and $D(x, y) = 0$ if and only if $x = y$,

(d_2) $D(x, y) = D(y, x)$ for all $x, y \in X$,

(d_3) $D(x, y) \leq D(x, z) + D(z, y)$ for all $x, y, z \in X$.

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Then D is called a *cone metric* on X, and (X, D) is called a *cone metric space*. It is obvious that a cone metric space is a generalization of an ordinary metric space [7].

***Example* 2.1.** Let $P = \big\{\{x_n\} \in l^1 \colon x_n \geq 0, \text{ for all } n\big\}$, (X, d) be any metric space and $D\colon X\times \times X \to l^1$ defined by $D(x,y) = \left\{\dfrac{\min\{1, d(x,y)\}}{n^2}\right\}$. Then (X, D) is a cone metric space.

Let (X, D) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* to x provided that, for every $c \in E$ with $0 \ll c$ there is a positive integer $N = N(c)$ such that $D(x_n, x) \ll c$ for all $n \geq N$. We denote this by $D - \lim_{n\to\infty} x_n = x$ or $x_n \overset{D}{\to} x$ as $n \to \infty$.

Let (X, D) be a cone metric space and $A \subseteq X$.

(i) A point $a \in A$ is called an *interior point* of A if there exists a point c with $0 \ll c$ such that $B_D(a, c) \subseteq A$, where $B_D(a, c) := \{y \in X : D(a, y) \ll c\}$ is called the *D-ball* of a.

(ii) A subset $A \subseteq X$ is called *D-open* if each element of A is an interior point of A. The family $\beta = \{B_D(x, e) : x \in X, 0 \ll e\}$ is a subbasis for a topology on X. We denote this cone topology by τ_c. The topology τ_c is Hausdorff and first countable [2, 3, 6].

Theorem 2.1 [2, 4]**.** *Let (X, D) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Define $\rho_D := \xi_e \circ D$. Then the following statements hold*:

(i) *$\{x_n\}$ converges to x in the cone metric space (X, D) if and only if $\rho_D(x_n, x) \to 0$ as $n \to \infty$,*

(ii) *$\{x_n\}$ is a Cauchy sequence in the cone metric space (X, D) if and only if $\{x_n\}$ is a Cauchy sequence in (X, ρ_D),*

(iii) *(X, D) is a complete cone metric space if and only if (X, ρ_D) is a complete metric space.*

3. Main results.

Definition 3.1. *Let D_1 and D_2 be cone metrics on a set X. If each D_1-open subset of X is D_2-open and each D_2-open subset of X is D_1-open, then D_1 and D_2 are said to be equivalent.*

Now we give an important result which characterizes the concept of equivalence of two cone metrics. Its proof is similar to the ordinary case (see [5], Proposition 1.32).

Proposition 3.1. *Let D_1 and D_2 be cone metrics on the same set X. Then a necessary and sufficient condition for D_1 to be equivalent to D_2 is that, given any point $x \in X$, each D_1-ball at x contains some D_2-ball at x, and each D_2-ball at x contains some D_1-ball at x.*

Proof. *Necessity.* Assume that D_1 is equivalent to D_2. Let $x \in X$. Consider an arbitrary D_1-ball $B_{D_1}(x, c)$ at x. Since $B_{D_1}(x, c)$ is D_1-open, by our assumption it must be D_2-open as well. Hence at the point $x \in B_{D_1}(x, c)$ there is some D_2-ball $B_{D_2}(x, e)$ with $B_{D_2}(x, e) \subset B_{D_1}(x, c)$. Similarly, for each D_2-ball $B_{D_2}(x, c')$, there is some D_1-ball $B_{D_1}(x, e')$ with $B_{D_1}(x, e') \subset B_{D_2}(x, c')$.

Sufficiency. Assume the given conditions hold. We will show that each D_1-open set is D_2-open and each D_2-open set is D_1-open. Let M be a D_1-open subset of X. Let $x \in M$. Since M is D_1-open, there is some $0 \ll c$ with $B_{D_1}(x, c) \subset M$. By assumption, there is some $0 \ll e$ with $B_{D_2}(x, e) \subset B_{D_1}(x, c)$. Then $B_{D_2}(x, e) \subset M$. Since x is an arbitrary point of M, we conclude that M is D_2-open.

Let N be a D_2-open subset of X. Let $x \in N$. Since N is D_2-open, there is some $0 \ll c'$ with $B_{D_2}(x, c') \subset N$. By assumption, there is some $0 \ll e'$ with $B_{D_1}(x, e') \subset B_{D_2}(x, c')$. Then $B_{D_1}(x, e') \subset N$. Since x is an arbitrary point of N, we conclude that N is D_1-open.

Proposition 3.1 is proved.

Now each D-ball at a point x in a cone metric space (X, D) is a D-neighborhood of x, and each D-neighborhood of x contains some D-ball at x. Hence we have the following criterion.

Theorem 3.1. *Let $\{x_n\}$ be a sequence in a cone metric space (X, D), and let $x \in X$. Then a necessary and sufficient condition for $\{x_n\}$ to converge to x in (X, D) is that for each D-neighborhood V of x there exists some $N \in \mathbb{N}$ with $x_n \in V$ whenever $n \geq N$.*

***Corollary* 3.1.** *In the notation of Theorem* 3.1, *let D_1 be a cone metric on X that is equivalent to D_2. Then $\{x_n\}$ converges to x in (X, D_1) if and only if it converges to x in (X, D_2).*

Now we define two cone metrics which are not equivalent. Let $P = \{(x, y) \in \mathbb{R}^2 \colon x, y \geq 0\} \subset \subset \mathbb{R}^2$ and $D_1, D_2 \colon X \times X \to \mathbb{R}^2$ such that

$$D_1(x, y) = (d_A(x, y), \alpha d_A(x, y)),$$

$$D_2(x, y) = (d_E(x, y), \alpha d_E(x, y)),$$

where $\alpha > 0$ is a constant and the metrics d_A and d_E denote the discrete and Euclidean metrics on $\mathbb{R}$, respectively. By Corollary 3.1, in order to show these cone metrics to be nonequivalent, it is enough to prove that convergence in these cone metric spaces do not require each other.

Definition 3.2. *Let D_1 and D_2 be two cone metrics on a set X. We say that D_1 and D_2 are Lipschitz equivalent on X if there exist two positive constants t_1 and t_2 such that*

$$t_1 D_2(x, y) \leq D_1(x, y) \leq t_2 D_2(x, y) \text{ for all } x, y \in X.$$

The next theorem says that Lipschitz equivalence implies the equivalence in the sense of Definition 3.1.

Theorem 3.2. *Let D_1 and D_2 be two cone metrics on a set X. If the cone metrics D_1 and D_2 are Lipschitz equivalent, then they are equivalent in the sense of Definition* 3.1.

Proof. Let $t_1 D_2(x, y) \leq D_1(x, y) \leq t_2 D_2(x, y)$. It suffices to show that $B_{D_1}(x, t_1 c) \subset \subset B_{D_2}(x, c)$ and $B_{D_2}\left(x, \dfrac{c}{t_2}\right) \subset B_{D_1}(x, c)$ for all $x \in X$ and $c \in \operatorname{int} P$. Take

$$y \in B_{D_1}(x, t_1 c) \ \Rightarrow \ D_1(x, y) \ \ll \ t_1 c \ \Rightarrow \ t_1 c - D_1(x, y) \in \operatorname{int} P. \tag{3.1}$$

On the other hand, we get

$$t_1 D_2(x, y) \leq D_1(x, y) \ \Rightarrow \ D_1(x, y) - t_1 D_2(x, y) \in P. \tag{3.2}$$

Combining the inequalities (3.1) and (3.2), we get $t_1 c - t_1 D_2(x, y) \in \operatorname{int} P + P$. Since the inclusion $\operatorname{int} P + P \subseteq \operatorname{int} P$ holds, we obtain

$$t_1 c - t_1 D_2(x, y) \in \operatorname{int} P. \tag{3.3}$$

Since $\lambda \operatorname{int} P \subset \operatorname{int} P$ for all $\lambda > 0$, we have $c - D_2(x, y) \in \operatorname{int} P$ from the expression (3.3). Then $D_2(x, y) \ll c$.

Now we prove that the inclusion $B_{D_2}\left(x, \dfrac{c}{t_2}\right) \subset B_{D_1}(x, c)$ is valid. Clearly,

$$y \in B_{D_2}\left(x, \frac{c}{t_2}\right) \ \Rightarrow \ D_2(x, y) \ \ll \ \frac{c}{t_2} \ \Rightarrow \ \frac{c}{t_2} - D_2(x, y) \in \operatorname{int} P. \tag{3.4}$$

Since $\lambda \operatorname{int} P \subset \operatorname{int} P$ for all $\lambda > 0$, using the expression (3.4) we have

$$c - t_2 D_2(x, y) \in \operatorname{int} P. \tag{3.5}$$

On the other hand, we get

$$D_1(x, y) \leq t_2 D_2(x, y) \Rightarrow t_2 D_2(x, y) - D_1(x, y) \in P. \tag{3.6}$$

Combining the inequalities (3.5) and (3.6) we get $c - D_1(x, y) \in \operatorname{int} P + P$. Since $\operatorname{int} P + P \subseteq \operatorname{int} P$, we have $c - D_1(x, y) \in \operatorname{int} P$, i.e., $D_1(x, y) \ll c$.

Theorem 3.2 is proved.

The converse of Theorem 3.2 is not true in general as can be seen in the example below.

***Example* 3.1.** Define $D_1,\ D_2 \colon X \times X \to l^1$ as $D_1(x, y) = \left\{ \dfrac{d(x, y)}{n^2} \right\}$ and $D_2(x, y) = \left\{ \dfrac{\min\{1, d(x, y)\}}{n^2} \right\}$. Let $X = \mathbb{R}$, $d \colon X \times X \to \mathbb{R}$ such that $d(x, y) = |x - y|$ and $P = \{\{x_n\} \in l^1 \colon x_n \geq 0 \text{ for all } n\}$. It is easily to show that the cone metrics D_1 and D_2 are equivalent but they are not Lipschitz equivalent. A cone metric space is a first-countable topological space [3]. That is why, in order to show that these cone metrics are equivalent it suffices to show that $x_k \overset{D_1}{\to} x_0$ iff $x_k \overset{D_2}{\to} x_0$ as $k \to \infty$.

Definition 3.3. *Let D_1 and D_2 be two cone metrics on a set X. We say that D_1 and D_2 are strong Lipschitz equivalent on X if there exist positive constants t_1 and t_2 such that*

$$t_1 D_2(x, y) \ll D_1(x, y) \ll t_2 D_2(x, y)$$

for all $x, y \in X$.

Theorem 3.3. *Let D_1 and D_2 be two cone metrics on a set X. If the cone metrics D_1 and D_2 are strong Lipschitz equivalent, then these cone metrics are equivalent in the sense of Definition* 3.1.

This theorem can be proved using the similar arguments in the proof of Theorem 3.2.

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