UDC 517.9

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ON TWO-DIMENSIONAL MODEL REPRESENTATIONS OF ONE CLASS OF COMMUTING OPERATORS

ПРО ДВОВИМІРНІ МОДЕЛЬНІ ЗОБРАЖЕННЯ ОДНОГО КЛАСУ КОМУТУЮЧИХ ОПЕРАТОРІВ

In the work by Zolotarev V. A. “On triangular models of systems of twice commuting operators” (Dokl. Akad. Nauk ArmSSR. – 1976. – **63**, № 3. – P. 136 – 140 (in Russian)), a triangular model is constructed for the system of twice-commuting linear bounded completely nonself-adjoint operators $\{A_1, A_2\}$ ($[A_1, A_2] = 0$, $[A_1^*, A_2] = 0$) such that $\operatorname{rank}(A_1)_I(A_2)_I = 1$ ($2i(A_k)_I = A_k - A_k^*$, $k = 1, 2$) and the spectrum of each operator A_k, $k = 1, 2$, is concentrated at zero. This triangular model has the form of a system of operators of integration over the independent variable in L^2_Ω where the domain $\Omega = [0, a] \times [0, b]$ is a compact set in $\mathbb{R}^2$ bounded by the lines $x = a$ and $y = b$ and by a decreasing smooth curve L connecting the points $(0, b)$ and $(a, 0)$.

У статті Золотарьова В. О. „Про трикутні моделі систем двічі переставних операторів” (Докл. АН АрмССР. – 1976. – **63**, № 3. – С. 136 – 140) для системи двічі переставних лінійних обмежених цілком несамоспряжених операторів $\{A_1, A_2\}$ ($[A_1, A_2] = 0$, $[A_1^*, A_2] = 0$) такої, що $\operatorname{rank}(A_1)_I(A_2)_I = 1$ ($2i(A_k)_I = A_k - A_k^*$, $k = 1, 2$) і спектр кожного із операторів A_k, $k = 1, 2$, зосереджено в нулі, побудовано трикутну модель, яка є системою операторів інтегрування по незалежній змінній в L^2_Ω, де $\Omega = [0, a] \times [0, b]$. В даній статті одержано узагальнення цього результату на випадок, коли область Ω модельного простору є компактом у $\mathbb{R}^2$, обмеженим прямими $x = a$, $y = b$ і спадною гладкою кривою L, що з’єднує точки $(0, b)$ і $(a, 0)$.

Triangular model [3 – 5] of nonself-adjoint bounded operator constructed first by M. S. Livšic plays an important role in several problems of spectral analysis for this operator class. In the simplest case, this model represents the integration operator [3 – 5] acting in the space $L^2_{(0,l)}$. Generalization of this result by M. S. Livšic for the systems of twice-commuting nonself-adjoint operators $\{A_1, A_2\}$, $A_1A_2 = A_2A_1$, $A_1^*A_2 = A_2A_1^*$ is obtained in work [6]. Namely, it is specified that this class of operator systems is realized by operators of integration by different variables in L^2_Ω, where $\Omega = [0, a] \times [0, b]$ is a rectangle ($0 < a < \infty$, $0 < b < \infty$). This line of investigation receives its development in the works [7, 8], where systems of nonself-adjoint operators, the commutators of which $C = [A_1, A_2]$ and $D = [A_1^*, A_2]$ are nilpotent ($D^m = 0$, $C^n = 0$, $n, m \in \mathbb{Z}_+$), are studied. In this case, the model operators are again the integrations by different variables in L^2_Ω, besides, the domain Ω represents the rectangle from which the series of rectangles adjoining the coordinate origin and point $(a, 0)$ are withdrawn. The problem of the construction of many-dimensional triangular models when the domain Ω of model space is given by the smooth descending curve connecting the points $(0, b)$ and $(a, 0)$ so far remained unsolved.

This paper is devoted to the solution of this problem, besides, we obtain generalization of the well-known result by M. S. Livšic (see Theorem 5).

I. Consider the continuous curve L in $\mathbb{R}^2_+$,

$$L = \{(x, \alpha(x)) \colon \alpha(0) = b, \ \alpha(a) = 0\}; \tag{1}$$

specified by the smooth, monotonously decreasing function $\alpha(x) \in C^1_{[0,a]}$ on $[0, a]$ ($0 < a, b < \infty$). Denote by Ω_L the compact in $\mathbb{R}^2_+$ bounded by the curve L (1) and the lines $x = a$, $y = b$. Define the

Hilbert space $L^2_{\Omega_L}$ formed by the quadratically summable functions $f(x,y)$,

$$L^2_{\Omega_L} \stackrel{\text{df}}{=} \left\{ f : \iint\limits_{\Omega_L} |f(x,y)|^2 dxdy < \infty \right\}. \tag{2}$$

Specify the commutative system of linear bounded operators in $L^2_{\Omega_L}$,

$$\left(\tilde{A}_1 f\right)(x,y) = i\int\limits_x^a f(t,y)dt, \qquad \left(\tilde{A}_2 f\right)(x,y) = i\int\limits_y^b f(x,s)\,ds. \tag{3}$$

It is easy to see that

$$\begin{aligned} 2\left(\left(\tilde{A}_1\right)_I f\right)(x,y) &= \chi_{\Omega_L} \int\limits_{\alpha^{-1}(y)}^a f(t,y)dt, \\ 2\left(\left(\tilde{A}_2\right)_I f\right)(x,y) &= \chi_{\Omega_L} \int\limits_{\alpha(x)}^b f(x,s)\,ds, \end{aligned} \tag{4}$$

where $\alpha^{-1}(y)$ is a smooth monotonously decreasing function on $[0,b]$, which is reciprocal to $\alpha(x)$, and χ_{Ω_L} is the characteristic function of the set Ω_L. (4) yields

$$\begin{aligned} L_1 &\stackrel{\text{df}}{=} \overline{\left(\tilde{A}_1\right)_I L^2_{\Omega_L}} = \left\{ f(y)\chi_{\Omega_L} \in L^2_{\Omega_L} \right\}, \\ L_2 &\stackrel{\text{df}}{=} \overline{\left(\tilde{A}_2\right)_I L^2_{\Omega_L}} = \left\{ g(x)\chi_{\Omega_L} \in L^2_{\Omega_L} \right\}. \end{aligned} \tag{5}$$

It is obvious that

$$\dim L_0 = 1, \qquad L_0 \stackrel{\text{df}}{=} L_1 \cap L_2, \tag{6}$$

and, besides, (3) yields that

$$\tilde{A}_1 L_2 \subseteq L_2, \qquad \tilde{A}_2 L_1 \subseteq L_1. \tag{7}$$

Specify smooth monotonously increasing functions

$$\lambda(y) \stackrel{\text{df}}{=} a - \alpha^{-1}(y), \qquad \mu(x) \stackrel{\text{df}}{=} b - \alpha(x). \tag{8}$$

The equalities

$$\iint\limits_{\Omega_L} \left|f(y)\chi_{\Omega_L}\right|^2 dxdy = \int\limits_0^b |f(y)|^2 \lambda(y)dy,$$

$$\iint\limits_{\Omega_L} \left|g(x)\chi_{\Omega_L}\right|^2 dxdy = \int\limits_0^a |g(x)|^2 \mu(x)dx$$

imply that the subspaces L_1, L_2 (5) are isomorphic to the weighted spaces

$$L^2_{(0,b)}(\lambda(y)dy) \stackrel{\text{df}}{=} \left\{ f : \int\limits_0^b |f(y)|^2 \lambda(y) dy < \infty \right\},$$
$$L^2_{(0,a)}(\mu(x)dx) \stackrel{\text{df}}{=} \left\{ g : \int\limits_0^a |g(x)|^2 \mu(x) dx < \infty \right\}. \tag{9}$$

The biunique correspondences between subspaces (5) and spaces (9) are realized by the mappings $f(y)\chi_{\Omega_L} \to f(y)$ and $g(x)\chi_{\Omega_L} \to g(x)$. Taking into account (4), we obtain that the operators $2\big(\tilde{A}_1\big)_I$, $2\big(\tilde{A}_2\big)_I$ after this mapping act in the spaces (9) via the multiplication by the functions (8),

$$2\left(\big(\tilde{A}_1\big)_I f\right)(y) = \lambda(y) f(y) \qquad \left(f \in L^2_{(0,b)}(\lambda(y)dy)\right),$$
$$2\left(\big(\tilde{A}_2\big)_I g\right)(x) = \mu(x) g(x) \qquad \left(g \in L^2_{(0,a)}(\mu(x)dx)\right). \tag{10}$$

It is easy to show that the commutator $\tilde{C} = \big[\tilde{A}_2, \tilde{A}_1^*\big]$ and its adjoint $\tilde{C}^*$ are given by

$$\big(\tilde{C} f\big)(x,y) = \int\limits_y^b ds \int\limits_{\alpha^{-1}(s)}^{\alpha^{-1}(y)} dt f(t,s) = \int\limits_0^{\alpha^{-1}(y)} dt \int\limits_{\alpha(t)}^b ds f(t,s),$$
$$\left(\tilde{C}^* f\right)(x,y) = \int\limits_x^a dt \int\limits_{\alpha(t)}^{\alpha(x)} ds f(t,s) = \int\limits_0^{\alpha(x)} ds \int\limits_{\alpha^{-1}(s)}^a dt f(t,s). \tag{11}$$

Theorem 1. *The operator $\tilde{C}$ is completely continuous, belongs to the Hilbert – Schmidt class and its spectrum is concentrated at zero, $\sigma\big(\tilde{C}\big) = \{0\}$. Moreover, the equalities*

$$\overline{\tilde{C} L^2_{\Omega_L}} = L_1, \qquad \overline{\tilde{C}^* L^2_{\Omega_L}} = L_2, \tag{12}$$

are true, where L_1 and L_2 are given by (5).

Proof. The operator $\tilde{C}$ (11) is an integral operator,

$$\big(\tilde{C} f\big)(x,y) = \iint\limits_{\Omega_L} K(x,y,t,s) f(t,s)\, dt ds,$$

the kernel of which is equal $K(x,y,t,s) = \chi_{\Omega_{x,y}}(t,s)$, where $\chi_{\Omega_{x,y}}(t,s)$ is the characteristic function of the set $\Omega_{x,y} = \big\{(t,s) \in \Omega_L \colon 0 \le t \le \alpha^{-1}(y)\big\}$. The quadratic summability of $K(x,y,t,s)$ in $L^2_{\Omega_L} \times L^2_{\Omega_L}$ implies [1] the complete continuity of $\tilde{C}$ and the Hilbert – Schmidt class membership of $\tilde{C}$.

To prove that $\overline{\tilde{C} L^2_{\Omega_L}} = L_1$, it is sufficient to ascertain that $\operatorname{Ker} \tilde{C}^* = \operatorname{Ker} \big(\tilde{A}_1\big)_I$. (4) implies that $\operatorname{Ker} \big(\tilde{A}_1\big)_I$ consists of such functions $f(x,y) \in L^2_{\Omega_L}$ that

$$\int\limits_{\alpha^{-1}(y)}^{a} f(t,y)dt = 0 \quad (\forall y \in [0,b]). \tag{13}$$

If $f(x,y) \in \operatorname{Ker} \tilde{C}^*$, then

$$\int\limits_{0}^{\alpha(x)} ds \int\limits_{\alpha^{-1}(s)}^{a} dt f(t,s) = 0 \quad (\forall x \in [0,a]), \tag{14}$$

in view of (11). Since (13) implies (14), then $\operatorname{Ker} \left(\tilde{A}_1\right)_I \subseteq \operatorname{Ker} \left(\tilde{C}^*\right)$. To prove the truth of the inverse inclusion $\operatorname{Ker} \tilde{C}^* \subseteq \operatorname{Ker} \left(\tilde{A}_1\right)$, differentiate equality (14), then

$$\alpha'(x) \int\limits_{x}^{a} f(t,\alpha(x))dt = 0.$$

Taking into account $\alpha'(x) < 0$ as $x \in [0,a)$, we obtain relation (13) after the substitution $x = \alpha^{-1}(y)$.

To show that $\sigma\left(\tilde{C}\right) = \{0\}$, it is necessary to establish that the function $\left(I - z\tilde{C}\right)^{-1}$ is holomorphic for all $z \in \mathbb{C}$. Let $\left(I - z\tilde{C}\right)^{-1} g = f$, then $f(x,y)$ is the solution of the integral equation

$$f(x,y) - z \int\limits_{y}^{b} ds \int\limits_{\alpha^{-1}(s)}^{\alpha^{-1}(y)} dt f(t,s) = g(x,y). \tag{15}$$

Since $\tilde{C}f$ depends only on y (in view of (11)), then

$$f(x,y) = g(x,y) + \psi(y)\chi_{\Omega_L},$$

where $\psi(y) \in L^2_{(0,b)}(\lambda(y)dy)$. (15) yields that $\psi(y)$ satisfies the equation

$$\psi(y) - z \int\limits_{y}^{b} (\alpha^{-1}(y) - \alpha^{-1}(s))\psi(s)\, ds = z \int\limits_{y}^{b} ds \int\limits_{\alpha^{-1}(s)}^{\alpha^{-1}(y)} dt g(t,s).$$

The function

$$\varphi(y) \stackrel{\mathrm{df}}{=} \int\limits_{y}^{b} ds \int\limits_{\alpha^{-1}(s)}^{\alpha^{-1}(y)} dt g(t,s)$$

is continuous and is expressed via the known function $g(x,y)$. Thus we obtain the integral equation

$$\psi(y) - zK\psi(y) = z\varphi(y, \tag{16}$$

where the operator K is given by

$$(Kf)(y) = \int_y^b (\alpha^{-1}(y) - \alpha^{-1}(s))f(s)\,ds = \int_y^b (\lambda(s) - \lambda(y))f(s)\,ds. \tag{17}$$

(16) implies that

$$\psi = z\varphi + z^2K\varphi + \ldots + z^{n+1}K^n\varphi + \ldots. \tag{18}$$

Show that this series converges for all $z \in \mathbb{C}$. Since $\lambda(y)$ is a monotonously increasing function, then it is obvious that

$$|(K\varphi)(y)| \leq \int_y^b \lambda(s)|\varphi(s)|ds \leq \int_0^b \lambda(s)|\varphi(s)|ds \leq d\|\varphi\|_{L^2_{(0,b)}(\lambda(y)dy)},$$

where

$$d^2 = \int_0^b \lambda(s)\,ds.$$

Taking into account this estimation and that $|\alpha^{-1}(y) - \alpha^{-1}(s)| \leq a$ $(\forall y,\, s \in [0,b])$, we obtain

$$\left|\left(K^2\right)(y)\right| \leq \left|\int_y^b (\alpha^{-1}(y) - \alpha^{-1}(s))(K\varphi)(s)\,ds\right| < ad\|\varphi\|(b-y).$$

Repeating this procedure n times, we obtain

$$\left|\left(K^n\varphi\right)(y)\right| \leq a^{n-1}d\|\varphi\|\frac{(b-y)^{n-1}}{(n-1)!} \quad (\forall n \in \mathbb{N}).$$

Therefore, for $\psi(y)$ (18), we have

$$|\psi(y)| \leq |z||\varphi(y)| + |z|^2d\|\varphi\| + |z|^3ad\|\varphi\|(b-y) + \ldots$$

$$\ldots + |z|^{n+1}a^{n-1}d\|\varphi\|\frac{(b-y)^{n-1}}{(n-1)!} + \ldots = |z||\varphi(y)| + |z|^2d\|\varphi\|\exp\{a\cdot|z|(b-y)\}.$$

Since the series in the right-hand side converges uniformly for all $z \in \mathbb{C}$, the function $\left(I - z\tilde{C}\right)^{-1}$ is holomorphic in the plane $\mathbb{C}$.

Theorem 1 is proved.

Note that (12) implies that

$$\overline{\tilde{C}L_2} = L_1, \qquad \overline{\tilde{C}^*L_1} = L_2, \tag{19}$$

besides,

$$\begin{aligned} \tilde{C}g(x)\chi_{\Omega_L} &= \chi_{\Omega_L}\int_0^{\alpha^{-1}(y)} g(t)\mu(t)dt, \\ \tilde{C}^*f(y)\chi_{\Omega_L} &= \chi_{\Omega_L}\int_0^{\alpha(x)} f(s)\lambda(s)\,ds, \end{aligned} \tag{20}$$

in view of (11).

Denote by $\widetilde{E}_s^1 \overset{\text{df}}{=} E_{\lambda(s)}^1$ and $\widetilde{E}_t^2 \overset{\text{df}}{=} E_{\mu(t)}^2$ the resolutions of identity of the self-adjoint operators $2\big(\tilde{A}_1\big)_I\Big|_{L_1}$ and $2\big(\tilde{A}_2\big)_I\Big|_{L_2}$. Then, in view of (4), (10),

$$\begin{aligned} \widetilde{E}_s^1 f(y)\chi_{\Omega_L} &= \chi_{[0,s]}(y) f(y)\chi_{\Omega_L} \qquad (s \in [0,b]), \\ \widetilde{E}_t^2 g(x)\chi_{\Omega_L} &= \chi_{[0,t]}(x) g(x)\chi_{\Omega_L} \qquad (t \in [0,a]), \end{aligned} \tag{21}$$

where $\chi_{[0,c]}(\xi)$ is the characteristic function of the set $[0,c]$. It is easy to see that

$$\widetilde{E}_s^1 L_1 \perp \widetilde{E}_{\alpha^{-1}(s)}^2 L_2 \quad (\forall s \in [0,b]). \tag{22}$$

Obviously, the subspaces $L_1(s) \overset{\text{df}}{=} E_s^1 L_1$ and $L_2(t) \overset{\text{df}}{=} E_t^2 L_2$ form the maximal chains of invariant subspaces $\tilde{A}_1\Big|_{L_2}$ and $\tilde{A}_2\Big|_{L_1}$ correspondingly,

$$\tilde{A}_1 L_1(s) \subseteq L_1(s) \quad (\forall s \in [0,b]), \qquad \tilde{A}_2 L_1(t) \subseteq L_1(t) \quad (\forall t \in [0,a]). \tag{23}$$

Note that the commutator $\tilde{C}$ has the property

$$\overline{\tilde{C}\left(I_{L_2} - E_t^2\right) L_2} = E_{\alpha(t)}^1 L^1 \quad (\forall t \in [0,a]). \tag{24}$$

Equality (24) can be written in the following form:

$$\left(I_{L_1} - E_{\alpha(t)}^1\right) \tilde{C}\left(I_{L_2} - E_t^2\right) = 0 \quad (\forall t \in [0,a]). \tag{25}$$

***Remark* 1.** The orthogonality condition (22), as well as equality (25), can be taken as a definition of the function $\alpha^{-1}(y)$ (and so of $\alpha(x)$) specifying the domain Ω_L.

Denote by P_{L_1} the orthoprojection on L_1 (5) in $L_{\Omega_L}^2$,

$$\left(P_{L_1} f\right)(x,y) \overset{\text{df}}{=} \frac{\chi_{\Omega_L}}{\lambda(y)} \int\limits_{\alpha^{-1}(y)}^{a} f(t,y)\,dt, \tag{26}$$

and specify the self-adjoint operator σ_1 in L_1,

$$\sigma_1 f(y)\chi_{\Omega_L} \overset{\text{df}}{=} \int\limits_{\alpha^{-1}(y)}^{a} f(y)\chi_{\Omega_L}\,dt = \lambda(y) f(y)\chi_{\Omega_L}. \tag{27}$$

(4) implies that $2\big(\tilde{A}_1\big)_I = P_{L_1}\sigma_1 P_{L_1}$, and so the family

$$\tilde{\Delta}_1 = \left(\tilde{A}_1; L_{\Omega_L}^2; P_{L_1}; L_1; \sigma_1\right) \tag{28}$$

is a colligation [4, 5], where the operators $\tilde{A}_1$, P_{L_1}, σ_1 are given by (3), (26), (27), and the spaces $L_{\Omega_L}^2$ and L_1 are equal to (2), (5) correspondingly.

Theorem 2. *The characteristic function* $S_{\tilde{\Delta}_1}(z)$ (7) *of the colligation* $\tilde{\Delta}_1$ (28) *is a scalar operator in* L_1,

$$S_{\tilde{\Delta}_1}(z) = e^{\frac{i\lambda(y)}{z}} I_{L_1}, \tag{29}$$

where $\lambda(y)$ *equals* (8).

Proof. The function $f(x,y) = \left(\tilde{A}_1 - zI\right)^{-1} P_{L_1}\sigma_1 f(y)\chi_{\Omega_L}$ satisfies the integral equation

$$i\int_x^a f(t,y)dt - zf(x,y) = \lambda(y)f(y)\chi_{\Omega}, \tag{30}$$

which is equivalent to the Cauchy problem

$$if(x,y) + z\frac{\partial}{\partial x}f(x,y) = 0,$$

$$f(a,y) = -\frac{\lambda(y)}{z}f(y).$$

This implies

$$f(x,y) = -\frac{\lambda(y)}{z}f(y)e^{\frac{i(a-x)}{z}}\chi_{\Omega_L}.$$

Therefore

$$S_{\tilde{\Delta}_1}(z)f(y)\chi_{\Omega_L} - i\frac{\chi_{\Omega_L}}{\lambda(y)}\int_{\alpha^{-1}(y)}^{a} f(t,y)dt =$$

$$= \left\{f(y) - \frac{1}{\lambda(y)}[zf(x(y),y) + \lambda(y)f(y)]\right\}\chi_{\Omega_L} = e^{\frac{i\lambda(y)}{z}}f(y)\chi_{\Omega_L},$$

in view of equation (30).

Theorem 2 is proved.

***Remark* 2.** The operator-function $S_{\tilde{\Delta}_1}(z)$ (29) commutes with the operator σ_1 (27) for all $z \in \mathbb{C}$, $z \neq 0$.

***Remark* 3.** Consider the restriction of the operator $\tilde{A}_2$ (3) on the invariant (7) subspace L_1 (5). In spite of the fact that L_0 (6) is one-dimensional, nevertheless, the closure of the operator image $P_{L_1}2\left(\tilde{A}_2\right)_I\Big|_{L_1}$ coincides with the whole of L_1. Really, since

$$\tilde{A}_2^* f(y)\chi_{\Omega_L} = -i\int_{\alpha(x)}^{y} f(s)\chi_{\Omega_L}ds,$$

then, taking into account the form of the orthoprojection P_{L_1} (26), it is easy to show that

$$P_{L_1}\tilde{A}_2^* f(y)\chi_{\Omega_L} = -i\frac{\chi_{\Omega_L}}{\lambda(y)}\int_0^y f(s)\lambda(s)\,ds.$$

Therefore

$$P_{L_1}2\big(\tilde{A}_2\big)_I f(y)\chi_{\Omega_L} = \chi_{\Omega_L}\left\{\int\limits_y^b f(s)\,ds + \frac{1}{\lambda(y)}\int\limits_0^y f(s)\lambda(s)\,ds\right\}.$$

Let $f(y)\chi_{\Omega_L} \in \operatorname{Ker} P_{L_1}2\big(\tilde{A}_2\big)\Big|_{L_1}$, then

$$\lambda(y)\int\limits_y^b f(s)\,ds + \int\limits_0^y f(s)\lambda(s)\,ds = 0.$$

Differentiating we obtain

$$\lambda'(y)\int\limits_y^b f(s)\,ds = 0,$$

and since $\lambda'(y) \neq 0$, this implies $f(y) = 0$. Thus $\operatorname{Ker} P_{L_1}2\big(\tilde{A}_2\big)_I\Big| = 0$ and so $\overline{P_{L_1}2\big(\tilde{A}_2\big)_I L_1} = L_1$.

II. Consider a bounded self-adjoint operator in a Hilbert space H with the simple spectrum in the segment $[0,a]$. Then [1] the operator B is unitary equivalent to the operator of multiplication by an independent variable

$$\big(\hat{B}f\big)(\lambda) = \lambda f(\lambda) \quad \Big(f(\lambda) \in L^2_{(0,a)}(d\sigma(\lambda))\Big), \tag{31}$$

where $\sigma(\lambda) = \langle E_\lambda u, u\rangle$ is nondecreasing on $[0,a]$; E_λ is the resolution of identity of B; and $u \in H$ is the generating vector of the operator B. This unitary equivalence is given by the map U [1],

$$Uf(\lambda) = f, \qquad f \overset{\text{df}}{=} \int\limits_0^a f(\lambda)dE_\lambda u, \tag{32}$$

besides, $f(\lambda) \in L^2_{(0,a)}(d\sigma(\lambda))$ and $f \in H$. Suppose that the measure $d\sigma(\lambda)$ is absolutely continuous by the Lebesgue measure,

$$d\sigma(\lambda) = m(\lambda)d\lambda \qquad (m(\lambda) = \sigma'(\lambda) \geq 0). \tag{33}$$

Definition 1. *An absolutely continuous measure $d\sigma(\lambda)$ (33) is said to have the AC_0-property if*

$$\int\limits_0^\lambda \frac{d\sigma(t)}{t} < \infty, \tag{34}$$

for all $\lambda \in [0,a]$.

Requirement (34), per se, is conditioned by the convergence of the given improper integral at zero. Define the smooth monotonously increasing function $y(\lambda)$,

$$y(\lambda) \overset{\text{df}}{=} \int\limits_0^\lambda \frac{d\sigma(t)}{t}. \tag{35}$$

***Remark* 4.** 'A priori' we can suppose that the function $y(\lambda)$ maps $[0,a]$ onto $[0,b]$, where b is a preset finite positive number. If $y(\lambda)\colon [0,a] \to [0,b]$ $(d > 0)$, then setting the measure $d\sigma_1(t) = \frac{b}{d} d\sigma(t)$ $(b > 0)$ and realizing the substitution $f(\lambda) \to f(\lambda)\sqrt{\frac{d}{b}}$ in $L^2_{(0,a)}$ $(d\sigma(\lambda))$ we obtain the Hilbert space $L^2_{(0,a)}$ $(d\sigma_1(\lambda))$ isomorphic to $L^2_{(0,a)}(d\sigma(\lambda))$, besides, the function $y_1(\lambda)$ constructed by $d\sigma_1(\lambda)$ (35) already possesses the values on $[0,b]$. This procedure signifies renormalization of the generating vector $u \to \sqrt{\frac{b}{d}}u$ since $\sigma(\lambda) = \langle E_\lambda u, u\rangle$.

Denote by $\lambda^{-1}(y)$ the function reciprocal to $y(\lambda)$ (35). Since $d\sigma(\lambda) = \lambda dy(\lambda)$, then the change of variable $\lambda \to \lambda(y)$ translates the space $L^2_{(0,a)}(d\sigma(\lambda))$ into $L^2_{(0,b)}(\lambda(y)dy)$ where the operator $\hat{B}$ (31) acts as a multiplication by the function $\lambda(y)$,

$$\big(\hat{B}f\big)(y) = \lambda(y)f(y) \quad \Big(f(y) \in L^2_{(0,b)}(\lambda(y)dy)\Big). \tag{36}$$

Theorem 3. *Let B be a bounded self-adjoint operator with the simple spectrum in H, besides, the spectrum of B belongs to the segment $[0,a]$. If the spectral measure $\sigma(\lambda)$ of the operator B is absolutely continuous* (33) *and has the AC_0-property* (34), *then the operator B is unitary equivalent to the operator of multiplication $\tilde{B}$* (36) *by the smooth monotonously increasing function $\lambda(y)$* (*reciprocal to $y(\lambda)$* (35)) *in $L^2_{(0,b)}(\lambda(y)dy)$, besides, the finite positive number b can be chosen arbitrarily.*

The following statement gives the description of the commutant of the operator $\hat{B}$ (31).

Theorem 4. *An arbitrary linear bounded operator $\hat{A}$ in $L^2_{(0,a)}$ $(d\sigma(\lambda))$ commuting with $\hat{B}$* (31) *is the operator of multiplication,*

$$\Big(\hat{A}f\Big)(\lambda) = a(\lambda)f(\lambda) \quad \Big(f \in L^2_{(0,a)}(d\sigma(\lambda))\Big), \tag{37}$$

where $a(\lambda)$ is a complex-valued function from $L^2_{(0,a)}(d\sigma(\lambda))$, besides, $\|A\| = \|a(\lambda)\|_{L^2_{(0,a)}(d\sigma(\lambda))}$.

Proof. Let A be a linear bounded operator in H commuting with B where B is a self-adjoint operator with the simple spectrum and $\sigma(B) \subseteq [0,a]$. The permutability of A and B implies [1] that $[A, E_\lambda] = 0$ $(\forall\lambda \in [0,a])$ where E_λ is the resolution of identity of the operator B. (32) yields that

$$Af = \int_0^a f(\lambda) dE_\lambda Au.$$

To the vector Au from H in view of the mapping U (32) there corresponds such function $a(\lambda) \in \in L^2_{(0,a)}(d\sigma(\lambda))$ that

$$Au = \int_0^a a(\lambda) dE_\lambda u,$$

and so

$$\langle E_\lambda Au, u\rangle = \int_0^\lambda a(t) d\sigma(t).$$

This implies that

$$\langle Au, E_\lambda u\rangle = \int_0^\lambda f(t)a(t)d\sigma(t) = \int_0^a f(t)d_t\,\langle E_t u, E_\lambda u\rangle = \left\langle \int_0^a f(t)a(t)dE_t u, E_\lambda u\right\rangle.$$

And since the linear manifold of the vectors $E_\lambda u$ is dense in H, we obtain that

$$Af = \int_0^a f(\lambda)a(\lambda)dE_\lambda u.$$

Thus $AU = U\hat{A}$ where $\hat{A}$ is given by (37), and U is given by formula (32).

Theorem 4 is proved.

So, the commutant $\hat{B}'$ coincides with the set of the operators given by (37) and is isomorphic to the space $L^2_{(0,a)}(d\sigma(\lambda))$.

The following generalization of the M. S. Livšic Theorem is true.

Theorem 5. *Let a linear bounded dissipative completely nonself-adjoint operator with the spectrum at zero, $\sigma(A) = \{0\}$, be given in a Hilbert space H, and the following conditions be met*:

1) *operator $2A_I$ restricted on $H_1 = \overline{A_I H}$ has a simple spectrum filling the finite segment $[0, a]$, $0 < a < \infty$, and its spectral function $\sigma(\lambda)$ is absolutely continuous* (33) *and has the AC_0-property* (34);

2) *for all $z \in \mathbb{C}$ $(z \neq 0)$, $\left[P_{H_1}(A - zI)^{-1}P_{H_1}, A_I\right] = 0$ takes place, where P_{H_1} is the orthoprojection on H_1.*

Then the operator A is unitary equivalent to the integration operator,

$$\left(\tilde{A}f\right)(x, y) = i\int_x^a f(t, y)dt, \tag{38}$$

in the space $L^2_{\Omega_L}$ (2), *besides, the curve L* (1) *is given by the function $\alpha^{-1}(y) = a - \lambda^{-1}(y)$, where $\lambda^{-1}(y)$ is the reciprocal to $y(\lambda)$* (35) *function.*

Proof. Theorem 3 implies that there exists a unitary operator $U\colon H_1 \to L^2_{(0,b)}(\lambda(y)dy)$ such that $U2A_I = \tilde{B}U$, where $\tilde{B}$ is given by (36). Construct a colligation

$$\Delta = \left(A; H; UP_{H_1}; L^2_{(0,b)}(\lambda(y)dy); \tilde{B}\right).$$

Condition 2 of the Theorem implies that the characteristic function $S_\Delta(z)$ of this colligation commutes with $\tilde{B}$. Using Theorem 4, we obtain that $S_\Delta(z)$ is an operator of multiplication by the function $\exp\left\{iz^{-1}c(y)\right\}$ in the space $L^2_{(0,b)}(\lambda(y)dy)$, in view of the standard type of the characteristic function, if one takes into account that $\sigma(A) = \{0\}$ and $\tilde{B} \geq 0$. Note that $c(y) = \lambda(y)$ since $\lim_{z\to\infty} iz\,(I - S_\Delta(z)) = \tilde{B}$.

Knowing $\lambda(y)$, from formula (8) we find the smooth decreasing function $\alpha^{-1}(y)$ specifying the curve L (1) and so the domain Ω_L also, for the functions from the space $L^2_{\Omega_L}$. After this we construct the colligation Δ_1 (28) and observe that the characteristic functions of the colligations Δ_1 and Δ coincide in view of Theorem 2. Application of the Theorem on unitary equivalence [4] concludes the proof.

Note that condition 2 in the M. S. Livšic Theorem is met automatically since $\operatorname{rank} A_I = 1$.

III. Consider two absolutely continuous measures

$$d\sigma(\lambda) = m(\lambda)d\lambda, \qquad d\omega(\mu) = n(\mu)d\mu \tag{39}$$

defined on the finite segments, $\lambda \in [0, a]$, $\mu \in [0, b]$, where $m(\lambda) \geq 0$ and $n(\mu) \geq 0$. Supposing that $d\sigma(\lambda)$ and $d\omega(\lambda)$ has the AC_0-property (34) and taking into account (35), we define the positive increasing functions

$$y(\lambda) \stackrel{\text{df}}{=} \int\limits_0^\lambda \frac{d\sigma(t)}{t}, \qquad x(\mu) \stackrel{\text{df}}{=} \int\limits_0^\mu \frac{d\omega(s)}{s}. \tag{40}$$

Remark 4 yields that we can suppose that $y(\lambda)$: $[0, a] \to [0, b]$ and $x(\mu)$: $[0, b] \to [0, a]$. Denote by $\lambda^{-1}(y)$ and $\mu^{-1}(x)$ the functions reciprocal to $y(\lambda)$ and $x(\mu)$ (40). Differentiating (40), we obtain that

$$m(\lambda) = \lambda y'(\lambda), \qquad n(\mu) = \mu x'(\mu). \tag{41}$$

Suppose that the functions $\lambda(y)$ and $\mu(x)$ are given by (8), then taking into account that $\alpha^{-1}(y)$ is the function reciprocal to $\alpha(x)$, we have

$$x = a - \lambda(b - \mu(x)),$$

and after the substitution $x = x(\mu)$ we obtain that

$$a - x(\mu) = \lambda(b - \mu). \tag{42}$$

This implies

$$x'(\mu) = \lambda'(b - \mu),$$

or, using (41), we obtain the equality

$$\lambda'(b - \mu) = \frac{n(\mu)}{\mu}. \tag{43}$$

Since $y = y(\lambda^{-1}(y))$, then

$$1 = \frac{dy(\lambda^{-1}(y))}{d\lambda}\frac{d\lambda^{-1}(y)}{dy} = \frac{m(\lambda^{-1}(y))}{\lambda(y)}\frac{n(b-y)}{b-y},$$

in view of (41) and (43). Thus

$$n(\mu)m(\lambda^{-1}(b - \mu)) = \mu\lambda(b - \mu) \quad (\forall \mu \in [0, b]). \tag{44}$$

Lemma 1. *Let two absolutely continuous measures* $d\sigma(\lambda)$ *and* $d\omega(\mu)$ (39) *have the* AC_0-*property* (34). *Then in order that* (42) *take place, where* $\lambda^{-1}(y)$ *is the function reciprocal to* $y(\lambda)$ (40) *and* $x(\mu)$ *is given by* (40), *it is necessary and sufficient that the fitting condition* (44) *is met.*

***Remark* 5.** The fitting condition (44) for the measures $d\sigma(\lambda)$ and $d\omega(\mu)$ (39) provides realization of the functions $\lambda^{-1}(y)$ and $\mu^{-1}(x)$ in the form of (8) where $\alpha(x)$ and $\alpha^{-1}(y)$ are mutually reciprocal functions. Since $\lambda(y)$ is explicitly constructed by $m(\lambda)$ in view of (40), then (44) implies that $n(\mu)$ is uniquely defined by the function $m(\lambda)$. Finally, the truth of the AC_0-property for $d\omega(\mu)$ in this case signifies that

$$\int\limits_0^\mu \frac{n(s)}{s}ds = \int\limits_0^\mu \frac{\lambda(b-s)}{m(\lambda^{-1}(b-s))}ds < \infty$$

for all $\mu \in [0, b]$.

IV. Consider a commutative system of linear bounded operators $\{A_1, A_2\}$ in a Hilbert space H. Denote by H_1 and H_2 two subspaces in H,

$$H_1 \stackrel{\text{df}}{=} \overline{(A_1)_I H}, \qquad H_2 \stackrel{\text{df}}{=} \overline{(A_2)_I H}. \tag{45}$$

Theorem 6. *Let two self-adjoint bounded nonnegative operators $2(A_1)_I$ and $2(A_2)_I$ be given in a Hilbert space H such that*:

1) *the restrictions $2(A_k)_I|_{H_k}$ on the subspaces H_k, $k = 1, 2$,* (45) *are operators with simple absolutely continuous spectrum, besides, $\sigma\left(2(A_1)_I|_{H_1}\right) = [0, a]$ and $\sigma\left(2(A_2)_I|_{H_2}\right) = [0, b]$;*

2) *the spectrum measures $\sigma(\lambda)$ and $\omega(\mu)$ corresponding to $2(A_1)_I|_{H_1}$ and $2(A_2)_I|_{H_2}$ respectively are absolutely continuous* (39), *have the AC_0-property* (34), *and* (44) *takes place, where $\lambda^{-1}(y)$ is the function reciprocal to $y(\lambda)$* (40).

Then there exist the Hilbert space $L^2_{\Omega_L}$ (2) *and the isometric operator $U\colon H_1{+}H_2 \to L_1{+}L_2$ (L_k are given by* (5), *$k = 1, 2$) realizing unitary equivalence between the operators $P_{H_1+H_2}2(A_k)_I|_{H_1+H_2}$ and $P_{L_1+L_2}2(\tilde{A}_k)_I\Big|_{L_1+L_2}$, $k = 1, 2$, besides,*

$$\left(2(\tilde{A}_1)_I f\right)(x, y) = \int\limits_{\alpha^{-1}(y)}^{a} f(t, y)dt, \qquad \left(2((\tilde{A}_2)_I f\right)(x, y) = \int\limits_{\alpha(x)}^{b} f(x, s)\, ds, \tag{46}$$

where $f(x, y) = [f(y) + g(x)]\,\chi_{\Omega_L} \in L_1 + L_2$.

Proof. The conditions 1, 2, and Theorem 3 imply that the operator $2(A_1)|_{H_1}$ restricted on H_1 is unitary equivalent to the operator of multiplication by the function $\lambda(y)$ in the function space $L^2_{(0,b)}(\lambda^{-1}(y)dy)$. Similarly, $2(A_2)_I|_{H_2}$ on H_2 is unitary equivalent to the operator of multiplication by the function $\mu(x)$ in the space $L^2_{(0,a)}(\mu^{-1}(x)dx)$. The functions $\lambda^{-1}(y)$: $[0, b] \to [0, a]$ and $\mu^{-1}(x)$: $[0, a] \to [0, b]$ are the reciprocal to $y(\lambda)$ and $x(\mu)$ (40). The fitting condition (44) implies that $\lambda^{-1}(y)$ and $\mu^{-1}(x)$ are given by (8), where $\alpha^{-1}(y)$ and $\alpha(x)$ are mutually reciprocal functions.

Knowing $\alpha(x)$, we construct the curve L (1) and define the Hilbert space $L^2_{\Omega_L}$ (2) of the functions $f(x, y)$ in the domain Ω_L. As was noted before (see Section I), the mappings $f(y) \to f(y)\chi_{\Omega_L}$, $g(x) \to g(x)\chi_{\Omega_L}$ set an isomorphism between the spaces

$$L^2_{(0,b)}(\lambda^{-1}(y)dy) \leftrightarrow L_1, \qquad L^2_{(0,a)}(\mu^{-1}(x)dx) \leftrightarrow L_2,$$

where L_1, L_2 are given by (5). Besides, the operator of multiplication by $\lambda(y)$ in $L^2_{(0,b)}(\lambda^{-1}(y)dy)$ transforms into the operator $2(\tilde{A}_1) = \int_{\alpha^{-1}(y)}^{a} .dt$ on L_1, and the operator of multiplication by $\mu(x)$ in $L^2_{(0,a)}(\mu^{-1}(x)dx)$ transforms correspondingly into the operator $2(\tilde{A}_2)_I = \int_{\alpha(x)}^{b} .ds$ on L_2. So, each subspace H_k (45) from H is isomorphic to L_k (5) in $L^2_{\Omega_L}$, $k = 1, 2$, and the operators $2(A_k)_I|_{H_k}$ appear to be unitary equivalent to $2(\tilde{A}_k)_I\Big|_{L_k}$, $k = 1, 2$,

$$2(\tilde{A}_1)_I f(y)\chi_{\Omega_L} = \int\limits_{\alpha^{-1}(y)}^{a} f(y)\chi_{\Omega_L} dt, \qquad 2(\tilde{A}_2)_I g(x)\chi_{\Omega_L} = \int\limits_{\alpha(x)}^{b} g(x)\chi_{\Omega_L} ds, \tag{47}$$

where $f(y)\chi_{\Omega_L} \in L_1$ and $g(x)\chi_{\Omega_L} \in L_2$.

Show that knowing these mappings $H_k \to L_k$, $k = 1, 2$, one can realize the unitary equivalence between $H_1 + H_2$ from H and $L_1 + L_2$ from $L^2_{\Omega_L}$. To do this, consider

$$\langle h_1, h_2 \rangle = \langle h_1, P_{H_1} h_2 \rangle ,$$

where $h_k \in H_k$, $k = 1, 2$. Let $h_1 \to f(y)\chi_{\Omega_L} \in L_1$ and $h_2 \to g(x)\chi_{\Omega_L} \in L_2$, then

$$\langle f(y)\chi_{\Omega_L}, g(x)\chi_{\Omega_L} \rangle_{L^2_{\Omega_L}} = \int\limits_0^b dy \int\limits_{\alpha^{-1}(y)}^a dx f(y)\overline{g(x)} = \int\limits_0^b dy f(y) \int\limits_{\alpha^{-1}(y)}^a \overline{g(x)} dx.$$

Since operator (26)

$$(P_{L_1} f)(x, y) = \frac{\chi_{\Omega_L}}{\lambda(y)} \int\limits_{\alpha^{-1}(y)}^a f(t, y) dt$$

is the orthoprojection on L_1 in $L^2_{\Omega_L}$,

$$\langle f(y)\chi_{\Omega_L}, g(x)\chi_{\Omega_L} \rangle_{L^2_{\Omega_L}} = \left\langle f(y), \frac{1}{\lambda(y)} \int\limits_{\alpha^{-1}(y)}^a g(x) dx \right\rangle_{L^2_{(0,b)}(\lambda^{-1}(y)dy)} =$$

$$= \langle f(y)\chi_{\Omega_L}, P_{L_1} g \chi_{\Omega_L} \rangle_{L^2_{\Omega_L}} .$$

Thus

$$\langle h_1, h_2 \rangle_H = \langle f(y)\chi_{\Omega_L}, g(x)\chi_{\Omega_L} \rangle_{L^2_{\Omega_L}} ,$$

and so the correspondence $h_1 + h_2 \to [f(y) + g(x)]\chi_{\Omega_L}$ is a unitary isomorphism between $H_1 + H_2$ and $L_1 + L_2$.

To complete the proof of the theorem, it is left for us to ascertain that the formulas (46) are true. For $2\big(\tilde{A}_1\big)_I$ (for example) formula (46) on functions of the type $f(y)\chi_{\Omega_L}$ is already proved (47). It is left to ascertain the truth of (46) on the functions $g(x)\chi_{\Omega_L}$. Really, since

$$2(A_1)_I h_2 = 2(A_1) P_{H_1} h_2, \quad h_2 \in H,$$

then

$$2\big(\tilde{A}_1\big)_I P_{L_1} g(x)\chi_{\Omega_L} = 2\big(\tilde{A}_1\big)_I \frac{\chi_{\Omega_L}}{\lambda(y)} \int\limits_{\alpha^{-1}(y)}^a g(t) dt = \int\limits_{\alpha^{-1}(y)}^a g(t)\chi_{\Omega_L} dt,$$

in view of (47), where $h_2 \to g(x)\chi_{\Omega_L}$.

Theorem 6 is proved.

Define now the class of linear operators K_∞, which in some sense is close to the class K_n [6] when $n = \infty$, but cannot be obtained from K_n as $n \to \infty$.

The class K_∞. *A system of linear bounded operators $\{A_1, A_2\}$ in a Hilbert space H is said to belong to the class K_∞ if*

1) $[A_1, A_2] = 0$; (48)

2) $\overline{CH} = H_1$, $\overline{C^* H} = H_2$, *where* $C = [A_2, A_1^*]$, *and* H_k, $k = 1, 2$, *are given by* (45);

3) *the operator C is completely continuous and belongs to the Hilbert–Schmidt class, and its spectrum lies at zero, $\sigma(C) = \{0\}$.*

Theorem 1 implies that the operator system $\{\tilde{A}_1, \tilde{A}_2\}$ (3) in $L^2_{\Omega_L}$ (2) belongs to the class K_∞.

Theorem 7. *Suppose that the operator system $\{A_1, A_2\}$ belongs to the class K_∞ (48) and conditions 1, 2 of Theorem 6 are met. Let*

$$\left[I_{H_1} - iE^1_{\lambda^{-1}(\alpha(t))}\right] C \left[I_{H_2} - E^2_{\mu^{-1}(t)}\right] = 0; \tag{49}$$

takes place for all t, where E^1_λ and E^2_μ are the resolutions of identity of the operators $2(A_1)_I|_{H_1}$ and $2(A_2)|_{H_2}$, the functions $\mu^{-1}(x)$ and $\lambda^{-1}(y)$ are reciprocal to $x(\mu)$ and $y(\lambda)$ (40) correspondingly. Then the operator C is unitary equivalent to the operator $\tilde{C}\colon L_2 \to L_1$,

$$\tilde{C}g(x)\chi_{\Omega_L} = \chi_{\Omega_L} \int\limits_0^{\alpha^{-1}(y)} dtg(t)\mu(t) = \chi_{\Omega_L} \int\limits_y^b ds \int\limits_{\alpha^{-1}(s)}^{\alpha^{-1}(y)} dtg(t), \tag{50}$$

for all $g(x)\chi_{\omega_L} \in L_2$.

Proof. The isometric mapping $H_1 + H_2$ on $L_1 + L_2$ constructed in the proof of Theorem 6 transforms the operator C into the operator $\tilde{C}$ mapping surjectively L_2 onto L_1. The Hilbert–Schmidt operator $\tilde{C}$ always can be represented as

$$\tilde{C} = \sum_{k=1}^{\infty} s_k \left\langle ., \varphi_k(x)\chi_{\Omega_L}\right\rangle \psi_k(y)\chi_{\Omega_L}, \quad s_k > 0,$$

besides, the series converges by the norm of the space $L^2_{\Omega_L} \times L^2_{\Omega_L}$, the functions $\varphi_k(x)\chi_{\Omega_L}$ form the complete orthonormal system in L_2 of the eigenvectors of the operator $\sqrt{\tilde{C}^*\tilde{C}}$, and $\psi_k(y)\chi_{\Omega_L} = U\varphi_k(x)\chi_{\Omega_L}$ is the orthonormal basis in L_1, where U is a unitary operator from L_2 onto L_1 corresponding to the polar decomposition $\tilde{C} = U\sqrt{\tilde{C}^*\tilde{C}}$. The last formula implies that

$$\tilde{C}g(x)\chi_{\Omega_L} = \iint\limits_{\Omega_L} K(t,y)g(t)\chi_{\Omega_L} dtds = \chi_{\Omega_L} \int\limits_0^a dtK(t,y)g(t)\mu(t), \tag{51}$$

besides, the kernel $K(t,y)$ is given by

$$K(t,y) = \sum_{k=1}^{n} s_k\overline{\varphi_k(t)}\chi_{\Omega_L}(t,s)\psi_k(y)\chi_{\Omega_L}(x,y).$$

Since the condition of theorem (49) can be represented as (25) where the spectral projectors E^1_t and E^2_s are equal to (21), (49) implies that the operator $\tilde{C}$ (51) equals

$$\tilde{C}g_x\chi_{\Omega_L} = \chi_{\Omega_L} \int\limits_0^{\alpha^{-1}(y)} dtK(t,y)g(t)\mu(t) = \int\limits_0^{\alpha^{-1}(y)} dt \int\limits_{\alpha(t)}^b dsK(t,y)g(t), \tag{52}$$

and so

$$\tilde{C}^* f_y \chi_{\Omega_L} = \chi_{\Omega_L} \int\limits_0^{\alpha(x)} ds \overline{K(x,s)} f(s)\lambda(s) = \int\limits_0^{\alpha(x)} ds \int\limits_{\alpha^{-1}(s)}^{a} dt \overline{K(x,s)} f(s). \tag{53}$$

Use the following formula:

$$4\left[\left(\tilde{A}_1\right)_I, \left(\tilde{A}_2\right)_I\right] = \tilde{C}^* - \tilde{C}. \tag{54}$$

It is obvious that

$$\tilde{C}^* f(y)\chi_{\Omega_L} - \tilde{C} f(y)\chi_{\Omega_L} = \chi_{\Omega_L} \int\limits_0^{\alpha(x)} ds \overline{K(x,s)} f(s)\lambda(s) - \tilde{C} P_{L_2} f(y)\chi_{\Omega_L} =$$

$$= \chi_{\Omega_L} \int\limits_0^{\alpha(x)} ds \overline{K(x,s)} f(s)\lambda(s) - \chi_{\Omega_L} \int\limits_0^{\alpha^{-1}(y)} dt K(t,y) \int\limits_{\alpha(t)}^{b} ds f(s),$$

in view of (52), (53) and the fact that

$$P_{L_2} f(y)\chi_{\Omega_L} = \frac{\chi_{\Omega_L}}{\mu(x)} \int\limits_{\alpha(x)}^{b} ds f(s).$$

Since

$$P_{L_1} g(x)\chi_{\Omega_L} = \frac{\chi_{\Omega_L}}{\lambda(y)} \int\limits_{\alpha^{-1}(y)}^{a} dt g(t),$$

it is easy to see that

$$4\left\{\left(\tilde{A}_1\right)_I\left(\tilde{A}_2\right)_I - \left(\tilde{A}_2\right)_I\left(\tilde{A}_1\right)_I\right\} f(y)\chi_{\Omega_L} =$$

$$= 4\left(\tilde{A}_1\right)_I\left(\tilde{A}_2\right)_I P_{L_2} f(y)\chi_{\Omega_L} - 2\left(\tilde{A}_2\right)_I \lambda(y) f(y)\chi_{\Omega_L} =$$

$$= 2\left(\tilde{A}_1\right)_I P_{L_1}\chi_{\Omega_L} \int\limits_{\alpha(x)}^{b} ds f(s) - 2\left(\tilde{A}_2\right)\frac{\chi_{\Omega_L}}{\mu(x)} \int\limits_{\alpha(x)}^{b} ds f(s)\lambda(s) =$$

$$= \chi_{\Omega_L} \int\limits_{\alpha^{-1}(y)}^{a} dt \int\limits_{\alpha(t)}^{b} ds f(s) - \chi_{\Omega_L} \int\limits_{\alpha(x)}^{b} ds f(s)\lambda(s).$$

Using equality (54), we obtain that

$$\int\limits_0^{\alpha(x)} ds \overline{K(x,s)} f(s)\lambda(s) + \int\limits_{\alpha^{-1}(y)}^{b} ds f(s)\lambda(s) =$$

$$= \int\limits_{\alpha^{-1}(y)}^{a} dt \int\limits_{\alpha(t)}^{b} dsf(s) + \int\limits_{0}^{\alpha^{-1}(y)} dtK(t,y) \int\limits_{\alpha(t)}^{b} dsf(s).$$

In connection with the fact that the left-hand side of this equality does not depend on y, then supposing that $y = b$ and taking into account that $\alpha^{-1}(b) = 0$, we obtain the relation

$$\int\limits_{0}^{\alpha(x)} ds \left\{ \overline{K(x,s)} - \chi_{[0,b]}(s) \right\} f(s)\lambda(s) + \int\limits_{0}^{b} dsf(s)\lambda(s) =$$

$$= \int\limits_{0}^{a} dt \int\limits_{\alpha(t)}^{a} dsf(s) = \int\limits_{0}^{b} dsf(s)\lambda(s).$$

So,

$$\int\limits_{0}^{a} ds \left\{ \overline{K(x,s)} - \chi_{[0,b]}(s) \right\} f(s)\lambda(s) = 0,$$

for all $f(y)\chi_{\Omega_L} \in L_1$ and all $x \in [0,a]$. This easily implies that $K(x,y) = \chi_{\Omega_L}$, this concludes the proof of the theorem.

(50) and (53) imply that

$$\tilde{C}^* f(y)\chi_{\Omega_L} = \chi_{\Omega_L} \int\limits_{0}^{\alpha(x)} dsf(s)\lambda(s) = \chi_{\Omega_L} \int\limits_{\alpha(t)}^{\alpha(x)} dsf(s), \tag{55}$$

where $f(y)\chi_{\Omega_L} \in L_1$.

Formulate the main theorem.

Theorem 8. *Suppose that the operator system* $\{A_1, A_2\}$ *belongs to the class* K_∞ (48), *besides, each operator* A_k *is completely nonself-adjoint, dissipative, and the spectrum* $\sigma(A_k) = \{0\}$, $k = 1,2$, *and let the operator system* $\{A_1, A_2\}$ *be such that*

1) *the subspace* $H_0 = H_1 \cap H_2$ *is one-dimensional, where* H_k, $k = 1,2$, *are given by* (45);

2) *restrictions* $2(A_k)_I|_{H_k}$ *on the subspaces* H_k, $k = 1,2$, (45) *are operators with the simple completely continuous spectrum, and* $\sigma\left(P_{H_1} 2(A_1)_I|_{H_1}\right) = [0,a]$, $\sigma\left(P_{H_2} 2(A_2)_I|_{H_2}\right) = [0,b]$;

3) *the spectral measures* $\sigma(\lambda)$ *and* $\omega(\lambda)$ *corresponding to* $P_{H_1} 2(A_1)_I|_{H_1}$ *and* P_{H_2} $2(A_2)_I|_{H_2}$ *are absolutely continuous* (39) *and have the* AC_0*-property* (34), *besides,* (44) *takes place, where* $\lambda^{-1}(y)$ *is the function reciprocal to* $y(\lambda)$ (40);

4) *for all* $t \in [0,a]$ *condition* (49) *takes place where* E^1_λ *and* E^2_μ *are the resolutions of identity of the operators* $2(A_1)_I|_{H_1}$ *and* $2(A_2)_I|_{H_2}$, *the functions* $\mu^{-1}(x)$ *and* $\lambda^{-1}(y)$ *are reciprocal to* $x(\mu)$ *and* $y(\lambda)$ (40).

Then the operator system $\{A_1, A_2\}$ *in* H *is unitary equivalent to the system* $\left\{\tilde{A}_1, \tilde{A}_2\right\}$ (3) *in the function space* $L^2_{\Omega_L}$ (2).

Proof. First of all, the equality

$$iC = A_2 2(A_1)_I - 2(A_1)_I A_2 \tag{56}$$

implies that $A_2 H_1 \subseteq H_1$, in view of 2 (48). And similarly $A_1 H_2 \subseteq H_2$. Denote by $\tilde{A}_2$ and $\tilde{A}_1$ the operators restricted on L_1 and L_2 (5) correspondingly, which are unitary equivalent to $A_2|_{H_1}$ and $A_1|_{H_2}$, besides, this equivalence specifies the isometric mapping from $H_1 + H_2$ onto $L_1 + L_2$ constructed in the proof of Theorem 6. Theorem 7 implies that this correspondence transforms the commutator C into the operator $\tilde{C}$ (50), therefore

$$i\tilde{C} f(y)\chi_{\Omega_L} = i\tilde{C} P_{L_2} f(y)\chi_{\Omega_L} = i\chi_{\Omega_L} \int\limits_0^{\alpha^{-1}(y)} dt \int\limits_{\alpha(t)}^{b} ds f(s) =$$

$$= i\chi_{\Omega_L} \int\limits_y^b ds f(s) \int\limits_{\alpha^{-1}(s)}^{\alpha^{-1}(y)} dt = i\chi_{\Omega_L} \int\limits_y^b ds f(s)[\lambda(s) - \lambda(y)],$$

in view of (8). Equality (56) for $\tilde{C}$, $\tilde{A}_2$ and $2\big(\tilde{A}_1\big)_I$ is

$$i\int\limits_y^b ds f(s)[\lambda(s) - \lambda(y)] = \tilde{A}_2 f(y)\lambda(y) - 2\big(\tilde{A}_1\big)_I \tilde{A}_2 f(y).$$

Taking into account that $\tilde{A}_2 f(y) \in L_1$ and that the operator $2\big(\tilde{A}_1\big)_I$ on L_1 acts as a multiplication by $\lambda(y)$, we obtain that

$$\lambda(y)\left\{\tilde{A}_2 f(y) - i\int\limits_y^b ds f(s)\right\} = \tilde{A}_2 f(y)\lambda(y) - i\int\limits_y^b ds f(s)\lambda(s).$$

Thus the operator

$$B_2 f(y) \stackrel{\text{df}}{=} \tilde{A}_2 f(y) - i\int\limits_y^b ds f(s) \quad (f(y)\chi_{\Omega_L} \in L_1)$$

maps L_1 onto L_1 and commutes with the operator of multiplication by $\lambda(y)$. Theorem 4 implies that the operator B_2 is the operator of multiplication by the function $B_2 f(y) = \Phi(\lambda(y))f(y)$, therefore

$$\tilde{A}_2 f(y) = i\int\limits_y^b ds f(s) + \varphi(y)f(y),$$

where $\varphi(y) = \Phi(\lambda(y))$. Elementary calculations show that

$$\left(\tilde{A}_2 - zI_{L_1}\right)^{-1} f(y) = \frac{f(y)}{\varphi(y) - z} - \frac{1}{\varphi(y) - z}\int\limits_y^b ds \frac{f(s)}{\varphi(s) - z} \exp\left\{ i\int\limits_y^s \frac{d\xi}{z - \varphi(\xi)} \right\}.$$

Thus spectrum of the operator $\tilde{A}_2$ consists of the range of values of the function $\varphi(y)$ and since $\sigma\big(\tilde{A}_2\big) = \{0\}$, then $\varphi(y) = 0$, and we finally obtain that

$$\tilde{A}_2 f(y) = i \int\limits_y^b ds f(s). \tag{57}$$

Similarly,

$$\tilde{A}_1 g(x) = i \int\limits_x^a dt g(t). \tag{58}$$

So the restrictions $A_2|_{H_1}$ and $A_1|_{H_2}$ are unitary equivalent to the integration operators: $\tilde{A}_2$ (57), on L_1, and $\tilde{A}_1$ (58), correspondingly, on L_2.

The one-dimensional subspace $L_0 = L_1 \cap L_2$ isomorphic to $H_0 = H_1 \cap H_2$ is formed by the constant functions from $L^2_{\Omega_L}$. The form of the operator $\tilde{A}_2$ (57) obviously implies that the linear span of the vectors $\tilde{A}_2^n f_0$ ($n \in \mathbb{Z}_+$, $f_0 \in L_0$) is dense in L_1 (5). And since H_1 and L_1 are unitary isomorphic and the operators $A_2|_{H_1}$ and $\tilde{A}_2\Big|_{L_1}$ are unitary equivalent, then

$$H_1 = \operatorname{span}\{A_2^n h_0 \colon n \in \mathbb{Z}_+; h_0 \in H_0\}, \tag{59}$$

and the equalities

$$\langle A_2^n h_0, A_2^m h_0\rangle = \left\langle \tilde{A}_2^n f_0, \tilde{A}_2^m f_0\right\rangle \quad (\forall n, m \in \mathbb{Z}_+) \tag{60}$$

take place, where f_0 is the image of h_0 under the correspondence $H_1 \to L_1$. Similar considerations for H_2 and L_2, and the operators $A_1\Big|_{H_2}$ and $\tilde{A}_1\Big|_{L_2}$ (58) give to us

$$H_2 = \operatorname{span}\{A_1^n h_0 : n \in \mathbb{Z}_+; h_0 \in H_0\}, \tag{61}$$

besides,

$$\langle A_1^n h_0, A_1^m h_0\rangle = \left\langle \tilde{A}_1^n f_0, \tilde{A}_1^m f_0\right\rangle \quad (\forall n, m \in \mathbb{Z}_+). \tag{62}$$

Moreover, the equalities

$$\langle A_1^n h_0, A_2^m h_0\rangle = \left\langle \tilde{A}_1^n f_0, \tilde{A}_2^m f_0\right\rangle \quad (\forall n, m \in \mathbb{Z}_+) \tag{63}$$

are true in view of unitary isomorphism (Theorem 6) between the subspaces $H_1 + H$ and $L_1 + L_2$.

Taking into account complete nonself-adjointness of the operator A_1,

$$H = \operatorname{span}\{A_1^n h_1 \colon n \in \mathbb{Z}_+;\ h_1 \in H_1\},$$

and form of H_1 (59), we obtain that

$$H = \operatorname{span}\{A_1^n A_2^m h_0 \colon n, m \in \mathbb{Z}_+;\ h_0 \in H_0\}. \tag{64}$$

Continue the operators $\tilde{A}_2$ (57) and $\tilde{A}_1$ (58) on the whole $L^2_{\Omega_L}$ (2) using the formulas

$$\big(\tilde{A}_1 f\big)(x,y) = i\int\limits_x^a f(t,y)dt, \qquad \big(\tilde{A}_2 f\big)(x,y) = i\int\limits_y^b f(x,s)\,ds. \tag{65}$$

It is obvious that operators (65) commute and

$$L^2_{\Omega_L} = \operatorname{span}\left\{\tilde{A}_1^n \tilde{A}_2^m f_0 \colon n,m\in\mathbb{Z}_+;\ f_0\in L_0\right\}. \tag{66}$$

To conclude the proof of the theorem it is necessary to ascertain that

$$\langle A_1^n A_2^m h_0, A_1^p A_2^q h_0\rangle = \left\langle \tilde{A}_1^n \tilde{A}_2^m f_0, \tilde{A}_1^p \tilde{A}_2^q f_0\right\rangle \quad (\forall n,m,p,q\in\mathbb{Z}_+). \tag{67}$$

Really, if (67) takes place, we can specify unitary in view of (67) operator U,

$$UA_1^n A_2^m h_0 = \tilde{A}_1^n \tilde{A}_2^m f_0 \quad (n,m\in\mathbb{Z}_+),$$

mapping the whole H (64) onto $L^2_{\Omega_L}$ (66), besides, it is obvious that $UA_k = \tilde{A}_k U$, $k=1,2$.

Proof of the relations (67) can be realized in several stages. Show first that the equalities (60), (62), and (63) imply that

$$\langle A_1^n A_2^m h_0, A_1^p h_0\rangle = \left\langle \tilde{A}_1^n \tilde{A}_2^m f_0, \tilde{A}_1^p f_0\right\rangle \quad (\forall n,m,p\in\mathbb{Z}_+) \tag{68}$$

take place. Use the method of induction by the parameter $n\in\mathbb{Z}_+$, for all m, $p\in\mathbb{Z}_+$. When $n=0$, equality (68) follows from (63). Let $n=1$, then

$$\langle A_1 A_2^m h_0, A_1^p h_0\rangle = \langle (A_1^* + 2i(A_1)_I)\, A_2^m h_0, A_1^p h_0\rangle =$$

$$= \left\langle A_2^m h_0, A_1^{p+1} h_0\right\rangle + 2i\,\langle (A_1)_I A_2^m h_0, A_1^p h_0\rangle\,.$$

Since for the first summand (63) is true, then one ought to consider the second summand, which can be written in the form

$$i\,\langle 2(A_1)_I A_2^m h_0, P_{H_1} A_1^p h_0\rangle\,.$$

Note that $\langle P_{H_1} A_1^p h_0, h_1\rangle = \left\langle P_{L_1}\tilde{A}_1^p f_0, f_1\right\rangle$ in view of (63) and (59), where $h_1\in H_1$, and f_1 is the image of h_1 under the correspondence $H_1\to L_1$ and $f_1\in L_1$. And in accordance with $2(A_1)A_2^m h_0\in H_1$, the equality $2\big\langle (A_1)_I, A_2^m h_0, h_1\big\rangle = \left\langle 2\big(\tilde{A}_1\big)_I \tilde{A}_2^m f_0, f_1\right\rangle$ also holds in view of Theorem 6 and formulas (46), (65). Thus

$$i\big\langle 2(A_1)_I A_2^m h_0, A_1^p h_0\big\rangle = i\left\langle 2\big(\tilde{A}_1\big)_I \tilde{A}_2^m f_0, \tilde{A}_1^p f_0\right\rangle,$$

and so the equalities (68) for $n=1$ are proved. Let for all $n=0,1,\ldots,q$ the statement be proved, show that it also is true for $n=q+1$. Consider

$$\left\langle A_1^{q+1} A_2^m h_0, A_1^p h_0\right\rangle = \langle (A_1^* + 2i(A_1)_I)\, A_1^q A_2^m h_0, A_1^p h_0\rangle =$$

$$= \left\langle A_1^q A_2^m h_0, A_1^{p+1} h_0\right\rangle + 2i\,\langle (A_1)_I A_1^q A_2^m h_0, A_1^p h_0\rangle\,.$$

For the first summand, (68) takes place in view of the induction supposition, as for the second summand, we write it as

$$\langle A_1^q A_2^m h_0, h_p \rangle ,$$

where $h_p = 2(A_1)_I A_1^p h_0 \in H_1$. If $m = 0$, then Theorem 6 and (63) imply that

$$\langle A_1^p h_0, 2(A_1)_I A_1^p h_0 \rangle = \left\langle \tilde{A}_1^p f_0, 2(\tilde{A}_1)_I \tilde{A}_1^p f_0 \right\rangle .$$

When $m > 0$, we obtain that

$$\langle A_1^q A_2^m h_0, h_p \rangle = \langle A_1^q A_2^{m-1} h_0, A_2 h_p \rangle + i \langle A_1^q A_2^{m-1} h_0, 2(A_2)_s h_p \rangle .$$

Since $2(A_2)_I h_p \in H_2$ and H_2 is given by (61), for the second summand (68) is true by the supposition of induction. In accordance with $A_2 h_p \in H_1$, repeating this process, 'transfer' of the operator A_2 on the second place in the scalar product, proper number of times, we at last receive the expression $\langle A_1^q h_0, A_2^m h_p \rangle$, for which (63) is true. Thus truth of the equalities (68) is proved.

To prove that (67) take place, consider

$$\langle A_1^n A_2^m h_0, A_1^p A_2^q h_0 \rangle = \left\langle A_1^n A_2^m h_0, (A_2^* + 2i(A_2)_I)\, A_2^{q-1} A_1^p h_0 \right\rangle =$$

$$= \left\langle A_1^n A_2^{m+1} h_0, A_2^{q-1} A_1^p h_0 \right\rangle - 2i \left\langle (A_2)_I A_1^n A_2^m h_0, A_2^{q-1} A_1^p h_0 \right\rangle.$$

Taking into account that $(A_2)_I A_1^n A_2^m h_0 \in H_2$ and H_2 is given by (61), we in view of (68) have that

$$\left\langle 2(A_2)_I A_1^n A_2^m h_0, A_2^{q-1} A_1^p h_0 \right\rangle = \left\langle 2(\tilde{A}_2)_I \tilde{A}_1^n \tilde{A}_2^m f_0, \tilde{A}_2^{q-1} \tilde{A}_1^p f_0 \right\rangle.$$

For the first summand, again repeat this procedure of 'transfer' of grades of the operator A_2 from the second place in the scalar product to the first. After the finite number of steps, we obtain the expression $\left\langle A_1^n A_2^{m+q} h_1, A_1^p h_0 \right\rangle$, for which the truth of equalities (68) is already proven.

Theorem 8 is proved.

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Received 25.09.11,
after revision — 14.10.13