UDC 512.5

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A NOTE ON A BOUND OF ADAN-BANTE *

ОДНЕ ЗАУВАЖЕННЯ ЩОДО ГРАНИЦІ АДАН-БАНТЕ

Let G be a finite solvable group and let χ be a nonlinear irreducible (complex) character of G. Also let $\eta(\chi)$ be the number of nonprincipal irreducible constituents of $\chi\bar{\chi}$, where $\bar{\chi}$ denotes the complex conjugate of χ. Adan-Bante proved that there exist constants C and D such that $\mathrm{dl}\,(G/\ker\chi) \leq C\eta(\chi) + D$. In the present work, we establish a bound lower than the Adan-Bante bound for $\eta(\chi) > 2$.

Нехай G — скінченна розв'язна група, а χ — нелінійний незвідний (комплексний) характер групи G. Також нехай $\eta(\chi)$ — число неголовних незвідних складових $\chi\bar{\chi}$, де $\bar{\chi}$ позначає величину, комплексно спряжену до χ. Як доведено Адан-Банте, існують сталі C та D такі, що $\mathrm{dl}\,(G/\ker\chi) \leq C\eta(\chi) + D$. В даній роботі встановлено оцінку нижчу, ніж оцінка Адан-Банте для $\eta(\chi) > 2$.

Let G be a finite solvable group and χ be a nonlinear irreducible (complex) character of G. Let $\eta(\chi)$ be the number of nonprincipal irreducible constituents of $\chi\bar{\chi}$, where $\bar{\chi}$ means the complex conjugate of χ. In her paper [1], E. Adan-Bante utilized a key lemma to yield a bound for the derived length of $G/\ker\chi$. That is the following lemma.

Lemma 1. *Let $n > 1$ be an integer and $\mathbb{N} = \{1, 2, \ldots\}$ be the set of all positive integers. Define*

$$p(n) = \max\{n_1 n_2 \ldots n_s \mid n_1, n_2, \ldots, n_s \in \mathbb{N} \text{ and } n_1 + n_2 + \ldots + n_s = n\}.$$

Hence

$$p(n) \leq 2^{n-1}.$$

Adan-Bante's inequality above can be improved slightly. In fact, we have the following lemma.

Lemma 1$'$. *Let $n > 1$ be an integer and $\mathbb{N} = \{1, 2, \ldots\}$ be the set of all positive integers. Define*

$$p(n) = \max\{n_1 n_2 \ldots n_s \mid n_1, n_2, \ldots, n_s \in \mathbb{N} \text{ and } n_1 + n_2 + \ldots + n_s = n\}.$$

Then

$$p(n) = \begin{cases} 3^{n/3}, & n \equiv 0 \pmod 3, \\ 4 \cdot 3^{(n-4)/3}, & n \equiv 1 \pmod 3, \\ 2 \cdot 3^{(n-2)/3}, & n \equiv 2 \pmod 3. \end{cases}$$

Hence

$$p(n) \leq 3^{n/3}.$$

Proof. By the relation of congruence, then for $n \geq 2$ we have that one of the following:

$$n \equiv 0 \pmod 3, \qquad n \equiv 1 \pmod 3, \qquad \text{or} \qquad n \equiv 2 \pmod 3.$$

* Supported by the Doctor Foundation of Henan University of Technology (2010BS048), the Project of Zhengzhou Municipal Bureau of Science and Technology (20130790), the Key Project of Education Department of Henan Province (14B110001), and the Project of Science and Technology Department of Henan Province (142300410133).

By the definition of $p(n)$ and computation, it follows that

$$n = 2, \quad p(n) = 2, \quad n = 5, \quad p(n) = 2 \cdot 3,$$

$$n = 3, \quad p(n) = 3, \quad n = 6, \quad p(n) = 3 \cdot 3,$$

$$n = 4, \quad p(n) = 4, \quad n = 7, \quad p(n) = 4 \cdot 3.$$

We prove that the factors of $p(n)$ are 2 or 3.

Let $n = m_1 + m_2 + \ldots + m_t$, $t \geq 1$, such that

$$p(n) = m_1 m_2 \ldots m_t.$$

We assert that

(i) $m_i > 1$ for every $i = 1, 2, \ldots, t$.

Otherwise, it is no loss to assume that $m_1 = 1$. Thus,

$$(1 + m_2) m_3 \ldots m_t > m_1 m_2 m_3 \ldots m_t = p(n),$$

a contradiction.

(ii) $m_i \leq 4$ for each $i = 1, 2, \ldots, t$.

Otherwise, it is no loss to assume that $m_1 > 4$ and then

$$2 \cdot (m_1 - 2) > m_1.$$

Hence,

$$2 \cdot (m_1 - 2) m_2 m_3 \ldots m_t > m_1 m_2 m_3 \ldots m_t = p(n),$$

a contradiction.

So, m_i, $i = 1, 2, \ldots, t$, are 2 or 3 since $4 = 2 \cdot 2$ and then

$$p(n) = 2^a 3^b,$$

where a, b are nonnegative integers and $2a + 3b = n$.

Now, since $2 \cdot 2 \cdot 2 < 3 \cdot 3$, it follows that the number of factor 3 in $p(n)$ should be as many as possible. That is,

$$0 \leq a \leq 2.$$

Therefore, we have that

$$p(n) = \begin{cases} 3^{n/3}, & n \equiv 0 \ (\mathrm{mod}\ 3), \\ 4 \cdot 3^{n-4/3}, & n \equiv 1 (\mathrm{mod}\ 3), \\ 2 \cdot 3^{n-2/3}, & n \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

It follows that

$$p(n) \leq 3^{n/3}.$$

Lemma $1'$ is proved.

Utilizing the inequality $p(n) \leq 3^{n/3}$ in Adan-Bante's proof in [1], we have that the bound of Adan-Bante can be improved as follows.

Theorem 1. *Let G be a finite solvable group and $\chi \in \mathrm{Irr}\,(G)$, where $\mathrm{Irr}\,(G)$ denotes the set of irreducible characters of G. Then there exists a constant c such that*

$$\mathrm{dl}\,(G/\ker\chi) \leq c\eta(\chi) + 1.$$

Remark. In particular, if $\chi \in \mathrm{Irr}(G)$ is faithful, we would have that $\mathrm{dl}\,(G) \leq c\eta(\chi) + 1$. Note that E. Adan-Bante has studied the finite solvable groups with $\eta(\chi) \leq 2$ in [2, 3].

Keller [4] obtained that there exist universal constants C_1 and C_2 such that $\mathrm{dl}\,(\mathrm{G}) \leq \leq \mathrm{C}_1 \log(\mathrm{m}(\mathrm{G}, \mathrm{V})) + \mathrm{C}_2$ for any finite solvable group G acting faithfully and irreducibly on a finite vector space V. In fact, the author proved the result with $\log = \log_2$, $C_1 = 24$ and $C_2 = 364$. And the author says in [4] that these constants are far from being best possible. Notice that the constants C and D in [1] are related to the constants in [4]. Actually, $C = C_1 \log 2 + C_2 + 1$ and $D = 1 - C_1 \log 2$ (By the way, that Adan-Bante wrote $D = 1 + C_1 \log 2$ in [1] is a typo). Also, our constant $c = \dfrac{\log 3}{3} C_1 + C_2 + 1$. If $\eta(\chi) > 2$, that is, $\eta(\chi) \geq 3$, and since

$$3 > \frac{\log 2}{\log 2 - \dfrac{\log 3}{3}},$$

then we have that $c\eta(\chi) + 1 < C\eta(\chi) + D$. So our bound is lower than Adan-Bante's if $\eta(\chi) > 2$. (It can be seen that the specific values of C_1 and C_2 are not used in the comparison.)

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Received 22.08.12,
after revision — 22.11.13