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ON A CLASS OF NONUNIFORMLY NONLINEAR SYSTEMS WITH DIRICHLET BOUNDARY CONDITIONS

ПРО ОДИН КЛАС НЕОДНОРІДНО НЕЛІНІЙНИХ СИСТЕМ З ГРАНИЧНИМИ УМОВАМИ ТИПУ ДІРІХЛЕ

The existence and multiplicity of weak solutions for some nonuniformly nonlinear elliptic systems are obtained by using the minimum principle and the Mountain pass theorem.

Існування та кратність слабких розв'язків деяких нерівномірно нелінійних еліптичних систем досліджено за допомогою принципу мінімуму та теореми про гірський перевал.

1. Introduction. We study the nonuniformly nonlinear elliptic system

$$\begin{aligned} &-\Delta_p u - \operatorname{div}(h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = F_u(x,u,v) \quad \text{in} \quad \Omega, \\ &-\Delta_q v - \operatorname{div}(h_2(|\nabla v|^q)|\nabla v|^{q-2}\nabla v) = F_v(x,u,v) \quad \text{in} \quad \Omega, \qquad (1.1) \\ &u = v = 0 \quad \text{in} \quad \partial\Omega, \end{aligned}$$

where Ω is a bounded smooth open set in $\mathbb{R}^N$, $-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $2 \leq p \leq q$ and $h_1,\, h_2 \in C_+(\mathbb{R},\mathbb{R})$. If $h_1(t) = h_2(t) = 1 + \dfrac{t}{\sqrt{1+t^2}}$, $t \geq 0$, then (1.1) is called a capillarity system. Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e., the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e., the attractive force between the molecules of the liquid. The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as motion of drops, bubbles, and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems, see [11, 12].

It should be noticed that the proof of the existence results for nonlinear elliptic systems is a long-standing question, see [7] and the references therein. To our knowledge, elliptic equations of (1.1) type has been firstly investigated by J. M. Bezerra do Ó [13], in which the author extended the existence results by D. G. Costa et al. [4] (for the p-Laplacian) to a more general class of operators. He also achieved a multiplicity result using Morse theory. On this topic, we refer to recent interesting papers [5, 6, 8 – 10, 15]. There, the authors have used different methods to prove the existence of a nontrivial solution or the existence of infinitely many solutions. In [1, 16], the authors studied the existence of a solution for (1.1) using the minimum principle. The purpose of this note is to deal with the multiplicity of solutions for system (1.1) by using the minimum principle combined with the mountain pass theorem. Thus, our result is a natural extension from the previous ones [1, 13, 16].

Through this paper for $(u,v)\in\mathbb{R}^2$, denote $|(u,v)|^2=|u|^2+|v|^2$. We assume that $F:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is of C^1 class such that $F(x,0,0)=0$ for all $x\in\overline{\Omega}$ and $(F_u,F_v)=\left(\dfrac{\partial F}{\partial u},\dfrac{\partial F}{\partial v}\right)$, F_u and F_v are Carathéodory functions satisfying the following growth conditions:

(H_1) $\lim_{|u|\to\infty}\dfrac{|F_u(x,u,v)|}{|u|^{p-1}}=0$, uniformly in $(x,v)\in\overline{\Omega}\times\mathbb{R}$, $\lim_{|v|\to\infty}\dfrac{|F_v(x,u,v)|}{|v|^{q-1}}=0$, uniformly in $(x,v)\in\overline{\Omega}\times\mathbb{R}$;

(H_2) $\lim_{|(u,v)|\to 0}\dfrac{|F(x,u,v)|}{|u|^{\delta+1}|v|^{\gamma+1}}=0$, $\lim_{|(u,v)|\to\infty}\dfrac{|F(x,u,v)|}{|u|^{\delta+1}|v|^{\gamma+1}}=\infty$, uniformly in $x\in\Omega\times\mathbb{R}$, where $\delta,\gamma\geq 0$, $\dfrac{\delta+1}{p}+\dfrac{\gamma+1}{q}=1$;

(H_3) let h_1 and $h_2\in C_+(\mathbb{R},\mathbb{R})$; we assume that h_1 and h_2 are the continuous and nondecreasing functions satisfying the following growth conditions: there exist α_1, α_2, β_1 and $\beta_2\in\mathbb{R}$ such that

$$0<\alpha_1\leq h_1(t)\leq\beta_1,$$

$$0<\alpha_2\leq h_2(t)\leq\beta_2.$$

The main result of this paper is given by the following theorem:

Theorem 1.1. *Suppose that* $(H_1)-(H_3)$ *hold. Then system* (1.1) *has at least two nontrivial weak solutions.*

This paper is organized as follows. In Section 2, we present some notations and relevant lemmas. We reserve the Section 3 for the proof of the main result.

2. Notations and preliminary lemmas. Let the product space $H=H_0^{1,p}(\Omega)\times H_0^{1,q}(\Omega)$ with the norm

$$\big\|(u,v)\big\|_H=\|u\|_{1,p}+\|v\|_{1,q}=\left(\int\limits_\Omega|\nabla u|^p\,dx\right)^{\frac{1}{p}}+\left(\int\limits_\Omega|\nabla u|^q\,dx\right)^{\frac{1}{q}}.$$

Let us define the mappings

$$J_1(u,v)=\frac{1}{p}\int\limits_\Omega|\nabla u|^p\,dx+\frac{1}{q}\int\limits_\Omega|\nabla v|^q\,dx$$

and $J_1'\colon H\to H^*$ by

$$\langle J_1'(u,v),(\xi,\eta)\rangle=\int\limits_\Omega(|\nabla u|^{p-2}\nabla u\nabla\xi+|\nabla v|^{q-2}\nabla v\nabla\eta)\,dx$$

for any $(u,v),(\xi,\eta)\in H$.

Let us define the mappings

$$h(u,v)=\frac{1}{p}\int\limits_0^u h_1(s)\,ds+\frac{1}{q}\int\limits_0^v h_2(s)\,ds,\qquad J_2(u,v)=\int\limits_\Omega h(|\nabla u|^p,|\nabla v|^q)\,dx$$

and $J_2'\colon H\to H^*$ by

$$\langle J_2'(u,v),(\xi,\eta)\rangle=\int\limits_\Omega\Big[h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla\xi+h_2(|\nabla v|^q)|\nabla v|^{q-2}\nabla v\nabla\eta\Big]\,dx$$

for any $(u,v),(\xi,\eta)\in H$.

Let us define the mapping

$$\widehat{W}(u,v) = \int\limits_{\Omega} F(x,u,v)\,dx$$

and $\widehat{W}'\colon H \to H^*$ by

$$\langle \widehat{W}'(u,v),(\xi,\eta)\rangle = \int\limits_{\Omega} \big[F_u(x,u,v)\xi + F_v(x,u,v)\eta\big]\,dx$$

for any $(u,v),(\xi,\eta) \in H$.

We need certain properties of the functional $J = J_1 + J_2\colon H \to \mathbb{R}$ defined by

$$J(u,v) = \frac{1}{p}\int\limits_{\Omega} |\nabla u|^p\,dx + \frac{1}{q}\int\limits_{\Omega} |\nabla v|^q\,dx + \frac{1}{p}\int\limits_{\Omega}\int\limits_{0}^{|\nabla u|^p} h_1(s)\,ds + \frac{1}{q}\int\limits_{\Omega}\int\limits_{0}^{|\nabla v|^q} h_2(s)\,ds$$

for all $(u,v) \in H$.

Definition 2.1. *We say that $w = (u,v)$ is a weak solution of system* (1.1) *if and only if*

$$\langle J'(u,v),(\xi,\eta)\rangle = \langle \widehat{W}'(u,v),(\xi,\eta)\rangle$$

for any $(\xi,\eta) \in H$.

Definition 2.2. *An operator $J\colon H \to H^*$ verifies the (S_+) condition if for any sequence $\{(u_n,v_n)\} \in H$ such that $\{(u_n,v_n)\} \rightharpoonup (u,v)$ weakly and*

$$\limsup_{n\to\infty} \big\langle J'(u_n,v_n),\ (u_n - u, v_n - v)\big\rangle \le 0$$

we have that $\{(u_n,v_n)\} \to (u,v)$ strongly in H.

Lemma 2.1. *The functional J is weakly lower semicontinuous.*

Proof. Let $(u,v) \in H$ and $\epsilon > 0$ be fixed. Using the properties of lower semicontinuous function (see [3], Section I.3), it is enough to prove that there exists $\delta > 0$ such that

$$J(u,v) \ge J(u_1,v_1) - \epsilon, \qquad \forall\,(u,v) \in H\colon \|(u,v) - (u_1,v_1)\| < \delta. \tag{2.1}$$

Using the hypothesis (H_3), it is easy to check that J is convex. Hence, we have

$$J(u,v) \ge J(u_1,v_1) + \langle J'(u_1,v_1),(u - u_1, v - v_1)\rangle \quad \forall\,(u,v) \in H.$$

Using condition (H_3) and Hölder's inequality we deduce there exists a positive constant $c > 0$ such that

$$J(u,v) \ge J(u_1,v_1) - \int\limits_{\Omega} |\nabla u_1|^{p-2}|\nabla u_1|\;|\nabla u - \nabla u_1|\,dx -$$

$$-\int\limits_{\Omega} |\nabla v_1|^{q-2}|\nabla v_1|\;|\nabla v - \nabla v_1|\,dx -$$

$$-\int_{\Omega} |h_1(|\nabla u_1|^p)|\, |\nabla u_1|^{p-2}|\nabla u_1|\, |\nabla u - \nabla u_1| dx -$$

$$-\int_{\Omega} |h_2(|\nabla v_1|^q)|\, |\nabla v_1|^{q-2}|\nabla v_1|\, |\nabla v - \nabla v_1| dx \geq$$

$$\geq J(u_1, v_1) - (\beta_1 + 1)\|u_1\|_{1,p}^{p-1}\|u - u_1\|_{1,p} - (\beta_2 + 1)\|v_1\|_{1,q}^{q-1}\|v - v_1\|_{1,q} \geq$$

$$\geq J(u_1, v_1) - c\|(u - u_1, v - v_1)\|_H \quad \forall\,(u, v) \in H.$$

It is clear that taking $\delta = \dfrac{\epsilon}{c}$ relation (2.1) holds true for all $(u_1, v_1) \in H$ with $\|(u, v) - (u_1, v_1)\|_H < \delta$. Thus we proved that J is strongly lower semicontinous. Taking into account the fact that J is convex then by [2] (Corollary III.8) we conclude that J is weakly lower semicontinous.

Lemma 2.1 is proved.

Lemma 2.2. *The functional* $\widehat{W}$ *is weakly continuous.*

Proof. Let $\{w_n\} = \{(u_n, v_n)\}$ be a sequence that converges weakly to $w = (u, v)$ in H. We will show that

$$\lim_{n\to\infty} \int_{\Omega} F(x, u_n, v_n)\, dx = \int_{\Omega} F(x, u, v)\, dx. \tag{2.2}$$

From (H_1) and the continuity of the potential F, for any $\epsilon > 0$, there exists a positive constant $M = M(\epsilon)$ such that

$$\big|F_u(x, u, v)\big| \leq \epsilon |u|^{p-1} + M_\epsilon, \qquad \big|F_v(x, u, v)\big| \leq \epsilon |v|^{q-1} + M_\epsilon \tag{2.3}$$

for all $(x, u, v) \in \overline{\Omega} \times \mathbb{R}^2$. Hence,

$$\int_{\Omega} \big[F(x, u_n, v_n) - F(x, u, v)\big]\, dx =$$

$$= \int_{\Omega} \nabla F(x, w + \theta_n(w_n - w))(w_n - w)\, dx =$$

$$= \int_{\Omega} F_u(x, u + \theta_{1,n}(u_n - u), v + \theta_{2,n}(v_n - v))\,(u_n - u)\, dx +$$

$$+ \int_{\Omega} F_v(x, u + \theta_{1,n}(u_n - u), v + \theta_{2,n}(v_n - v))\,(v_n - v)\, dx,$$

where $\theta_n = (\theta_{1,n}, \theta_{2,n})$ and $0 \leq \theta_{1,n}(x),\ \theta_{2,n}(x) \leq 1$ for all $x \in \Omega$. Now, using (2.3) and Hölder's inequality we conclude that

$$\left| \int_{\Omega} [F(x, u_n, v_n) - F(x, u, v)]\, dx \right| \leq$$

$$\leq \int\limits_{\Omega} \left|F_u(x, u + \theta_{1,n}(u_n - u), v + \theta_{2,n}(v_n - v)\right| |u_n - u| \, dx +$$

$$+ \int\limits_{\Omega} \left|F_v(x, u + \theta_{1,n}(u_n - u), v + \theta_{2,n}(v_n - v)\right| |v_n - v| \, dx \leq$$

$$\leq \int\limits_{\Omega} \big(\epsilon \, |u + \theta_{1,n}(u_n - u)|^{p-1} + M_\epsilon\big) |u_n - u| \, dx +$$

$$+ \int\limits_{\Omega} \big(\epsilon | \, v + \theta_{1,n}(v_n - v)|^{q-1} + M_\epsilon\big) \, |v_n - v| \, dx \leq$$

$$\leq M_\epsilon |\Omega|^{\frac{p-1}{p}} \|u_n - u\|_{L^p(\Omega)} + \epsilon \|u + \theta_{1,n}(u_n - u)\|^{p-1}_{L^p(\Omega)} \, \|u_n - u\|_{L^p(\Omega)} +$$

$$+ M_\epsilon |\Omega|^{\frac{q-1}{q}} \|v_n - v\|_{L^q(\Omega)} + \epsilon \|v + \theta_{2,n}(v_n - v)\|^{q-1}_{L^q(\Omega)} \, \|v_n - v\|_{L^q(\Omega)}. \tag{2.4}$$

On the other hand, since $H \hookrightarrow L^i(\Omega) \times L^j(\Omega)$ is compact for all $i \in [p, p^*)$ and $j \in [p, p^*)$ the sequence $\{w_n\}$ converges to $w = (u, v)$ in the space $L^p(\Omega) \times L^q(\Omega)$, i.e., $\{u_n\}$ converges strongly to u in $L^p(\Omega)$ and $\{v_n\}$ converges strongly to v in $L^q(\Omega)$. Hence, it is easy to see that the sequences $\big\{\|u + \theta_{1,n}(u_n - u)\|_{L^p(\Omega)}\big\}$ and $\big\{\|v + \theta_{2,n}(v_n - v)\|_{L^q(\Omega)}\big\}$ are bounded. Thus, it follows from (2.4) that relation (2.2) holds true.

Lemma 2.2 is proved.

Lemma 2.3. *The functional* $J' : H \to H^*$ *verifies the* (S_+) *condition.*

Proof. Assume that $(u_n, v_n) \rightharpoonup (u, v)$ in H and

$$\limsup_{n\to\infty} \big\langle J'(u_n, v_n), \, (u_n - u, v_n - v)\big\rangle \leq 0. \tag{2.5}$$

Since $\{(u_n, v_n)\}$ is weakly convergent to (u, v) in H it follows that $\{(u_n, v_n)\}$ is bounded in H. By the condition (H_3) we have

$$J(u_n, v_n) \leq \frac{1}{p}(\beta_1 + 1) \int\limits_{\Omega} |\nabla u_n|^p \, dx + \frac{1}{q}(\beta_2 + 1) \int\limits_{\Omega} |\nabla v_n|^q \, dx.$$

So $\{J(u_n, v_n)\}$ is bounded. Then we may assume that $J\{(u_n, v_n)\} \to \alpha$. Using Lemma 2.1, we find

$$J(u, v) \leq \liminf_{n\to\infty} J(u_n, v_n) = \alpha.$$

Since J is convex the following inequality holds true:

$$J(u, v) \geq J(u_n, v_n) + \big\langle J'(u_n, v_n), (u - u_n, v - v_n)\big\rangle \quad \text{for all} \quad n. \tag{2.6}$$

Using (2.5), (2.6), we have

$$J(u, v) - \limsup_{n\to\infty} J(u_n, v_n) = \liminf_{n\to\infty} (J(u, v) - J(u_n, v_n)) \geq$$

$$\geq \liminf_{n\to\infty} \langle J'(u_n, v_n),\ (u - u_n, v - v_n)\rangle =$$

$$= -\limsup_{n\to\infty} \langle J'(u_n, v_n),\ (u_n - u, v_n - v)\rangle \geq 0,$$

which implies that $J(u, v) \geq \alpha$ and thus $J(u, v) = \alpha$.

We also have $\left(\frac{u_n + u}{2}, \frac{v_n + v}{2}\right)$ converges weakly to (u, v) in H. Using again Lemma 2.1 we deduce that

$$\alpha = J(u, v) \leq \liminf_{n\to\infty} J\left(\frac{u_n + u}{2}, \frac{v_n + v}{2}\right). \tag{2.7}$$

If we suppose that $\{(u_n, v_n)\}$ does not converges to (u, v), then there exists $\epsilon > 0$ and a subsequence $\{(u_{n_k}, v_{n_k})\}$ such that $\|(u_{n_k} - u, v_{n_k} - v)\|_H = \|u_{n_k} - u\|_{H_0^{1,p}} + \|v_{n_k} - v\|_{H_0^{1,q}} \geq \epsilon$.

If $\|u_{n_k} - u\|_{H_0^{1,p}} \geq \frac{\epsilon}{2}$ we know that $T_1(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx$ is p-uniformly convex, i.e., there exists a positive constant $k_1 > 0$ such that

$$T_1\left(\frac{u + v}{2}\right) \leq \frac{1}{2}T_1(u) + \frac{1}{2}T_1(v) - k_1\|u - v\|^p_{H_0^{1,p}}.$$

Hence, we have

$$\frac{1}{2p}\int\limits_\Omega |\nabla u|^p\, dx + \frac{1}{2p}\int\limits_\Omega |\nabla u_{n_k}|^p\, dx - \frac{1}{p}\int\limits_\Omega \left|\frac{\nabla u + \nabla u_{n_k}}{2}\right|^p dx \geq k_1 \int\limits_\Omega |\nabla u - \nabla u_{n_k}|^p\, dx.$$

That fact and convexity of J_2, $\|\cdot\|_{H_0^{1,q}}$ imply

$$\frac{1}{2}J(u, v) + \frac{1}{2}J(u_{n_k}, v_{n_k}) - J\left(\frac{u_{n_k} + u}{2}, \frac{v_{n_k} + v}{2}\right) =$$

$$= \frac{1}{2p}\int\limits_\Omega |\nabla u|^p\, dx + \frac{1}{2p}\int\limits_\Omega |\nabla u_{n_k}|^p\, dx - \frac{1}{p}\int\limits_\Omega \left|\frac{\nabla u + \nabla u_{n_k}}{2}\right|^p dx +$$

$$+ \frac{1}{2q}\int\limits_\Omega |\nabla v|^q\, dx + \frac{1}{2q}\int\limits_\Omega |\nabla v_{n_k}|^q\, dx - \frac{1}{q}\int\limits_\Omega \left|\frac{\nabla v + \nabla v_{n_k}}{2}\right|^q dx +$$

$$+ \frac{1}{2}J_2(u, v) + \frac{1}{2}J_2(u_{n_k}, v_{n_k}) - J_2\left(\frac{u_{n_k} + u}{2}, \frac{v_{n_k} + v}{2}\right) \geq$$

$$\geq k_1 \int\limits_\Omega |\nabla u - \nabla u_{n_k}|^p\, dx \geq k_1\left(\frac{\epsilon}{2}\right)^p.$$

Letting $k \to \infty$ we find

$$\limsup_{n\to\infty} J\left(\frac{u_{n_k} + u}{2}, \frac{v_{n_k} + v}{2}\right) \leq \alpha - k_1\left(\frac{\epsilon}{2}\right)^p$$

which contradicts (2.7).

If $\|v_{n_k} - v\|_{H_0^{1,q}} \geq \frac{\epsilon}{2}$ we know that $T_2(v) = \frac{1}{q}\int_\Omega |\nabla v|^q\, dx$ is q-uniformly convex. That fact and convexity of J_2, $\|\cdot\|_{H_0^{1,p}}$ imply that

$$\limsup_{n\to\infty} J\left(\frac{u_{n_k}+u}{2}, \frac{v_{n_k}+v}{2}\right) \leq \alpha - k_1\left(\frac{\epsilon}{2}\right)^q,$$

which contradicts (2.7).

Similarly if $\|u_n\|_{H_0^{1,p}} \geq \frac{\epsilon}{2}$ and $\|v_n\|_{H_0^{1,q}} \geq \frac{\epsilon}{2}$ we obtain contradictions.

Lemma 2.3 is proved.

In our proof, we use the mountain pass theorem stated in [2]. For the reader's convenience, we recall it as follows.

Definition 2.3. *Let $(X, \|\cdot\|)$ be a real Banach space, $J \in C^1(X, \mathbb{R})$. We say that J satisfies the $(PS)_c$ condition if any sequence $\{u_m\} \subset X$ such that $J(u_m) \to c$ and $J'(u_m) \to 0$ as $m \to \infty$ has a convergent subsequence.*

Proposition 2.1 (see [2]). *Let $(X, \|\cdot\|)$ be a real Banach space, $J \in C^1(X, \mathbb{R})$ satisfies the $(PS)_c$ condition for any $c > 0$, $J(0) = 0$ and the following conditions hold*:

(i) *There exists a function $\phi \in X$ such that $\|\phi\| > \rho$ and $J(\phi) < 0$.*

(ii) *There exist two positive constants ρ and R such that $J(u) \geq R$ for any $u \in X$ with $\|u\| = \rho$. Then the functional J has a critical value $c \geq R$, i.e., there exists $u \in X$ such that $J'(u) = 0$ and $J(u) = c$.*

3. Proof of the main theorem. In this section we give the proof of Theorem 1.1. Let $J(u,v) = \int_\Omega h(|\nabla u|^p, |\nabla v|^q)\, dx$ as in Section 2, and let the energy $E\colon H \to \mathbb{R}$ given by

$$E(u,v) = J(u,v) - \int\limits_\Omega F(x,u,v)\, dx$$

for any $(u,v) \in H$. Then weak solutions of system (1.1) are exactly the critical points of $E(u,v)$ in H. Lemmas 2.1 and 2.2 imply that E is weakly lower semicontinous.

By Hölder's inequality, (2.4), we have

$$F(x,u,v) = \int\limits_0^u \frac{\partial F}{\partial s}(x,s,v)\, ds + F(x,0,v) =$$

$$= \int\limits_0^u \frac{\partial F}{\partial s}(x,s,v)\, ds + \int\limits_0^v \frac{\partial F}{\partial s}(x,0,s)\, ds + F(x,0,0) \leq$$

$$\leq \int\limits_0^u (\epsilon|u|^{p-1} + M_\epsilon)\, ds + \int\limits_0^v (\epsilon|v|^{q-1} + M_\epsilon)\, ds =$$

$$= \frac{\epsilon}{p}|u|^p + M_\epsilon u + \frac{\epsilon}{p}|v|^q + M_\epsilon v,$$

so

$$\left|\int_\Omega F(x,u,v)\,dx\right| \leq \int_\Omega |F(x,u,v)|\,dx \leq$$

$$\leq \epsilon \left(\frac{1}{p}\int_\Omega |u|^p\,dx + \frac{1}{q}\int_\Omega |v|^q\,dx\right) + M_\epsilon \left(\int_\Omega u\,dx + \int_\Omega v\,dx\right) \leq$$

$$\leq \frac{\epsilon}{p}\, S_1^p \int_\Omega |\nabla u|^p\,dx + \frac{\epsilon}{q}\, S_2^q \int_\Omega |\nabla v|^q\,dx +$$

$$+ M_\epsilon |\Omega|^{\frac{p-1}{p}} S_1 \left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{1}{p}} + M_\epsilon |\Omega|^{\frac{q-1}{q}} S_2 \left(\int_\Omega |\nabla v|^q\,dx\right)^{\frac{1}{q}} \leq$$

$$\leq \frac{\epsilon}{p}\, S_1^p\, \|u\|_{1,p} + \frac{\epsilon}{q}\, S_2^q\, \|v\|_{1,q}^q + A\left(\|u\|_{1,p} + \|v\|_{1,q}\right),$$

where $S_1,\ S_2$ are the embedding constants of $H_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega),\ H_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$ and $A = \max\left\{M_\epsilon |\Omega|^{\frac{p-1}{p}} S_1, M_\epsilon |\Omega|^{\frac{q-1}{q}} S_2\right\}$.

Hence

$$E(u,v) \geq \frac{1}{p}(\alpha_1 + 1 - \epsilon S_1^p) \int_\Omega |\nabla u|^p\,dx + \frac{1}{q}(\alpha_2 + 1 - \epsilon S_2^q) \int_\Omega |\nabla v|^q\,dx - A\|(u,v)\|_H.$$

Letting $\epsilon = \frac{1}{2}\min\left\{\frac{\alpha_1+1}{S_1^p}, \frac{\alpha_2+1}{S_2^q}\right\}$. Note that

$$\int_\Omega |\nabla u|^p\,dx + \int_\Omega |\nabla v|^q\,dx \geq \frac{1}{2^p}\left[\left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla v|^q\,dx\right)^{\frac{1}{q}}\right]^p - 1.$$

Since $p \leq q$, we obtain that

$$E(u,v) \geq \frac{1}{2q}\min\{\alpha_1+1, \alpha_2+1\} \times \left[\frac{1}{2^p}\|(u,v)\|_H^p - 1\right] - A\|(u,v)\|_H.$$

It follows that E is coercive in H. By (H_1), (H_3), E is continuously differentiable on H and

$$\left\langle E'(u,v), (\epsilon,\eta)\right\rangle =$$

$$= \int_\Omega \Big[h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla \xi + h_2(|\nabla v|^q)|\nabla u|^{q-2}\nabla v \nabla \eta - F_u(x,u,v)\xi - F_v(x,u,v)\eta\Big]\,dx =$$

$$= \left\langle J'(u,v), (\epsilon,\eta)\right\rangle - \left\langle \widehat{W}'(u,v), (\epsilon,\eta)\right\rangle$$

for any $(u,v)\in H$. By Lemmas 2.1, 2.2 and the coercivity of E, applying the Minimum principle (see [14, p. 4], Theorem 1.2), the functional E has a global minimum and system (1.1) admits a weak solution $w_1=(u_1,v_1)\in H$. Moreover, we shall prove that $E(w_1)<0$.

Indeed, let φ_1 be first eigenfunction of $-\Delta_p$ associated with first eigenvalue λ_1 and ψ_1 be first eigenfunction of $-\Delta_q$ associated with first eigenvalue μ_1. In view of (H_2), we get for $M>0$ and $t>0$ sufficiently large,

$$E\left(t^{\frac{1}{p}}\varphi_1, t^{\frac{1}{q}}\psi_1\right)\leq$$

$$\leq\left(\frac{\lambda_1}{p}\,(\beta_1+1)\int\limits_{\Omega}|\varphi_1|^p\,dx+\frac{\mu_1}{q}\,(\beta_2+1)\int\limits_{\Omega}|\psi_1|^q\,dx\right)t-\int\limits_{\Omega}F\left(x,t^{\frac{1}{p}}\varphi_1,t^{\frac{1}{q}}\psi_1\right)dx<$$

$$<\left(\frac{\lambda_1}{p}(\beta_1+1)\int\limits_{\Omega}|\varphi_1|^p\,dx+\frac{\mu_1}{q}(\beta_2+1)\int\limits_{\Omega}|\psi_1|^q\,dx\right)t-Mt^{\frac{\delta+1}{p}+\frac{\gamma+1}{q}}\int\limits_{\Omega}|\varphi_1|^{\delta+1}|\psi_1|^{\gamma+1}\,dx.$$

Letting

$$M=\frac{\dfrac{\lambda_1}{p}(\beta_1+1)\displaystyle\int_\Omega|\varphi_1|^p\,dx+\dfrac{\mu_1}{q}\,(\beta_2+1)\displaystyle\int_\Omega|\psi_1|^q dx}{\displaystyle\int_\Omega|\varphi_1|^{\delta+1}|\psi_1|^{\gamma+1}\,dx}$$

we have $E(u_0,v_0)<0$, where $w_0=(u_0,v_0)=\left(t^{\frac{1}{p}}\varphi_1,t^{\frac{1}{q}}\psi_1\right)$. This means that

$$-\infty<E(w_1)=\inf\{E(u,v)\colon (u,v)\in H\}<0 \tag{3.1}$$

and $w_1\not\equiv 0$.

In the next parts, we shall show the existence of the second weak solution $w_2=(u_2,v_2)\in H$ $(w_2\neq w_2)$ of system (1.1) by applying the Mountain Pass theorem in [2]. To this purpose, we first show that J has the geometry of the Mountain Pass theorem.

It is clear that $E(0,0)=0$. By using Young's inequality, (H_2) we get for $\epsilon>0$

$$E(u,v)\geq\frac{1}{p}(\alpha_1+1)\int\limits_{\Omega}|\nabla u|^p\,dx+\frac{1}{q}(\alpha_2+1)\int\limits_{\Omega}|\nabla v|^q dx-$$

$$-\frac{\epsilon pS_1^p}{(\delta+1)\lambda_1}\int\limits_{\Omega}|\nabla u|^p\,dx-\frac{\epsilon qS_2^q}{(\gamma+1)\mu_1}\int\limits_{\Omega}|\nabla v|^q dx=$$

$$=\left(\frac{1}{p}(\alpha_1+1)-\frac{\epsilon pS_1^p}{(\delta+1)\lambda_1}\right)\int\limits_{\Omega}|\nabla u|^p dx+\left(\frac{1}{q}(\alpha_2+1)-\frac{\epsilon qS_2^q}{(\gamma+1)\mu_1}\right)\int\limits_{\Omega}|\nabla v|^q dx.$$

Letting $0<\epsilon<\dfrac{1}{2}\min\left\{\dfrac{(\alpha_1+1)(\delta+1)}{p^2S_1^p},\dfrac{\mu_1(\alpha_2+1)(\gamma+1)}{q^2S_2^q}\right\}$. Hence there exists $r>0$ small enough and such that

$$\inf_{\|(u,v)\|=r} E(u,v) > 0 = E(u,v).$$

On the other hand, by (3.1) there exists $t > 0$ (large enough) that for $w_0 = \left(t^{\frac{1}{p}}\varphi_1, t^{\frac{1}{q}}\psi_1\right) \in H$ we have both $\|w_0\|_H > r$ and $E(w_0) < 0$.

In order to verify the $(PS)_c$ condition we proceed as follows. Let $\{(u_n, v_n)\} \in H$ be a sequence satisfying

$$E(u_n, v_n) \to c, \qquad \left\|E'(u_n, v_n)\right\|_{H^*} \to 0.$$

Since E is coercive, it follows that the sequence $\{(u_n, v_n)\}$ is bounded in H. Up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ weakly in H. From $E' = J' - \widehat{W}'$ we get

$$\left\langle J'(u_n, v_n), (u_n - u, v_n - v)\right\rangle = \left\langle E'(u_n, v_n),\ (u_n - u, v_n - v)\right\rangle +$$

$$+ \int\limits_\Omega \Big[F_u(x, u_n, v_n)(u_n - u) + F_v(x, u_n, v_n)(v_n - v)\Big] dx. \tag{3.2}$$

Since $\left\|E'(u_n, v_n)\right\|_{H^*} \to 0$ and $\left\{(u_n - u, v_n - v)\right\}$ is bounded in H, by the inequality

$$\left|\left\langle E'(u_n, v_n), (u_n - u, v_n - v)\right\rangle\right| \le \left\|E'(u_n, v_n)\right\|_{H^*} \|(u_n - u, v_n - v)\|_H$$

it follows that

$$\left\langle E'(u_n, v_n), (u_n - u, v_n - v)\right\rangle \to 0.$$

By (2.3), (3.2) we get

$$\int\limits_\Omega \Big(|F_u(x, u_n, v_n)||u_n - u| + |F_v(x, u_n, v_n)||v_n - v|\Big) dx \le$$

$$\le \epsilon \|u_n\|^{p-1}_{L^p(\Omega)} \|u_n - u\|_{L^p(\Omega)} + M_\epsilon |\Omega|^{\frac{p-1}{p}} \|u_n - u\|_{L^p(\Omega)} +$$

$$+ \epsilon \|v_n\|^{q-1}_{L^q(\Omega)} \|v_n - v\|_{L^q(\Omega)} + M_\epsilon |\Omega|^{\frac{q-1}{q}} \|v_n - v\|_{L^q(\Omega)}.$$

Since $(u_n, v_n) \to (u, v)$ strongly in $L^p(\Omega) \times L^q(\Omega)$, we get

$$\lim_{n\to\infty} \int\limits_\Omega \Big(|F_u(x, u_n, v_n)||u_n - u| + |F_v(x, u_n, v_n)||v_n - v|\Big) dx = 0.$$

In conclusion, relation (3.2) implies

$$\limsup_{n\to\infty} \left\langle J'(u_n, v_n), (u_n - u, v_n - v)\right\rangle \le 0.$$

Then applying Lemma 2.3 we deduce that $\{(u_n, v_n)\}$ converges strongly to (u_2, v_2) in H. Set

$$\overline{c} = \inf_{\chi\in\Gamma} \max_{w\in\chi([0,1])} E(w),$$

where $\Gamma := \{\chi \in C([0,1], H) \colon \chi(0) = 0, \chi(1) = w_0\}$. We know that all assumptions of Proposition 2.1 are satisfied. Therefore, there exists $0 \not\equiv w_2 \in H$ such that $E(w_2) = \overline{c}$ and $\langle E'(w_2), (\xi, \eta)\rangle = 0$ for all $(\xi, \eta) \in H$ or w_2 is a weak solution of (1.1). Moreover $w_2 \neq w_1$ since $E(w_2) = \overline{c} > 0 > E(w_1)$.

Theorem 1.1 is proved.

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