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s-CONDITIONALLY PERMUTABLE SUBGROUPS
AND p-NILPOTENCY OF FINITE GROUPS*

We study the \( p \)-nilpotency of a group such that every maximal subgroup of its Sylow \( p \)-subgroups is \( s \)-conditionally permutable for some prime \( p \). By using the classification of finite simple groups, we get interesting new results and generalize some earlier results.

Вивчено \( p \)-нільпотентність групи, для якої кожна максимальна підгрупа її силовських \( p \)-підгруп є \( s \)-умовно пе-
реставною для деякого простого \( p \). За допомогою класифікації скінчених простих груп отримано цікаві нові результати та узагальнено декі результати, що отримані раніше.

1. Notation and introduction. In this paper, all groups are finite and \( G \) stands for a finite group. Let \( \pi(G) \) be the set of all prime divisors of \(|G|\). Let \( G_p \) and \( \text{Syl}_p(G) \) be a Sylow \( p \)-subgroup and the set of Sylow \( p \)-subgroups of \( G \) respectively. Let \( F \) denote a formation, \( U \) the class of supersolvable groups. Let \( n_p \) be the \( p \)-part of a nature number \( n \), that is, \( n_p = p^a \) such that \( p^a | n \) but \( p^{a+1} \notin n \). Let \( G \) be a Lie-type simple group over the finite field \( F_q \). To collect some useful information and for convenience in narrating, we define \( n(G) \) in Table 1.1. The other notation and terminology are standard (see [11, 13]).

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<td>( C_n(q)(p \neq 2) )</td>
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<td>( ^2D_n(q) )</td>
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<td>( E_8(q) )</td>
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<td>( G_2(q) )</td>
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Many authors have investigated the structure of a group when maximal subgroups of Sylow subgroups of the group are well situated in the group. Srinivasan [28] showed that a group \( G \) is supersolvable if all maximal subgroups of every Sylow subgroup of \( G \) are normal. Later, several authors obtain the same conclusion if normality is replaced by some weaker property (see [25, 27]).

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In particular, these results indicate that the generalized normality of some maximal subgroups of Sylow subgroups give a lot of useful information on the structure of groups.

In this paper, we obtain some sufficient conditions on $p$-nilpotency and supersolvability of groups by using the $s$-conditional permutability of maximal subgroups of Sylow subgroups. Some earlier results on this topic are generalized.

2. Basic definitions and preliminary results. Let $H$ and $K$ be two subgroups of $G$. We say that $H$ permutes with $K$ if $HK = KH$. Recently, Huang and Guo [10] introduced a new embedding property, namely, the $s$-conditional permutability of subgroups of a group.

Definition. A subgroup $H$ of $G$ is $s$-conditionally permutable if for every prime $p \in \pi(G)$, there exists a Sylow $p$-subgroup $P$ of $G$ such that $HP = PH$.

For the sake of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 ([10], Lemma 2.3). Let $H$ and $K$ be subgroups of $G$. Then the following hold:

1. If $H$ is $s$-conditionally permutable in $G$ and $K$ is normal in $G$, then $HK/K$ is $s$-conditionally permutable in $G$.
2. If $H \leq K \triangleleft G$ and $H$ is $s$-conditionally permutable in $G$, then $H$ is $s$-conditionally permutable in $K$.

Lemma 2.2 ([24], Lemma 6). Suppose that $G$ is a non-Abelian simple group. Then there exists an odd prime $r \in \pi(G)$ such that $G$ has no Hall $\{2, r\}$-subgroup.

Lemma 2.3 ([29], Theorem 3.1). Let $F$ be a saturated formation containing $U$, and $G$ a group with a normal subgroup $N$ such that $G/N \in F$. If all Sylow subgroups of $F^*(N)$ are cyclic, then $G \in F$.

Lemma 2.4 ([26], Lemma 1.6). Let $P$ be a nilpotent normal subgroup of a group $G$. If $P \cap \Phi(G) = 1$, then $P$ is the direct product of some minimal normal subgroups of $G$.

Recall that a prime divisor $d$ of $a^m - 1$ is called primitive, if $d$ does not divide $a^i - 1$ for $1 \leq i \leq m - 1$. For primitive prime divisors, an important property is due to Zsigmondy, refer to [8].

Lemma 2.5 [8]. Let $b$ and $n$ be positive integers.

1. There are primitive prime divisors of $b^n - 1$ unless $(b, n) = (2, 6)$ or $b$ is a Mersenne prime and $n = 2$.
2. Each primitive prime divisor $p$ of $b^n - 1$ is at least $n + 1$. Moreover, if $p = n + 1$, then $p^2$ divides $b^n - 1$ except for the following cases:
   - (i) $n = 2$ and $b = 2^s - 1$ or $3 \cdot 2^s - 1$;
   - (ii) $b = 2$ and $n = 4, 6, 10, 12$ or $18$;
   - (iii) $b = 3$ and $n = 4$ or $6$;
   - (iv) $b = 5$ and $n = 6$.
3. For a positive integer $s$, if a primitive prime divisor of $b^s - 1$ divides $b^n - 1$, then $s$ divides $n$.

3. Main results and their proofs.

Theorem 3.1. Let $G$ be a non-Abelian simple group and $|G|_2 = 2^t$. If $G$ has a subgroup of order $2^{t-1}|G|_2$ for every $r \in \pi(G) \setminus \{2\}$, then $G \cong PSL_2(q)$, where $q$ is a power of an odd prime and $t = 2$.
Proof. Let \( r \in \pi(G) \setminus \{2\} \), \( H \) be a subgroup of \( G \) of order \( 2^{t-1}|G|_r \), \( A \in \text{Syl}_2(H) \) and \( R \in \text{Syl}_r(H) \). Then \( |A| = 2^{t-1} \) and \( R \in \text{Syl}_r(G) \) and \( H = AR \). Let \( M \) be a maximal subgroup of \( G \) containing \( H \). Then \( |M|_2 = 2^t \) or \( |M|_2 = 2^{t-1} \). If \( |M|_2 = 2^{t-1} \), then \( A \in \text{Syl}_2(M) \) and \( H \) is a Hall \((2, r)\)-subgroup of \( M \); if \( |M|_2 = 2^t \), then \( M_2 \in \text{Syl}_2(G) \), \( |G : M| \) is odd and so \( G \) has a faithful primitive permutation representation of odd degree and \( M \) is listed in [20] (Theorem). By the classification of finite simple groups, we divide the argument into the following cases.

1. \( G \) is a sporadic simple group.

Let \( r = \max \pi(G) \). Then by [5] and http://brauer.maths.qmul.ac.uk/Atlas/v3, \( 2^{t-1} \mid |M| \), a contradiction.

2. \( G \) is an alternating \( A_n \).

We have \( 2^t = \left( \frac{1}{2} n! \right)^2 \). Let \( r = \max \pi(G) \). By [3], \( R^4 = R \) and \( 2^{t-1} \mid \frac{1}{2}(r - 1)(n - r)! \), this is impossible.

3. \( G \) is a Lie-type simple group over \( GF(q) \), where \( q = p^f \) and \( p \) is a prime.

Suppose that \( G = PSL_2(q) \) and \( |G|_2 > 4 \). If \( q = 2^f \), then \( G \) has no subgroup of order \( \frac{1}{2}|G|_2|R| \) by [14], a contradiction. Hence \( q = p^f \) with \( p \) odd. Thus \((q - 1)_2 = 2 \) or \((q + 1)_2 = 2 \).

If \((q + 1)_2 = 2 \), let \( t = \max \pi(q + 1) \) and \( V \in \text{Syl}_q(G) \), then \( G \) has no subgroup of order \( \frac{1}{2}|G|_2|V| \) by [14]; if \((q - 1)_2 = 2 \), let \( u = \max \pi(q - 1) \) and \( U \in \text{Syl}_u(G) \), then \( G \) has no subgroup of order \( \frac{1}{2}|G|_2|U| \) by [14], a contradiction. Hence \( |G|_2 = 2^2 \), the result holds. From now, we assume that \( n(G) > 2f \).

Assume that \((n(G), p) = (6, 2) \). Then \((n(G)/f, f) \) is one of \((3, 2) \) and \((6, 1) \), and so \( G \) is one of the groups \( PSL_3(2^2), PSU_4(2), PSU_6(2), D_4(2) \). Suppose that \( G \in \{PSL_3(2^2), PSU_4(2), PSU_6(2), D_4(2)\} \). Let \( r = 3 \). Since \( M_r \in \text{Syl}_r(G) \), by [5, p.23, 26, and 85], \( M \in \{A_6, 3^2 : Q_8\} \) if \( G = PSL_3(2^2) \), \( M \in \{3^{1+2} : 2A_4, 3^3 : S_4\} \) if \( G = PSU_4(2) \) and \( M = 3^4 : 2^3 : S_4 \) if \( G = D_4(2) \), hence \( 4 \mid |G : M| \), a contradiction. Suppose that \( G = PSL_6(2) \). Let \( r = 7 \).

By http://brauer.maths.qmul.ac.uk/Atlas/lin/L62, \( M \in \{(2^3 : (L_3(2) \times L_3(2)), (L_3(2) \times L_3(2)) : 2, (L_2(8) \times 7) : 3\} \). If \( M \in \{(L_3(2) \times L_3(2)) : 2, (L_2(8) \times 7) : 3\} \), then \( 4 \mid |G : M| \); if \( M = 2^3 : (L_3(2) \times L_3(2)) \), since the maximal subgroup \( A \) of \( L_3(2) \) satisfying \( 7 \mid |A| \) is isomorphic to \( 7 \times 3 \), \( M \) has no the maximal subgroup of order \( 2^{14} \times 7^2 \), a contradiction. Hence \( (n(G), p) \neq (6, 2) \). By Lemma 2.5, \( p^{n(G)} - 1 \) has at least one primitive prime divisor. Let \( r \) be the largest primitive prime divisor of \( p^{n(G)} - 1 \) and \( M \) a maximal subgroup of \( G \) of order \( 2^{t-1}|G|_r \). Then \( M \) is not a parabolic subgroup of \( G \).

Suppose that \( G \in \{PSL_3(q), PSU_3(q), 2F_4(2^{2m+1}), Sz(q), 3D_4(q), D_4(2^f), 2G_2(q), G_2(q)\} \). The maximal subgroups or orders of maximal subgroups of \( 2B_2(2^{2m+1}) \), \( PSL_3(q) \) and \( PSU_3(q) \) are listed in the proof of Lemmas 1–4 in [7]; the maximal subgroups of \( 2F_4(2^{2m+1}) \), \( 2G_2(q) \), \( G_2(2^f) \), \( 3D_4(q) \) and \( D_4(2^f) \) are listed in [6, 15–17, 23]. A simple checking shows that \( 4 \mid |G : M| \), a contradiction. Suppose that \( G = G_2(q) \) with \( q \) odd. Since \( |M|_r = |G|_r \), by [16], the possibilities of \( M \) are \( SL_3(q) : 2, SU_3(q) : 2, L_2(13), G_2(2) \) and \( J_1 \). It is easy to prove that if \( M \in \{SL_3(q) : 2, SU_3(q) : 2, L_2(13), G_2(2), J_1\} \), then \( M \) has no the subgroup of order \( 2^{t-1}|G|_r \).

Next, we deal with the remaining Lie-type simple group \( G \) in the previous argument. Let \( H \) be maximal subgroups of \( G \) containing a subgroup of \( G \) of order \( 2^{t-1}|G|_r \). Then \( H \) is a parabolic subgroup of \( G \).
Suppose that $G$ is an exceptional Lie-type simple group and the notation $K(G)$ is defined in [21] (Theorem). Suppose that $p = 2$. It is easy to see that the maximal subgroup in Table 1 [21] don't contain a subgroup of order $2^{l-1}|G|$. Thus by [21] (Theorem), $|M| < 2^{|K(G)|}$. On the other hand, by [12], $|M| > (|M|_2)^2 \geq 2^{2(K(G)-1)j}$ if $G \neq E_8(2j)$ or $|M| > (|M|_2)^2 \geq 2^{2(K(G)-10)j}$ if $G = E_8(2j)$, a contradiction. Suppose that $p > 2$ and $G$ is one of simple groups $F_4(q), E_6(q), E_7(q), E_8(q)$. Then $4 \nmid |G:H|$, this is impossible. Thus we have proved that there is no exceptional Lie-type simple group satisfying the condition of Theorem 3.1.

Suppose that $G$ is a classical simple group on $n$-dimension vector space $V$ and $n > 3$. We shall use the notation of the book [13] in the following argument. Aschbacher [1] classified maximal subgroups of a classical simple group into 9 types: $C_i$, where $1 \leq i \leq 8$, and $S$, see [13] for the description.

Suppose that $p = 2$. If $3 < n < 12$, using [14] and [15], it is easy to see that $4 \nmid |G : M|$, $G$ doesn't satisfy the condition of Theorem 3.1. Hence we assume that $n \geq 12$. Assume that $M$ is an almost simple group. Since $2^{l-1} \nmid |M|_2$, by [18], $|M| < 2^{2n+2} < 2^{2(n-2j-2)} \leq (|M|_2)^2$. On the other hand, by [12], $|M| > (|M|_2)^2$, a contradiction. Suppose that $M$ is a $C_i$ subgroup. By [15] (Table A–E), a simple checking shows that $4 \nmid |G : M|$, $G$ doesn't satisfy the hypothesis.

Assume that $p > 2$. Since $4 \nmid |G : K|$, we have $4 \nmid n$ if $G = PSL_n(q)$; $2 \nmid n$ if $G = PSL_n(q)$ with $4 \nmid (q + 1)$; $4 \nmid (q + 1)$ if $G \neq PSL_n(q)$; $4 \nmid (n - 1)$ if $G \in \{U_n(q), PSp_n(q)\}$; $2 \nmid k$ if $G \in \{P\Omega_{2k}^+(q), P\Omega_{2k+1}^+(q)\}$; $2 \nmid (k - 1)$ if $G = P\Omega_{2k}^-(q)$. Suppose that $2 < n < 12$. From [14] and [15], it is easy to see that either $2^{l-1} \nmid |M|$ or $M \not\in \text{Syl}_n(G)$, a contradiction. Hence we may assume that $n \geq 12$. By Lemma 2.5, we may assume that $r > n(G) + 1$ or $r = n(G) + 1$ and $r^2 \nmid p^{(n(n-1)} - 1$. By [20], it is easy to see that $|G : M|$ is not odd, hence $M$ has a Hall $\{2, r\}$-subgroup. Suppose that $M$ is a $S$ subgroup of $G$. Then the covering group of $M$ is a subgroup of $GL_n(q)$ and there is a non-Abelian simple group $S$ such that $S \leq M \leq \text{Sut}(S)$. Moreover, if $N$ is the preimage of $S$ in $G$, then $N$ is absolutely irreducible on $V$ and $N$ is not a classical group defined over a subfield of $GF(q)$ (in its natural representation). All possibilities of $S$ have given in Examples 2.6–2.9 in [9]. For all possible $S$ either $2^{l-1} \nmid |M|$ or $r^2 \nmid |M|$ when $r = n(G) + 1$, this is impossible. Suppose that $M$ is not a $S$ subgroup of $G$. Since $r \nmid |M|$, by [14] (Table 3.5.A–F), it is easy to see that $M$ must be one of $C_3, C_6$ and $C_8$ subgroups of $G$. Since $r > n(G) + 1$ or $r^2 \nmid |M|$ if $r = n(G) + 1$, $M$ is not a $C_6$ subgroup. If $M$ is $C_3$ and $C_8$ subgroups, a simple calculation shows that $2^{l-1} \nmid |M|$, a final contradiction.

Theorem 3.1 is proved.

Let $M$ be a class of groups. If there is no the section in a group $G$ to be isomorphic to a member of $M$, then $G$ is called $M$-free. For the convenience of writing $\exists$ for the set of all $PSL_2(q)$, where $q = p^j$ is odd and the order of Sylow 2-subgroup of $PSL_2(q)$ is 4.

**Theorem 3.2.** Let $G$ be a group and $N$ a normal subgroup of $G$, $p \in \pi(G)$ and $P \in \text{Syl}_p(N)$. Suppose that $(|G|, p - 1) = 1$ and $G/N$ is $p$-nilpotent. If $G$ is 3-free and all maximal subgroups of $P$ are $s$-conditionally permutable in $G$, then $G$ is $p$-nilpotent.

**Proof.** Assume that the result is false. Let $(G, N)$ be a counterexample with $|G| + |N|$ minimal.

1. $G$ has a unique minimal normal subgroup $L$ contained in $N$, $G/L$ is $p$-nilpotent and $L \nleq \Phi(G)$, and so $L$ is not a $p'$-group.

Let $L$ be a minimal normal subgroup of $G$ contained in $N$. Consider the quotient group $\overline{G} = G/L$. Clearly $\overline{G}/\overline{N} \cong G/N$ is $p$-nilpotent and $\overline{P} = PL/L$ is a Sylow $p$-subgroup of $\overline{N}$, where $\overline{N} = N/L$. Let $\overline{P_1} = P_1L/L$ be a maximal subgroup of $\overline{P}$. We may assume that $P_1$ is a maximal subgroup of $P$. By Lemma 2.1(1), $\overline{P_1}$ is $s$-conditionally permutable in $\overline{G}$. The choice of $G$ implies that $\overline{G}$ is...
p-nilpotent. Since the class of p-nilpotent groups is a saturated formation, we may assume that L is a unique minimal normal subgroup of G contained in N and L $\not\leq \Phi(G)$, and so L is not a p'-group.

(2) $O_p(N) = 1$.

If not, then by (1), $L \leq O_p(N)$ and, there is a maximal subgroup $M$ of G such that $G = LM$ and $L \cap M = 1$, so $N = G \cap N = L(M \cap N)$ and $L \cap (M \cap N) = 1$. It is clear that $LM_p \in \text{Syl}_p(G)$ and we may let $(M \cap N)_p < P$, where $(M \cap N)_p \in \text{Syl}_p(M \cap N)$. Let $P_i$ be a maximal subgroup of $P$ containing $(M \cap N)_p$. Then $P = P_1$. By the hypothesis, $P_i$ is a s-conditionally permutable subgroup of G, then there exists a Sylow $q$-subgroup $Q$ of G such that $P_iQ = QP_i$ for any $q \in \pi(G)$, where $q \neq p$. Let $L_1 = L \cap P_i$. Then $|L : L_1| = |L : L \cap P_i| = |LP_i : P_i| = |P : P_i| = p$. So $L_1$ is a maximal subgroup of L. If $L \leq P_iQ$, then $P = LP_i \leq P_iQ$, a contradiction. Hence $L \cap P_iQ < L$ and $L_1 = L \cap P_iQ$. Consequently, $L_1 = L \cap P_iQ < P_iQ$, $P_iQ \leq N_G(L_1)$. It is clear that $L_1 \leq L$. So $P = LP_i \leq N_G(L_1)$. By the arbitrariness of $q \in \pi(G)$, we have $L_1 \triangleleft G$, hence $L_1 = 1$ by the minimal normality of $L$ in G. This means that L is a cyclic subgroup of prime order. Since $G/C_G(L)$ is isomorphic to a subgroup of $\text{Aut}(L)$ and $|\text{Aut}(L)| = p - 1$, by (|G|, p - 1) = 1, we have $C_G(L) = G$, and $L \leq Z(G)$. Hence $G = L \times M$. Since $M \cong G/L$, we get M is p-nilpotent by (1), so G is p-nilpotent, a contradiction.

(3) End of the proof.

By (1) and (2), L is not solvable and so $p = 2$ by the Odd Order Theorem. Let $L = T_1 \times T_2 \times \ldots \times T_s$, where $T_i$ are non-Abelian simple groups with $|T_i| \cong 2$, $1 \leq i \leq s$. Since $P \cap L \in \text{Syl}_2(L)$, we have $P \cap L = K_1 \times K_2 \times \ldots \times K_s$, where $K_i \in \text{Syl}_2(T_i)$. Now we claim that there exists a maximal subgroup $P_i$ of $P$ and $i$ such that $K_i \leq P_1$. If $P \cap L < P_1$, it is clear. Assume that $P \cap L = P_1$. Then $(L, L)$ satisfies the hypothesis by Lemma 2.1(2). If L is a non-Abelian simple group, then every maximal subgroup of P is s-conditionally permutable in L. By the hypothesis and Theorem 3.1, we get $L \in \mathcal{U}$, a contradiction. Hence L is not a non-Abelian simple group. Therefore, we can choose the maximal subgroup $P_1$ of $P$ and $i$ such that $K_i \leq P_1$. By the hypothesis, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $P_iQ = QP_i$ for any $q \in \pi(G)$, where $q \neq 2$. Hence $T_i \cap P_iQ$ is a Hall $\{2, q\}$-subgroup of $T_i$ for any $q \in \pi(T)$ with $q \neq 2$. This contradicts the Lemma 2.2.

Theorem 3.2 is proved.

**Corollary 3.1.** Suppose that G is $\mathfrak{S}$-free. If for every prime $p$ dividing the order of $G$ and $P \in \text{Syl}_p(G)$, every maximal subgroup of $P$ is s-conditionally permutable in G, then G is a Sylow tower group of supersolvable type.

**Theorem 3.3.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with a normal subgroup $N$ such that $G/N \in \mathcal{F}$. If $N$ is $\mathfrak{S}$-free and all maximal subgroups of every noncyclic Sylow subgroup $P$ of $N$ are s-conditionally permutable in $G$, then $G \in \mathcal{F}$.

**Proof.** Assume that the result is false and let $(G, N)$ be a counterexample with $|G| + |N|$ minimal.

If all Sylow subgroups of N are cyclic, then all Sylow subgroups of $F^*(N)$ are cyclic. By Lemma 2.3, $G \in \mathcal{F}$. Therefore, when we want to prove $G \in \mathcal{F}$ in the following arguments, we always assume that $\overline{N}$ has a noncyclic Sylow subgroup if $(\overline{G}, \overline{N})$ satisfies the hypothesis of $(G, N)$ in Theorem 3.3. By Lemma 2.1(2) and Corollary 3.1 $N$ is a Sylow tower group of supersolvable type. Let $r$ be the largest prime in $\pi(N)$ and $R \in \text{Syl}_r(N)$. Then $R$ is normal in $G$ and $(G/R)/(N/R) \cong G/N \in \mathfrak{S}$. By Lemma 2.1(1), every maximal subgroup of any Sylow subgroup of $N/R$ is s-conditionally permutable in $G/R$. Therefore, $G/R$ satisfies the hypotheses for the normal subgroup $N/R$. Thus, by induction, $G/R \in \mathcal{F}$, so $R$ is noncyclic by Lemma 2.3. By Lemma 2.1(1), we may
assume that $G$ has a unique minimal normal subgroup $L$ which is contained in $R$ and $G/L \in \mathcal{F}$. If $L \leq \Phi(G)$, then it follows that $G \in \mathcal{F}$, a contradiction. Thus, we may further assume that $R \cap \Phi(G) = 1$. Then, by Lemma 2.4, $R = F(R) = L$ is an elementary abelian minimal normal subgroup of $G$. Since $R = L \nsubseteq \Phi(G)$, we may choose a maximal subgroup $M$ of $G$ such that $R \nsubseteq M$. Let $M_r$ be a Sylow $r$-subgroup of $M$. Then $G = RM_r$ and $G \cap M = 1$ and $G_r = RM_r$ is a Sylow $r$-subgroup of $G$. Let $G_1$ be a maximal subgroup of $G_r$ containing $M_r$. Then $G \cap G_1$ is a maximal subgroup of $R$. By the hypothesis, $R \cap G_1$ is $s$-conditionally permutable in $G$, so there exists a $Q \in \text{Syl}_q(G)$ such that $(R \cap G_1)Q = Q(R \cap G_1)$ with $q \neq r$, thus $R \cap G_1 = (R \cap G_1)(R \cap Q) = R \cap (R \cap G_1)Q \lhd (R \cap G_1)Q$, hence $(R \cap G_1)Q \leq N_G(R \cap G_1)$, a contradiction. Therefore, $R \cap G_1 \leq G_r$. The minimal normality of $R$ in $G$, we have $R \cap G_1 = 1$. Hence $|R| = r$, $R$ is cyclic, a contradiction.

Theorem 3.3 is proved.

**Corollary 3.2** ([10], Theorem 4.2). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and $G$ a group with a solvable normal subgroup $N$ such that $G/N \in \mathcal{F}$. If all maximal subgroups of every noncyclic Sylow subgroup $P$ of $N$ are $s$-conditionally permutable in $G$, then $G \in \mathcal{F}$.


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