UDC 512.5

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# ON NEARLY $\mathcal{M}$-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS*

# ПРО МАЙЖЕ $\mathcal{M}$-ДОПОВНЕНІ ПІДГРУПИ СКІНЧЕННИХ ГРУП

A subgroup $H$ is called nearly $\mathcal{M}$-supplemented in a finite group $G$ if there exists a normal subgroup $K$ of $G$ such that $HK \trianglelefteq G$ and $TK < HK$ for every maximal subgroup $T$ of $H$. We obtain some new results on supersoluble groups and formation by using nearly $\mathcal{M}$-supplemented subgroups and investigate the structure of finite groups.

Підгрупу $H$ називаємо майже $\mathcal{M}$-доповненою в скінченній групі $G$, якщо існує нормальна підгрупа $K$ групи $G$ така, що $HK \trianglelefteq G$ і $TK < HK$ для кожної максимальної підгрупи $T$ групи $H$. Отримано деякі нові результати про суперрозв'язні групи та їх утворення за допомогою майже $\mathcal{M}$-доповнених підгруп та досліджено структуру скінченних груп.

**1. Introduction.** It is well-known that supplemented subgroups play an important role in the theory of finite groups. For instance, Hall [4] proved that a group $G$ is soluble if and only if every Sylow subgroup of $G$ is complemented in $G$. Srinivasan [10] proved that a finite group is supersoluble if every maximal subgroup of every Sylow subgroup is normal. Later on, by considering some special supplemented subgroups ($c$-supplemented subgroups), Wang [12] proved that $G$ is soluble if and only if every Sylow subgroup of $G$ is $c$-supplemented in $G$. Recently, Miao and Lempken [6] considered $\mathcal{M}$-supplemented subgroups of finite groups $G$ and obtained some characterization of saturated formations containing all supersoluble groups. More recently, Wang and Guo [13] introduced the concept of nearly $s$-normal subgroups and obtained some interesting results.

Now, we introduce the following concept of nearly $\mathcal{M}$-supplemented subgroups.

**Definition 1.1.** *A subgroup $H$ is called nearly $\mathcal{M}$-supplemented in group $G$, if there exists a normal subgroup $K$ of $G$ such that $HK \trianglelefteq G$ and $TK < HK$ for every maximal subgroup $T$ of $H$.*

The following examples indicate that the nearly $\mathcal{M}$-supplementation of subgroups can neither be deduced from $\mathcal{M}$-supplementation of subgroup nor from nearly $s$-normality of subgroup.

***Example* 1.1.** Let $G = S_4$. Since $A_4$ is normal in $G$, clearly, $A_4$ is nearly $\mathcal{M}$-supplemented in $G$, but $A_4$ is not $\mathcal{M}$-supplemented in $G$.

***Example* 1.2.** Let $G = S_4$ and $H = \langle(1234)\rangle$ be a cyclic subgroup of order 4. Then $G = HA_4$, where $A_4$ is the alternating group of degree 4. Clearly, since $A_4 \trianglelefteq G$, we have $H$ is nearly $\mathcal{M}$-supplemented in $G$, but $H$ is not nearly $s$-normal in $G$. Otherwise, there exists a normal subgroup $K$ of $G$ such that $HK \trianglelefteq G$ and $H \cap K \leqslant H_{sG}$, we have $H_{sG} = 1$. Otherwise, if $H_{sG} = H$ is $s$-permutable in $G$, then $H$ is normal in $G$, a contradiction. If $H_{sG} = \langle(13)(24)\rangle$ is $s$-permutable in $G$, then $\langle(13)(24)\rangle$ is normal in $G$, a contradiction. But $H \cap K \neq 1$. Therefore $H$ is not nearly $s$-normal in $G$.

All the groups in this paper are finite. Most of the notation is standard and can be found in [1] and [9].

**2. Preliminaries.** For the sake of convenience, we first list here some known results which will be useful in the sequel.

* This research is supported by the grant of NSFC (Grant #11271016).

**Lemma 2.1.** *Let $G$ be a group. Then*:

(1) *If $H$ is nearly $\mathcal{M}$-supplemented in $G$, $H \leq K \leq G$, then $H$ is nearly $\mathcal{M}$-supplemented in $K$.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. If $H$ is nearly $\mathcal{M}$-supplemented in $G$, then $H/N$ is nearly $\mathcal{M}$-supplemented in $G/N$.*

(3) *Let $\pi$ be a set of primes. Let $N$ be a normal $\pi'$-subgroup and let $H$ be a $\pi$-subgroup of $G$. If $H$ is nearly $\mathcal{M}$-supplemented in $G$, then $HN/N$ is nearly $\mathcal{M}$-supplemented in $G/N$.*

(4) *Let $R$ be a soluble minimal normal subgroup of a group $G$. If there exists a maximal subgroup $R_1$ of $R$ such that $R_1$ is nearly $\mathcal{M}$-supplemented in $G$, then $R$ is a cyclic group of prime order.*

***Proof.*** (1) – (4) follow from the definition of nearly $\mathcal{M}$-supplemented subgroups.

**Lemma 2.2** ([7], Lemma 2.6)**.** *If $H$ is a subgroup of a group $G$ with $|G : H| = p$, where $p$ is a prime divisor of $|G|$ and $(|G|, p-1) = 1$, then $H \trianglelefteq G$.*

**Lemma 2.3** ([8], Lemma 2.7)**.** *Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then $G$ is $p$-nilpotent if and only if $P$ is $\mathcal{M}$-supplemented in $G$.*

**Lemma 2.4** ([2], Theorem 1.8.17)**.** *Let $N$ be a nontrivial soluble normal subgroup of a group $G$. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of $N$ is the direct product of minimal normal subgroups of $G$ which are contained in $N$.*

**Lemma 2.5** ([14], Theorem 3.1)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, $G$ a group with a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$. If for any maximal subgroup $M$ of $G$, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

**Lemma 2.6** ([3], Main theorem)**.** *Suppose that a finite group $G$ has a Hall $\pi$-subgroup where $\pi$ is a set of primes not containing 2. Then all Hall $\pi$-subgroups of $G$ are conjugate.*

**Lemma 2.7.** *Let $\mathfrak{F}$ be a formation and $G$ be a group. Suppose that a subgroup $H$ of $G$ has a $\mathfrak{F}$-supplement in $G$. Then*:

(1) *If $N \trianglelefteq G$, then $HN/N$ has a $\mathfrak{F}$-supplement in $G/N$.*

(2) *If $H \leq K \leq G$, then $H$ has a $\mathfrak{F}$-supplement in $K$.*

**Lemma 2.8** ([8], Lemma 2.9)**.** *Let $G$ be a finite group and $P$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime divisor of $|G|$. Then $G$ is $p$-nilpotent if and only if every maximal subgroup of $P$ having no $p$-nilpotent supplement in $G$ is $\mathcal{M}$-supplemented in $G$.*

**Lemma 2.9.** *Let $G$ be a group and $N$ a subgroup of $G$. The generalized Fitting subgroup $F^*(G)$ of $G$ is the unique maximal normal quasinilpotent subgroup of $G$. Then*:

(1) *If $N$ is normal in $G$, then $F^*(N) \leq F^*(G)$.*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{Soc}\,(F(G)C_G(F(G))/F(G))$.*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

(4) *$C_G(F^*(G)) \leq F(G)$.*

(5) *If $P \trianglelefteq G$ with $P \leq O_p(G)$, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*

(6) *If $K \leq Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

**Lemma 2.10** ([6], Lemma 2.7)**.** *Let $G$ be a finite group with normal subgroups $H$ and $L$ and let $p \in \pi(G)$. Then the following hold*:

(1) *$\Phi(L) \leq \Phi(G)$.*

(2) *If $L \leq \Phi(G)$, then $F(G/L) = F(G)/L$.*

(3) *If $L \leq H \cap \Phi(G)$, then $F(H/L) = F(H)/L$.*

(4) *If $H$ is a $p$-group and $L \leq \Phi(H)$, then $F^*(G/L) = F^*(G)/L$.*

(5) *If $L \leq \Phi(G)$ with $|L| = p$, then $F^*(G/L) = F^*(G)/L$.*

(6) *If* $L \leq H \cap \Phi(G)$ *with* $|L| = p$, *then* $F^*(H/L) = F^*(H)/L$.

**3. Main results.**

**Theorem 3.1.** *Let* $G$ *be a finite group and* $P$ *be a Sylow* $p$*-subgroup of* $G$, *where* $p$ *is a prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *Then* $G$ *is* $p$*-nilpotent if and only if every maximal subgroup of* $P$ *having no* $p$*-nilpotent supplement is nearly* $\mathcal{M}$*-supplemented in* $G$.

***Proof.*** As the necessity part is obvious, we only need to prove the sufficiency part. Assume that the theorem is false and choose $G$ to be a counterexample of minimal order. Moreover, we have

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G)\neq 1$, then we consider the quotient group $G/O_{p'}(G)$. By Lemmas 2.1 and 2.7, it is easy to see that $G/O_{p'}(G)$ satisfies the hypotheses of our theorem. The minimal choice of $G$ implies that $G/O_{p'}(G)$ is $p$-nilpotent and hence $G$ is $p$-nilpotent, a contradiction.

(2) $O_p(G) \neq 1$.

Assume that $O_p(G)$=1. If every maximal subgroup $P_1$ of $P$ has a $p$-nilpotent supplement in $G$, then $G$ is $p$-nilpotent, a contradiction. So there at least exists a maximal subgroup $P_1$ of $P$ such that $P_1$ is nearly $\mathcal{M}$-supplemented in $G$. Then there exists a normal subgroup $K_1$ of $G$ such that $P_1K_1 \trianglelefteq G$ and $TK_1 < \cdot P_1K_1$ for every maximal subgroup $T$ of $P_1$. Furthermore, if $P_1K_1 < G$, then $(P_1K_1)_p = P_1$ or $(P_1K_1)_p = P$. If $(P_1K_1)_p = P_1$, then $P_1$ is $\mathcal{M}$-supplemented in $P_1K_1$. By Lemma 2.3, $P_1K_1$ is $p$-nilpotent, it follows from $(P_1K_1)_{p'}$ char $P_1K_1 \trianglelefteq G$ and (1) that $(P_1K_1)_{p'} = 1$. Therefore $P_1K_1 = P_1 \trianglelefteq G$, a contradiction. Hence we have $(P_1K_1)_p = P$. Obviously, $P_1K_1$ satisfies the hypotheses of the theorem and hence $P_1K_1$ is $p$-nilpotent by the choice of $G$. With the similar discussion as above, we also get a contradiction. So we assume that $P_1K_1 = G$. This means every maximal subgroup of $P$ having no $p$-nilpotent supplement in $G$ is $\mathcal{M}$-supplemented in $G$. By Lemma 2.8, $G$ is $p$-nilpotent also a contradiction.

(3) *$G$ has a unique minimal normal subgroup $N$ contained in $O_p(G)$ such that* $N = O_p(G) = F(G)$.

Let $N$ be a minimal normal subgroup contained in $O_p(G)$. Obviously, $G/N$ satisfies the condition of the theorem and hence $G/N$ is $p$-nilpotent by the choice of $G$. Since the class of all $p$-nilpotent groups is a saturated formation, we have $N$ is the unique minimal normal subgroup of $G$ contained in $O_p(G)$ and $\Phi(G) = 1$. By Lemma 2.4 $N = O_p(G) = F(G)$.

(4) *Final contradiction*.

Since $N$ is the unique minimal normal subgroup of $G$ contained in $O_p(G)$ and $N \nleq \Phi(G)$, there exists a maximal subgroup $M$ of $G$ such that $N \nleq M$. Then $G = NM$ and $N \cap M = 1$, $P = NM_p$ where $M_p$ is a Sylow $p$-subgroup of $M$. Since $G/N \cong M$ is $p$-nilpotent and (3), we have $G = NN_G(M_{p'}) = PN_G(M_{p'})$ where $M_{p'}$ is a Hall $p'$-subgroup of $M$ and of course of $G$. So we may assume $M_p \leq P \cap N_G(M_{p'}) \leq L_2 < \cdot L_1 < \cdot P$. Otherwise, if $P\cap N_G(M_{p'})$=$P$, then $M_{p'} \trianglelefteq G$, a contradiction. If $|P : P \cap N_G(M_{p'})| = |G : N_G(M_{p'})| = p$ and hence $N_G(M_{p'}) \trianglelefteq G$ by Lemma 2.2, a contradiction. If $L_1$ has a $p$-nilpotent supplement $K$ in $G$ such that $G = L_1N_G(K_{p'})$ where $K_{p'}$ is a Hall $p'$-subgroup of $K$ and of course of $G$. By Lemma 2.6, there exists an element $x$ of $L_1$ such that $N_G(M_{p'}) = (N_G(K_{p'}))^x$. Therefore $G = L_1N_G(K_{p'}) = (L_1N_G(K_{p'}))^x = L_1N_G(M_{p'})$. Moreover, $P = P \cap L_1N_G(M_{p'}) = L_1(P \cap N_G(M_{p'})) = L_1$, a contradiction. So we may assume $L_1$ is nearly $\mathcal{M}$-supplemented in $G$, there exists a normal subgroup $B$ of $G$ such that $L_1B \trianglelefteq G$ and $TB < L_1B$ for every maximal subgroup $T$ of $L_1$. We will divide into the following two cases.

(a) $L_1B < G$.

If $(L_1B)_p = L_1$, then $L_1$ is $\mathcal{M}$-supplemented in $L_1B$ and hence $L_1B$ is $p$-nilpotent by Lemma 2.3. It follows from $(L_1B)_{p'}$ char $L_1B \trianglelefteq G$ and (1) that $L_1B = L_1 \trianglelefteq G$, a contradiction. If $(L_1B)_p = P$,

$L_1B$ satisfies the condition of the theorem by Lemmas 2.1 and 2.7, the minimality of $G$ implies that $L_1B$ is $p$-nilpotent. So $(L_1B)_{p'}$ char $L_1B \trianglelefteq G$. With the similar discussion as above, $L_1B = P = N \trianglelefteq G$ by (3) and $L_1$ is the maximal subgroup of $N$ with $L_1$ is nearly $\mathcal{M}$-supplemented in $G$, hence we have $|N| = p$ by Lemma 2.1(4) and $G/N$ is $p$-nilpotent. Therefore $G$ is $p$-nilpotent, a contradiction.

(b) $L_1B = G$.

That is, $L_1$ is $\mathcal{M}$-supplemented in $G$. For every maximal subgroup $T$ of $L_1$, $|G : TB| = p$ and hence $TB \trianglelefteq G$ by Lemma 2.2. Set $T = L_2$ and $N \leq L_2B$ or $N \cap L_2B = 1$. If $N \cap L_2B = 1$, then $|N| = |G : L_2B| = p$, a contradiction. If $N \leq L_2B$, since $M_p \leq P \cap N_G(M_{p'}) \leq L_2$ and $P = NL_2$, it follows that $L_1B = PB = NL_2B = L_2B < G$, a contradiction.

Theorem 3.1 is proved.

**Theorem 3.2.** *Let $G$ be a finite group where $p$ is an odd prime divisor of $|G|$. Then $G$ is $p$-nilpotent if and only if $N_G(P)$ is $p$-nilpotent and every maximal subgroup of $P$ is nearly $\mathcal{M}$-supplemented in $G$.*

***Proof.*** As the necessity part is obvious, we only need to prove the sufficiency part. Assume that the assertion is false and choose $G$ to be a counterexample of minimal order. Then

(1) $O_{p'}(G) = 1$.

Suppose that $L = O_{p'}(G) \neq 1$, we consider the factor group $G/L$. Clearly, $P_1L/L$ is a maximal subgroup of Sylow $p$-subgroup of $G/L$ where $P_1$ is a maximal subgroup of Sylow $p$-subgroup of $P$. Since $P_1$ is nearly $\mathcal{M}$-supplemented in $G$, we have $P_1L/L$ is also nearly $\mathcal{M}$-supplemented in $G/L$ by Lemma 2.1(3). On the other hand, $N_{G/L}(PL/L) = N_G(P)L/L$ ([6], Lemma 3.6.10) and so it is $p$-nilpotent. Therefore $G/L$ satisfies the hypotheses of the theorem. The minimal choice of $G$ implies that $G/L$ is $p$-nilpotent, and hence $G$ is $p$-nilpotent, a contradiction.

(2) *If $M$ is a proper subgroup of $G$ with $P \leq M < G$, then $M$ is $p$-nilpotent.*

Clearly, $N_M(P) \leqslant N_G(P)$ and hence $N_M(P)$ is $p$-nilpotent. Applying Lemma 2.1(1), we find that $M$ satisfies the hypotheses of our theorem. Now, the minimal choice of $G$ implies that $M$ is $p$-nilpotent.

(3) $G = PQ$, *where $Q$ is a Sylow $q$-subgroup of $G$ with $q \neq p$.*

Since $G$ is not $p$-nilpotent, by Thompson ([11], Corollary), there exists a characteristic subgroup $H$ of $P$ such that $N_G(H)$ is not $p$-nilpotent. Since $N_G(P)$ is $p$-nilpotent, we may choose a characteristic subgroup $H$ of $P$ such that $N_G(H)$ is not $p$-nilpotent, but $N_G(K)$ is $p$-nilpotent for every characteristic subgroup $K$ of $P$ with $H < K \leq P$. Since $N_G(H) \geq N_G(P)$ and $N_G(H)$ is not $p$-nilpotent, we must have $N_G(P) < N_G(H)$. Then by our claim (2), we obtain $N_G(H) = G$. This leads to $O_p(G) \neq 1$ and $N_G(K)$ is $p$-nilpotent for every characteristic subgroup $K$ of $P$ satisfying $O_p(G) < K \leq P$. Now, by using the result of Thompson ([11], Corollary) again, we see that $G/O_p(G)$ is $p$-nilpotent and therefore $G$ is $p$-soluble. Since $G$ is $p$-soluble for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $G_1 = PQ$ is a subgroup of $G$. Invoking our claim (2) above, $G_1$ is $p$-nilpotent if $G_1 < G$. This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus we have proved that $G = PQ$.

(4) *$G$ has a unique minimal normal subgroup $N$ such that $N = O_p(G) = C_G(N) = F(G)$.*

Let $N$ be a minimal normal subgroup of $G$. By (1) and (3), $N$ is an elementary abelian $p$-group. Obviously $G/N$ satisfies the condition of the theorem, the minimal choice of $G$ implies that $G/N$ is $p$-nilpotent. Since the class of all $p$-nilpotent groups is a saturated formation, we have $N$ is the unique minimal normal subgroup and $\Phi(G) = 1$. By Lemma 2.4 $N = O_p(G) = C_G(N) = F(G)$ and $N \nleq \Phi(G)$.

(5) *Final contradiction*.

Since $N$ is the unique minimal normal subgroup of $G$ and $N \nleq \Phi(G)$, there exists a maximal subgroup $M$ of $G$ such that $N \nleq M$. Then $G = NM$ and $N \cap M = 1$. Clearly, $P = NM_p$ and $M_p \leq P_2 < P_1$ where $P_1$ is the maximal subgroup of $P$ and $P_2$ is the maximal subgroup of $P_1$. If $M_p = P_1$ then $|N| = p$ and hence $\mathrm{Aut}\,(N)$ is a cyclic group of order $p-1$. If $p < q$, by ([9], 10.1.9) then $NQ$ is $p$-nilpotent. Consequently, $Q \leq C_G(N) = O_p(G)$, a contradiction. If $q < p$, then $M \cong G/N \cong N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}\,(N)$. It follows that $Q$ is a cyclic group that $G$ is $q$-nilpotent by ([9], 10.1.9) and hence $P \trianglelefteq G$. Therefore $N_G(P) = G$ is $p$-nilpotent, a contradiction.

By hypotheses, $P_1$ is nearly $\mathcal{M}$-supplemented in $G$. There exists a normal subgroup $B_1$ such that $P_1B_1 \trianglelefteq G$ and $TB_1 < P_1B_1$ for every maximal subgroup $T$ of $P_1$. Furthermore,

(a) $P_1B_1 < G$.

Since $M_p \leq P_1$ and $N \leq P_1B_1$, we have $(P_1B_1)_p = P$. By Lemma 2.1(1) and the hypotheses, $P_1B_1$ satisfies the condition of the theorem, the minimal choice of $G$ implies that $P_1B_1$ is $p$-nilpotent. Since $(P_1B_1)_{p'}$ char $P_1B_1 \trianglelefteq G$ and $(P_1B_1)_{p'} \trianglelefteq G$. We have $(P_1B_1)_{p'} = 1$ by (1), and hence $(P_1B_1) = P \trianglelefteq G$, also is a contradiction.

(b) $P_1B_1 = G$.

That is, $P_1$ is $\mathcal{M}$-supplemented in $G$. For every maximal subgroup $T$ of $P_1$, $TB_1 < G$ and $|G : TB_1| = p$ by ([6], Lemma 2.2). Set $T = P_2$, if $N \leq P_2B_1$, then $P_2B_1 = NP_2B_1 = PB_1 = = P_1B_1 = G$, a contradiction. So we may have that $N \nleq P_2B_1$ and hence we have $G = P_2B_1N$ and $|N| = |G : P_2B_1| = p$, a contradiction.

**Theorem 3.3.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, suppose $G$ has a soluble normal subgroup N with $G/N \in \mathcal{F}$. If every maximal subgroup of noncyclic Sylow subgroup of $F(N)$ having no supersoluble supplement is nearly $\mathcal{M}$-supplemented in $G$, then $G \in \mathcal{F}$.*

***Proof.*** Assume that the theorem is false and let $G$ be a counterexample of minimal order. Furthermore, we have that

(1) $N \cap \Phi(G) = 1$.

If $N \cap \Phi(G) \neq 1$, then there exists a minimal normal subgroup $L$ of $G$ such that $L \leq N \cap \cap \Phi(G)$. Since $N$ is soluble, we know that $L$ is an elementary abelian $p$-group, moreover, we have $F(N/L) = F(N)/L$. By Lemmas 2.1(2) and 2.7, every maximal subgroup of noncyclic Sylow subgroup of $F(N/L)$ having no supersoluble supplement is nearly $\mathcal{M}$-supplemented in $G/L$ and $(G/L)/(N/L) \cong G/N \in \mathcal{F}$. Clearly, $G/L$ satisfies the condition of the theorem and hence $G/L \in \mathcal{F}$ by the minimal choice of $G$. Therefore $G \in \mathcal{F}$, a contradiction.

(2) *Every minimal normal subgroup of $G$ contained in $O_p(N)$ is cyclic of order p where p is a prime divisor of $|N|$.*

If $N = 1$, the assertion is ture. So we may assume that $N \neq 1$, the solubility of $N$ implies that $F(N) \neq 1$. By Lemma 2.4, $F(N)$ is the direct product of minimal normal subgroups of $G$ contained in $N$. There at least exists a maximal subgroup $W$ of $G$ not containing $F(N)$ and hence there at least exists a prime $p$ of $\pi(|N|)$ with $O_p(N) \nleq W$ by Lemma 2.5. Applying Lemma 2.5 again, we have $|G : W|$ is not prime order .

Denote $P = O_p(N)$. Then $P$ is the direct product of some minimal normal subgroups of $G$. We assume that $P = R_1 \times R_2 \times \ldots \times R_t$ where $R_i$ is a minimal normal subgroup of $G$, $i = 1, 2, \ldots, t$. Since $N \cap \Phi(G) = 1$, for every minimal normal subgroup $R$ of $G$ contained $P$, there exists a maximal subgroup $M$ of $G$ such that $G = RM = PM$ and $R \cap M$=1. Let $M_p$ be a Sylow $p$-subgroup of $M$ and $P = (P \cap M) \times R$. Then $G_p = PM_p$ is a Sylow $p$-subgroup of $G$. Now, let $P_1$ be a maximal subgroup of $G_p$ containing $M_p$ and set $P_2 = P_1 \cap P$. Since $|P : P_2| = |P : P_1 \cap P| = p$, that is, $P_2$ is a maximal subgroup of $P$. On the other hand, $P_2 = P_2 \cap P = P_2 \cap (P \cap M)R = (P \cap M) \times (P_2 \cap R)$. Similarly, we know that $P_2 \cap R$ is a maximal subgroup of $R$. If $P_2$ is nearly $\mathcal{M}$-supplemented in $G$, then there exists a normal subgroup $K$ of $G$ such that $P_2K \trianglelefteq G$ and $TK < P_2K$ for every maximal subgroup $T$ of $P_2$. Set $K_2 = (P \cap M)K \trianglelefteq G$, therefore $(P_2 \cap R)K_2 \trianglelefteq G$ and $T'K_2 < (P_2 \cap R)K_2 = P_2K$ for every maximal subgroup $T'$ of $P_2 \cap R$, that is, $P_2 \cap R$ is nearly $\mathcal{M}$-supplemented in $G$ and hence $|R| = p$ by Lemma 2.1(4). If $P_2$ has supersoluble supplement in $G$. Then there exists a subgroup $H$ of $G$ such that $G = P_2H$ and $H$ is supersoluble. Let $L = (P \cap M)H$, so we have $G = P_2H = P_2(P \cap M)H = P_2L = PL$. If $P \nleq L$, then $L < G$. Since $P \cap M \leq P_2 \cap L \leq P \cap L = (P \cap M)R \cap L = (P \cap M)(R \cap L) = P \cap M$, then $|P| = |P_2|$, a contradiction. So we assume $P \leq L$, then $L = G$, $G/(P \cap M) = L/(P \cap M) \cong H/(P \cap M \cap H)$ is supersoluble. Since $M/(P \cap M)$ is a maximal subgroup of $G/(P \cap M)$, and hence $|G : M| = p$ and $|R| = p$.

(3) *Final contradiction.*

For every $R_i$, $i = 1, 2, \ldots, t$, is of prime order, $G = R_iM$ and $R_i \cap M = 1$. It is clearly that $|G : M| = p$ and hence $G \in \mathcal{F}$ by Lemma 2.5, a contradiction.

Theorem 3.3 is proved.

The final contradiction completes our theorem.

***Corollary* 3.1.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $G$ be a soluble group. If every maximal subgroup of noncyclic Sylow subgroup of $F(G)$ having no supersoluble supplement is nearly $\mathcal{M}$-supplemented in $G$, then $G \in \mathcal{F}$.*

***Corollary* 3.2.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose $G$ has a soluble normal subgroup $N$ with $G/N \in \mathcal{F}$. If every maximal subgroup of noncyclic Sylow subgroup of $N$ having no supersoluble supplement is nearly $\mathcal{M}$-supplemented in $G$, then $G \in \mathcal{F}$.*

***Proof.*** By Theorem 3.1, we know that $N$ has supersoluble type Sylow tower. Let $P$ be the Sylow $p$-subgroup of $N$, where $p$ is the largest prime divisor of $|N|$. Then $P$ char $N$ and hence $P \trianglelefteq G$. It is easy to know $G/P$ satisfies the hypotheses, therefore $G/P \in \mathcal{F}$. Since every maximal subgroup of noncyclic Sylow subgroup of $F(P) = P$ having no supersoluble supplement is nearly $\mathcal{M}$-supplemented in $G$, then $G \in \mathcal{F}$ by Theorem 3.1.

**Theorem 3.4.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $G$ be a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F^*(H)$ having no supersoluble supplement is nearly $\mathcal{M}$-supplemented in $G$, then $G \in \mathcal{F}$.*

***Proof.*** Suppose that the theorem is false and choose $G$ to be a counterexample of the minimal order, so in particular, $H \neq 1$. Furthermore, we have

*Case* I. $\mathcal{F} = \mathcal{U}$.

By Corollary 3.2, we easily verify that $F^*(H)$ is supersoluble and hence $F(H) = F^*(H) \neq 1$. Since the pair $(H, H)$ satisfies the hypotheses of the theorem in place of $(G, H)$, the minimal choice of $G$ implies that $H$ is supersoluble if $H < G$; then $G \in \mathcal{U}$ by Theorem 3.3, a contradiction. Hence

(1) *$H = G$ is nonsoluble and $F^*(G) = F(G) \neq 1$.*

Let $N$ be a proper normal subgroup of $G$ containing $F^*(G)$. By Lemma 2.9, $F^*(G) = F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$, so $F^*(N) = F^*(G)$. Moreover, every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(N)$ having no supersoluble supplement is nearly $\mathcal{M}$-supplemented in $N$ by Lemma 2.1(1). Hence $N$ is supersoluble by the minimal choice of $G$. So we have

(2) *Every proper normal subgroup of $G$ containing $F^*(G)$ is supersoluble.*

Suppose now that $\Phi(O_p(G)) \neq 1$ for some $p \in \pi(F(G))$. By Lemma 2.9 we have $F^*(G/\Phi(O_p(G))) = F^*(G)/\Phi(O_p(G))$. Using Lemma 2.1 we observe that the pair $\big(G/\Phi(O_p(G)\big)$, $F^*(G)/\Phi(O_p(G)))$ satisfies the hypotheses of the theorem. The minimal choice of $G$ then implies $G/\Phi(O_p(G)) \in \mathcal{U}$. Since $\mathcal{U}$ is a saturated formation, we then get $G \in \mathcal{U}$, a contradiction. Thus we have

(3) *If* $p \in \pi(F(G))$, *then* $\Phi(O_p(G)) = 1$ *and so* $O_p(G)$ *is elementary abelian; in particular,* $F^*(G) = F(G)$ *is abelian and* $C_G(F(G)) = F(G)$.

Suppose that $L$ is a minimal normal subgroup of $G$ contained in $F(G)$ and that $|L| = p$ for some $p \in \pi(F(G))$; also set $C := C_G(L)$. Clearly, $F(G) \leq C \trianglelefteq G$. If $C < G$, then $C$ is solvable by $(2)$. Since $G/C$ is cyclic, we get that $G$ is solvable, a contradiction. So we have $C = G$ and hence $L \leq Z(G)$. Then we consider the group $G/L$. By Lemma 2.9, we have $F^*(G/L) = F^*(G)/L = F(G)/L$. In fact, $G/L$ satisfies the hypotheses of the theorem by Lemma 2.1. Therefore the minimal choice of $G$ implies that $G/L \in \mathcal{U}$ and hence $G$ is supersoluble, a contradiction. This proves

(4) *There is no minimal normal subgroup of prime order in* $G$ *contained in* $F(G)$.

If $F(G) = H_1 \times \ldots \times H_r$ with cyclic Sylow subgroups $H_1, \ldots, H_r$ of of $F(G)$, then $G/C_G(H_i)$ is abelian for any $i \in \{1, \ldots, r\}$ and so $G/\bigcap_{i=1}^r C_G(H_i) = G/C_G(F(G)) = G/F(G)$ is abelian. Therefore $G$ is solvable, a contradiction. This proves

(5) $P := O_p(G) \in Syl_p(F(G))$ *is non-cyclic for some* $p \in \pi(F(G))$.

Let $P_1$ be a maximal subgroup of $O_p(G)$. If $P_1$ has a supersoluble supplement in $G$, then there exists a supersoluble subgroup $K$ of $G$ such that $G = P_1K = O_p(G)K$. Clearly, $G/O_p(G) \cong K/K \cap O_p(G)$ is supersoluble and hence $G$ is soluble, a contradiction. So we obtain that

(6) *Every maximal subgroup of every noncyclic Sylow subgroup of* $F(G)$ *has no supersoluble supplement in* $G$.

Furthermore, if $P \cap \Phi(G) = 1$, then $P = R_1 \times \ldots \times R_t$ with minimal normal subgroups $R_1, \ldots, R_t$ of $G$ by Lemma 2.4. Clearly, $P_2 = R_1{}^*R$ is the maximal subgroup of $P$ where $R_1^*$ is the maximal subgroup of $R_1$ and $R = R_2 \times \ldots \times R_t$. By hypotheses, $P_2$ is nearly $\mathcal{M}$-supplemented in $G$. There exists a normal subgroup of $K$ of $G$ such that $P_2K \trianglelefteq G$ and $TK < P_2K$ for every maximal subgroup $T$ of $P_2$. Let $K_1 = RK$. Clearly, $K_1 \trianglelefteq G$ and $P_2K = R_1{}^*K_1$ and $T_1K_1 < R_1{}^*K_1$ for every maximal subgroup $T_1$ of $R_1{}^*$. Therefore $R_1{}^*$ is also nearly $\mathcal{M}$-supplemented in $G$ and hence $|R_1| = p$ by Lemma 2.1(4), contrary to $(4)$. So we get that

(7) $R := P \cap \Phi(G) \neq 1$.

Now suppose that $Q \in \mathrm{Syl}_q(F(G))$ for some prime $q \neq p$ and let $L$ be a minimal normal subgroup of $G$ contained in $R$. Then $Q$ is elementary abelian by $(3)$. By the definition of a generalized Fitting subgroup, $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G/L), E(G/L)] = 1$, where $E(G/L)$ is the layer of $G/L$. Since $L \leq \Phi(G)$, $F(G/L) = F(G)/L$ by Lemma 2.10. Now set $E/L = E(G/L)$. Since $Q$ is normal in $G$ and $[F(G)/L, E/L] = 1$, $[Q, E] \leq Q \cap L = 1$, i.e., $[Q, E] = 1$. Therefore $F(G)E \leq C_G(Q) \trianglelefteq G$. If $C_G(Q) < G$, then $C_G(Q)$ is supersoluble by (2); thus $E(G/L) = E/L$ is supersoluble and consequently $F^*(G/L) = F(G)/L$. Clearly, we see that $G/L$ satisfies the hypotheses of the theorem. By the minimal choice of $G$, $G/L$ is supersoluble and so is $G$, a contradiction. Henceforth we have $C_G(Q) = G$, i.e. $Q \leq Z(G)$. Obviously, using the same argument as in the proof of $(7)$, $G/Q \in \mathcal{U}$ and hence $G$ is supersoluble, also is a contradiction. Thus we have

(8) $F(G) = P$, *in particular*, $1 < R = \Phi(G) \leq P$.

On the other hand, let $X$ be a minimal normal subgroup of $G$ contained in $P$ with $X \neq L$. By the definition of a generalized Fitting subgroup, $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G/L), E(G/L)] = 1$, where $E(G/L)$ is the layer of $G/L$. Since $L \leq \Phi(G)$, $F(G/L) = F(G)/L$ by

Lemma 2.10. Now set $E/L = E(G/L)$. Since $X$ is normal in $G$ and $[F(G)/L, E/L] = 1$, $[X, E] \leq \leq X \cap L = 1$, i.e., $[X, E] = 1$. Therefore $F(G)E \leq C_G(X) \trianglelefteq G$. If $C_G(X) < G$, then $C_G(X)$ is supersoluble by (2); thus $E(G/L) = E/L$ is supersoluble and consequently $F^*(G/L) = F(G)/L$. Using the same argument as in the proof of (3) we see that $G/L$ satisfies the hypotheses of the theorem. By the minimal choice of $G$, $G/L$ is supersoluble and so is $G$, a contradiction. Henceforth we have $C_G(X) = G$, i.e., $X \leq Z(G)$. Clearly, this also violates (4). Thus we have

(9) *$L$ is the unique minimal normal subgroup of $G$ contained in $P$.*

By (3), there exists a maximal subgroup $P_1$ of $P$ with $L \not\leq P_1$. By hypotheses, $P_1$ is nearly $\mathcal{M}$-supplemented in $G$. So there exists a normal subgroup $K_1$ in $G$ such that $P_1K_1 \trianglelefteq G$ and $TK_1 < P_1K_1$ for every maximal subgroup $T$ of $P_1$. If $P_1K_1 = G$, since $L \cap P_1 \neq 1$, we may choose a maximal subgroup $P_2$ of $P_1$ with $L \cap P_1 \not\leq P_2$ and $P_2K_1 < G$. On the other hand, $L \leq \Phi(G)$ and hence $P_2K_1 = LP_2K_1 = G$, a contradiction.

So we may assume $P_1K_1 < G$. Since $L$ is the unique minimal normal subgroup, we have $L \cap P_1K_1 = 1$ or $L$. If $L \cap P_1K_1 = 1$, then $L \cap P_1 = 1$ and hence $|L| = p$, contrary to (4). Therefore $L \leq P_1K_1$ and $P \leq P_1K_1$. By Lemma 2.9 and (2), $P_1K_1$ is supersoluble. Particularly, $K_1$ is supersoluble and $(K_1)_q \trianglelefteq K$ where $q$ is the largest prime divisor of $|K_1|$. On the other hand, $P_1 \cap K_1 \leq \Phi(P_1) = 1$ and $P \cap K_1 = 1$, otherwise, $|P \cap K_1| = p$, contrary to (4). It follows from $(K_1)_q$ char $K_1$ and $K_1 \trianglelefteq G$ imply that $(K_1)_q \trianglelefteq G$. So we have $(K_1)_q \leq P$, a contradiction. Therefore $K_1 = 1$ and $P_1K_1 = P_1 \trianglelefteq G$, contrary to the choice of $P_1$.

*Case* II. $\mathcal{F} \neq \mathcal{U}$.

By Case I, $H$ is supersoluble. Particularly, $H$ is soluble and hence $F^*(H) = F(H)$. Therefore $G \in \mathcal{F}$ by Theorem 3.3.

Theorem 3.4 is proved.

***Corollary* 3.3.** *Let $G$ be a group with a normal subgroup $H$ such that $G/H \in \mathcal{U}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(H)$ is nearly $\mathcal{M}$-supplemented in $G$, then $G \in \mathcal{U}$.*

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Received 03.01.12