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A RING OF PYTHAGOREAN TRIPLES OVER QUADRATIC FIELDS*
КІЛЬЦЕ ПІФАГОРОВИХ ТРІЙОК НАД КВАДРАТНИМИ ПОЛЯМИ

Let K be a quadratic field and let be R the ring of integers of K such that R is a unique factorization domain. The set P of all Pythagorean triples in R is partitioned into P_η, sets of triples $\langle \alpha, \beta, \gamma \rangle$ in P where $\eta = \gamma - \beta$. This paper shows the ring structures of each P_η and P from the ring structure of R.

Нехай K — квадратне поле, а R — кільце цілих з K таких, що R — єдина факторизаційна область. Множина P всіх піфагорових трійок з R разбивається на P_η, множини трійок $\langle \alpha, \beta, \gamma \rangle$ в P, де $\eta = \gamma - \beta$. В роботі показано кільцеві структури для кожного P_η та P з кільцевої структури R.

1. Introduction. A triple $\langle \alpha, \beta, \gamma \rangle$ of elements of a ring is said to be a *Pythagorean triple* if $\alpha^2 + \beta^2 = \gamma^2$. B. Dawson [1] defined operations on the set of all Pythagorean triples in $\mathbb{Z}$ so that this set is a ring. J. T. Cross [2] displayed a method for generating all Pythagorean triples over the ring of Gaussian integers.

Let K be a quadratic extension of $\mathbb{Q}$ such that the ring of integers R of K is a unique factorization domain. Let P be the set of all Pythagorean triples in R, i.e.,

$$P = \{\langle \alpha, \beta, \gamma \rangle \in R^3 \mid \alpha^2 + \beta^2 = \gamma^2\}.$$

The set P is partitioned into sets

$$P_\eta = \{\langle \alpha, \beta, \gamma \rangle \in P \mid \gamma - \beta = \eta\}$$

for all $\eta \in R$. This paper shows how to find all elements of each P_η with all elements of P as the byproducts and define bijections between P_η and R, which construct a one-to-one correspondence between P and $R \times R$.

2. Preliminaries. Throughout this paper, all variables will be assumed to represent algebraic integers unless otherwise stated. The notation $\lceil r \rceil$ will be used for the smallest rational integer greater than or equal to the real number r.

The parity is significant in many theorems about Pythagorean triples. James T. Cross shows that $\delta := 1 + i$ plays a role in the ring of Gaussian integers like that played by 2 in $\mathbb{Z}$ [2]. We expand his idea by using the following theorem.

Theorem 2.1. *Let $K = \mathbb{Q}(\sqrt{d})$, where d is a squarefree integer, R be the ring of integers of K. Then*:

2 *is ramified in R if $d \equiv 2$ or $3 \pmod 4$.*

2 *splits completely in R if $d \equiv 1 \pmod 8$.*

2 *is inert in R if $d \equiv 5 \pmod 8$.*

We will separate each case of R into three sections. If 2 is ramified in R, there is a prime $\delta \in R$ such that $2 \sim \delta^2$ and $|R/\langle \delta \rangle| = 2$. For $\alpha \in R$, we may say that α is *even* if α is divisible by δ and α is *odd* otherwise. Moreover, the sum of two even or two odd algebraic integers gives an even

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one, the sum of an even algebraic integer and an odd one gives an odd one, the product of two odd ones gives an odd one, and the product of an even one and any algebraic integer gives an even one. Furthermore, since $\delta \mid 2$, 2 and all integers that are divisible by 2 are even algebraic integers. All units in R are odd. In case that 2 splits completely in R, there are non-associate primes δ, $\overline{\delta} \in R$ such that $2 \sim \delta\overline{\delta}$ and $|R/\langle\delta\rangle| = \left|R/\langle\overline{\delta}\rangle\right| = 2$. If 2 is inert in R, 2 is a prime in R.

Let π be a prime in R. The set $R \setminus \pi R$ contains all elements of R which are not divisible by π. We use the countability property of R to show a connection between $R \setminus \pi R$ and R which leads to a one-to-one correspondence between P_η and R.

Definition 2.1. *Let π be a prime in R. All non-associate primes in R can be put into order, say π, π_1, π_2, $\pi_3, \ldots$. Define $\Psi_\pi \colon (R \setminus \pi R) \to R$ by*

$$\Psi_\pi(u\pi_1^{a_1}\pi_2^{a_2}\pi_3^{a_3}\ldots) = u\pi^{a_1}\pi_1^{a_2}\pi_2^{a_3}\ldots,$$

where $\{a_1, a_2, \ldots\} \subset \mathbb{Z}_0^+$ and u is a unit in R. It is not difficult to see that the mapping Ψ_π is a one-to-one correspondence.

For the case that $\eta = 0$, $P_0 = \{\langle 0, \beta, \beta\rangle \mid \beta \in R\}$ and the mapping $\varphi \colon P_0 \to R$ defined by $\varphi(\langle 0, \beta, \beta\rangle) = \beta$ is a one-to-one correspondence. The following theorems consider the case where $\eta \neq 0$.

3. 2 is ramified in R. In this case, there is a prime $\delta \in R$ such that $2 \sim \delta^2$ and $|R/\langle\delta\rangle| = 2$. To show a ring structure of P_η and P, we characterize P_η and define bijections by considering two cases of η where $\delta^{a_0} || \eta$ for $a_0 = 0, 1$ and $\delta^2 \mid \eta$ in the following theorems.

Theorem 3.1. *Let η be an algebraic integer and $\eta = u\delta^{a_0}\pi_1^{a_1}\pi_2^{a_2}\ldots\pi_m^{a_m}$, where $a_0 = 0, 1$ and for $k \geq 1$, $a_k \in \mathbb{Z}_0^+$, u is a unit and $\pi_k \in R$ are non-associate odd primes. Set $\rho = \delta^{a_0}\pi_1^{b_1}\pi_2^{b_2}\ldots\pi_m^{b_m}$, where $b_k = \left\lceil \dfrac{a_k}{2} \right\rceil$. Then P_η is*

$$\left\{ \left\langle \alpha, \frac{\alpha^2 - \eta^2}{2\eta}, \frac{\alpha^2 + \eta^2}{2\eta} \right\rangle \;\middle|\; \alpha = \tau\rho \ \text{ for some odd } \ \tau \in R \right\}.$$

Moreover, the mapping $\varphi \colon P_\eta \to R$ defined by

$$\varphi(\langle \alpha, \beta, \gamma\rangle) = \Psi_\delta\left(\frac{\alpha}{\rho}\right)$$

is a one-to-one correspondence.

Proof. Suppose $\langle \alpha, \beta, \gamma\rangle \in P_\eta$. Since $\eta = \gamma - \beta$, we have $\langle \alpha, \beta, \gamma\rangle = \langle \alpha,\ (\alpha^2 - \eta^2)/2\eta,\ (\alpha^2 + \eta^2)/2\eta\rangle$. Then $2\eta \mid \alpha^2 + \eta^2$ and thus $\delta^{a_0+2}\pi_1^{a_1}\pi_2^{a_2}\ldots\pi_m^{a_m} \mid \alpha^2 + u^2\delta^{2a_0}\pi_1^{2a_1}\pi_2^{2a_2}\ldots\pi_m^{2a_m}$. Hence $\delta^{2a_0}\pi_1^{a_1}\pi_2^{a_2}\ldots\pi_m^{a_m} \mid \alpha^2$. Since for each $k = 1, \ldots, m$, $b_k = \left\lceil \dfrac{a_k}{2} \right\rceil$, we get $\delta^{a_0}\pi_1^{b_1}\pi_2^{b_2}\ldots\pi_m^{b_m} \mid \alpha$. Therefore, there exist an algebraic integer τ such that $\alpha = \tau\rho$. If τ is even, then $\delta^{a_0+2} \mid \alpha^2$ and thus $\delta^{a_0+2} \mid \delta^{2a_0}\pi_1^{2a_1}\pi_2^{2a_2}\ldots\pi_m^{2a_m}$. This is a contradiction, so τ is odd.

Conversely, suppose $\alpha = \tau\rho$ where τ is odd. We have $\alpha^2 - \eta^2 = \tau^2\delta^{2a_0}\pi_1^{2b_1}\pi_2^{2b_2}\ldots\pi_m^{2b_m} - u^2\delta^{2a_0}\pi_1^{2a_1}\pi_2^{2a_2}\ldots\pi_m^{2a_m} = \delta^{2a_0}(\tau^2\pi_1^{2b_1}\pi_2^{2b_2}\ldots\pi_m^{2b_m} - u^2\pi_1^{2a_1}\pi_2^{2a_2}\ldots\pi_m^{2a_m})$. Since $2b_k \geq a_k$, we obtain $\pi_1^{a_1}\pi_2^{a_2}\ldots\pi_m^{a_m} \mid \alpha^2 - \eta^2$. If $a_0 = 0$, i.e., η and ρ are odd, $\alpha + \eta$ and $\alpha - \eta$ are divisible by δ. Then $\delta^2 | \alpha^2 - \eta^2$. If $a_0 = 1$, since $\tau^2\pi_1^{2b_1}\pi_2^{2b_2}\ldots\pi_m^{2b_m}$ and $u^2\pi_1^{2a_1}\pi_2^{2a_2}\ldots\pi_m^{2a_m}$ are odd and the difference of these two numbers is even, $\delta^{a_0+2} \mid \alpha^2 - \eta^2$. Consequently, $2\eta \mid \alpha^2 - \eta^2$ and thus $2\eta \mid \alpha^2 + \eta^2$.

If $\langle \alpha, \beta, \gamma\rangle \in P_\eta$, then α/ρ is an odd algebraic integer and $\Psi_\delta(\alpha/\rho)$ makes the mapping φ injective and surjective.

Theorem 3.1 is proved.

Theorem 3.2. *Let η be an even algebraic integer and $\eta = u\delta^{a_0}\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m}$, where $a_0 \geq 2$ and for $k \geq 1$, $a_k \in \mathbb{Z}_0^+$, u is a unit and $\pi_k \in R$ are non-associate odd primes. Set $\rho = \delta^{b_0}\pi_1^{b_1}\pi_2^{b_2}\dots\pi_m^{b_m}$, where $b_0 = \left\lceil \dfrac{a_0+2}{2} \right\rceil$ and $b_k = \left\lceil \dfrac{a_k}{2} \right\rceil$. Then P_η is*

$$\left\{ \left\langle \alpha, \frac{\alpha^2-\eta^2}{2\eta}, \frac{\alpha^2+\eta^2}{2\eta} \right\rangle \;\middle|\; \alpha = \tau\rho \text{ for some } \tau \in R \right\}.$$

Moreover, the mapping $\varphi\colon P_\eta \to R$ defined by

$$\varphi(\langle \alpha, \beta, \gamma \rangle) = \frac{\alpha}{\rho}$$

is a one-to-one correspondence.

Proof. Suppose $\langle \alpha, (\alpha^2-\eta^2)/2\eta,\ (\alpha^2+\eta^2)/2\eta \rangle \in P_\eta$. Then $\delta^{a_0+2}\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m} | \alpha^2 + u^2\delta^{2a_0}\pi_1^{2a_1}\pi_2^{2a_2}\dots\pi_m^{2a_m}$. Therefore, $\delta^{a_0+2}\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m} \mid \alpha^2$. Hence $\delta^{b_0}\pi_1^{b_1}\pi_2^{b_2}\dots\pi_m^{b_m} \mid \alpha$. Thus $\alpha = \tau\rho$ for some $\tau \in R$.

Conversely, suppose $\alpha = \tau\rho$, where $\tau \in R$. We have $\alpha^2 = \tau^2\delta^{2b_0}\pi_1^{2b_1}\pi_2^{2b_2}\dots\pi_m^{2b_m}$ which is divisible by 2η. Moreover, η^2 is divisible by 2η because $2 \mid \eta$. Hence $2\eta \mid \alpha^2 + \eta^2$.

Since any algebraic integer can be written in the form α/ρ, the mapping φ is bijective.

Theorem 3.2 is proved.

4. 2 splits completely in *R*. There are non-associate primes $\delta, \overline{\delta} \in R$ such that $2 \sim \delta\overline{\delta}$ and $|R/\langle\delta\rangle| = \left|R/\langle\overline{\delta}\rangle\right| = 2$. Notice that the ideas of even and odd we used in the proofs of the previous theorems are also practical in this section where we consider three cases of η depending on the divisibility by δ and $\overline{\delta}$. Note that δ and $\overline{\delta}$ hold the same properties and can be switched around in the following theorem.

Theorem 4.1. *Let $\eta \in R$ and $\eta = u\overline{\delta}^{\overline{a}_0}\pi_1^{a_1}\dots\pi_m^{a_m}$, where $\overline{a}_0 \geq 1$, and for $k \geq 1$, $a_k \in \mathbb{Z}_0^+$, u is a unit and $\pi_k \in R$ are non-associate primes where $\pi_k \nsim \delta, \overline{\delta}$. Set $\rho = \overline{\delta}^{\overline{b}_0}\pi_1^{b_1}\dots\pi_m^{b_m}$, where $\overline{b}_0 = \left\lceil \dfrac{\overline{a}_0+1}{2} \right\rceil$ and $b_k = \left\lceil \dfrac{a_k}{2} \right\rceil$. Then P_η is*

$$\left\{ \left\langle \alpha, \frac{\alpha^2-\eta^2}{2\eta}, \frac{\alpha^2+\eta^2}{2\eta} \right\rangle \;\middle|\; \alpha = \tau\rho \text{ for some } \tau \in R, \text{ where } \delta \nmid \tau \right\}.$$

Moreover, the mapping $\varphi\colon P_\eta \to R$ defined by

$$\varphi(\langle \alpha, \beta, \gamma \rangle) = \Psi_\delta\left(\frac{\alpha}{\rho}\right)$$

is a one-to-one correspondence.

Proof. Suppose $\langle \alpha, (\alpha^2-\eta^2)/2\eta,\ (\alpha^2+\eta^2)/2\eta \rangle \in P_\eta$. Then $2\eta \mid \alpha^2 + \eta^2$ and thus $\delta\overline{\delta}^{\overline{a}_0+1}\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m} | \alpha^2 + u^2\overline{\delta}^{2\overline{a}_0}\pi_1^{2a_1}\pi_2^{2a_2}\dots\pi_m^{2a_m}$. Therefore, $\overline{\delta}^{\overline{a}_0+1}\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m} \mid \alpha^2$. Hence $\overline{\delta}^{\overline{b}_0}\pi_1^{b_1}\pi_2^{b_2}\dots\pi_m^{b_m} | \alpha$. Thus $\alpha = \tau\rho$ for some $\tau \in R$. If $\delta | \tau$, then $\delta \mid \alpha^2$ and $\delta \mid u^2\overline{\delta}^{2\overline{a}_0}\pi_1^{2a_1}\pi_2^{2a_2}\dots\pi_m^{2a_m}$, a contradiction. This means that $\delta \nmid \tau$.

Conversely, suppose $\alpha = \tau\rho$ where $\tau \in R$ and $\delta \nmid \tau$. We have $\overline{\delta}\eta \mid \alpha^2 + \eta^2$. Since $\delta \nmid \alpha^2$ (odd wrt δ) and $\delta \nmid \eta^2$, we have $\delta \mid \alpha^2 + \eta^2$ (even wrt δ). Since $2 \sim \delta\overline{\delta}$, $2\eta \mid \alpha^2 + \eta^2$.

Theorem 4.1 is proved.

The proofs of the next two theorems are left to the reader.

Theorem 4.2. *Let $\eta \in R$ and $\eta = u\delta^{a_0}\overline{\delta}^{\overline{a}_0}\pi_1^{a_1}\dots\pi_m^{a_m}$, where $a_0 \geq 1$, $\overline{a}_0 \geq 1$, and for $k \geq 1$, $a_k \in \mathbb{Z}_0^+$, u is a unit and $\pi_k \in R$ are non-associate primes, where $\pi_k \nsim \delta, \overline{\delta}$. Set $\rho = \delta^{b_0}\overline{\delta}^{\overline{b}_0}\pi_1^{b_1}\dots\pi_m^{b_m}$, where $b_0 = \left\lceil \dfrac{a_0+1}{2} \right\rceil$, $\overline{b}_0 = \left\lceil \dfrac{\overline{a}_0+1}{2} \right\rceil$ and $b_k = \left\lceil \dfrac{a_k}{2} \right\rceil$. Then P_η is*

$$\left\{ \left\langle \alpha, \frac{\alpha^2-\eta^2}{2\eta}, \frac{\alpha^2+\eta^2}{2\eta} \right\rangle \,\middle|\, \alpha = \tau\rho \text{ for some } \tau \in R \right\}.$$

Moreover, the mapping $\varphi\colon P_\eta \to R$ defined by

$$\varphi(\langle \alpha, \beta, \gamma \rangle) = \frac{\alpha}{\rho}$$

is a one-to-one correspondence.

The following theorem uses the idea that all non-associate primes in R can be put into order, say δ, $\overline{\delta}$, π_1, $\pi_2, \dots$.

Theorem 4.3. *Let $\eta \in R$ and $\eta = u\pi_1^{a_1}\dots\pi_m^{a_m}$, where for $k \geq 1$, $a_k \in \mathbb{Z}_0^+$, u is a unit and $\pi_k \in R$ are non-associate primes where $\pi_k \nsim \delta, \overline{\delta}$. Set $\rho = \pi_1^{b_1}\dots\pi_m^{b_m}$, where $b_k = \left\lceil \dfrac{a_k}{2} \right\rceil$. Then P_η is*

$$\left\{ \left\langle \alpha, \frac{\alpha^2-\eta^2}{2\eta}, \frac{\alpha^2+\eta^2}{2\eta} \right\rangle \,\middle|\, \alpha = \tau\rho \text{ for some } \tau \in R, \text{ where } \delta \nmid \tau, \ \overline{\delta} \nmid \tau \right\}.$$

Moreover, the mapping $\varphi\colon P_\eta \to R$ defined by

$$\varphi(\langle \alpha, \beta, \gamma \rangle) = \Psi_\delta\left(\Psi_{\overline{\delta}}\left(\frac{\alpha}{\rho}\right)\right)$$

is a one-to-one correspondence.

5. 2 is inert in R. By Theorem 2.1, $R = \left\{ \dfrac{x+y\sqrt{d}}{2} \,\middle|\, x, y \in \mathbb{Z} \text{ and } x \equiv y \pmod 2 \right\}$ and 2 is a prime in R. Notice that the norm of 2 in $\mathbb{Q}(\sqrt{d})$ is 4, this means that the parity is not as useful as in the previous sections.

Theorem 5.1. *Let $\eta \in R$ and $\eta = u\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m}$, where $a_k \in \mathbb{Z}_0^+$, u is a unit and $\pi_k \in R$ are non-associate primes such that $2 \nmid \pi_k$. Set $\rho = \pi_1^{b_1}\pi_2^{b_2}\dots\pi_m^{b_m}$, where $b_k = \left\lceil \dfrac{a_k}{2} \right\rceil$. Then P_η is*

$$\left\{ \left\langle \alpha, \frac{\alpha^2-\eta^2}{2\eta}, \frac{\alpha^2+\eta^2}{2\eta} \right\rangle \,\middle|\, \alpha = \tau\rho \text{ for some } \tau \in R, \text{ where } 2 \nmid \tau \right\}.$$

Moreover, the mapping $\varphi\colon P_\eta \to R$ defined by

$$\varphi(\langle \alpha, \beta, \gamma \rangle) = \Psi_2\left(\frac{\alpha}{\rho}\right)$$

is a one-to-one correspondence.

Proof. Suppose $\langle \alpha, (\alpha^2-\eta^2)/2\eta, (\alpha^2+\eta^2)/2\eta \rangle \in P_\eta$. Then $2\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m} \mid \alpha^2 + {} + u^2\pi_1^{2a_1}\pi_2^{2a_2}\dots\pi_m^{2a_m}$. Therefore, $\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m} \mid \alpha^2$. Hence $\rho \mid \alpha$, say $\alpha = \tau\rho$ for some $\tau \in R$. It is easy to see that $2 \nmid \tau$.

Conversely, suppose $\alpha = \tau\rho$, where $\tau \in R$ and $2 \nmid \tau$. We have $\eta \mid \alpha^2 - \eta^2$. Let $\alpha = (x+y\sqrt{d})/2$ and $\eta = (z+w\sqrt{d})/2$, where $x, y, z, w \in \mathbb{Z}$ and $x \equiv y, z \equiv w \pmod 2$. Since $2 \nmid \alpha$ and $2 \nmid \eta$, $x \equiv y \equiv z \equiv w \equiv 1 \pmod 2$. Hence $2 \mid \alpha - \eta$ and thus $2 \mid \alpha^2 - \eta^2$. Since $\gcd(\eta, 2) = 1$, $2\eta \mid \alpha^2 + \eta^2$.

Theorem 5.1 is proved.

Theorem 5.2. *Let $\eta \in R$ and $\eta = u2^{a_0}\pi_1^{a_1}\pi_2^{a_2}\dots\pi_m^{a_m}$, where $a_0 \geq 1$ and for $k \geq 1$, $a_k \in \mathbb{Z}_0^+$, u is a unit and $\pi_k \in R$ are non-associate primes such that $2 \nmid \pi_k$. Set $\rho = 2^{b_0}\pi_1^{b_1}\pi_2^{b_2}\dots\pi_m^{b_m}$, where $b_0 = \left\lceil \dfrac{a_0+1}{2} \right\rceil$ and $b_k = \left\lceil \dfrac{a_k}{2} \right\rceil$. Then P_η is*

$$\left\{ \left\langle \alpha, \frac{\alpha^2-\eta^2}{2\eta}, \frac{\alpha^2+\eta^2}{2\eta} \right\rangle \;\middle|\; \alpha = \tau\rho \text{ for some } \tau \in R \right\}.$$

Moreover, the mapping $\varphi\colon P_\eta \to R$ defined by

$$\varphi(\langle \alpha, \beta, \gamma \rangle) = \frac{\alpha}{\rho}$$

is a one-to-one correspondence.

Proof. The proof is similar to the proof of Theorem 3.2.

6. The ring structure. We combine results from Sections 3 – 5 and define operations addition and multiplication on P_η and P to establish rings of Pythagorean triples. The ring structures of P_η and P are constructed from the ring structure of R.

***Corollary* 6.1.** *Let η be an algebraic integer. $\langle P_\eta, \oplus, \odot \rangle$ is a commutative ring with identity, where $\oplus$ and $\odot$ are operations on P_η defined by*

$$\langle \alpha, \beta, \gamma \rangle \oplus \langle \mu, \nu, \lambda \rangle = \varphi^{-1}(\varphi(\langle \alpha, \beta, \gamma \rangle) + \varphi(\langle \mu, \nu, \lambda \rangle))$$

and

$$\langle \alpha, \beta, \gamma \rangle \odot \langle \mu, \nu, \lambda \rangle = \varphi^{-1}(\varphi(\langle \alpha, \beta, \gamma \rangle) \cdot \varphi(\langle \mu, \nu, \lambda \rangle)).$$

***Corollary* 6.2.** *The mapping $\Phi\colon P \to R \times R$ given by*

$$\Phi(\langle \alpha, \beta, \gamma \rangle) = (\gamma - \beta, \varphi(\langle \alpha, \beta, \gamma \rangle))$$

is a bijection. Consequently, $\langle P, \boxplus, \boxdot \rangle$ is a commutative ring with identity where $\boxplus$ and $\boxdot$ are operations on P defined by

$$\langle \alpha, \beta, \gamma \rangle \boxplus \langle \mu, \nu, \lambda \rangle = \Phi^{-1}(\Phi(\langle \alpha, \beta, \gamma \rangle) + \Phi(\langle \mu, \nu, \lambda \rangle))$$

and

$$\langle \alpha, \beta, \gamma \rangle \boxdot \langle \mu, \nu, \lambda \rangle = \Phi^{-1}(\Phi(\langle \alpha, \beta, \gamma \rangle) \cdot \Phi(\langle \mu, \nu, \lambda \rangle)).$$

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