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GENERALIZED SEMICOMMUTATIVE AND SKEW ARMENDARIZ IDEALS

УЗАГАЛЬНЕНІ НАПІВКОМУТАТИВНІ ТА КОСІ ІДЕАЛИ АРМЕНДАРІЗА

We generalize the concepts of semicommutative, skew Armendariz, Abelian, reduced, and symmetric left ideals and study the relations between them.

Узагальнено поняття напівкомутативних косих абелевих зведених та симетричних лівих ідеалів Армендаріза та вивчено співвідношення між ними.

1. Introduction. Throughout this paper R denotes an associative ring with identity 1 and α denotes a nonzero and nonidentity endomorphism of a given ring with $\alpha(1) = 1$, and 1 denotes identity endomorphism, unless specified otherwise.

We write $R[x]$, for the polynomial ring, moreover, $R[x,\alpha] = \left\{\sum_{i=0}^{n} a_i x^i \,\middle|\, n \geq 0, a_i \in R\right\}$ becomes a ring under the following operation:

$$f(x) = \sum_{i=0}^{n} a_i x^i, \qquad g(x) = \sum_{j=0}^{m} b_j x^j \in R[x,\alpha], \qquad f(x)g(x) = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i \alpha^i(b_j) \right) x^k.$$

The ring $R[x,\alpha]$ is called the *skew polynomial extension of* R.

In [4], *Baer-rings* are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [5], a ring R is said to be *quasi-Baer* ring if the right annihilator of every right ideal of R is generated (as a right ideal) by an idempotent. A ring R is called right *principally quasi-Baer ring* if the right annihilator of a principally right ideal of R is generated (as a right ideal) by an idempotent. Finally, a ring R is called *right principally projective ring* if the right annihilator of an element of R is generated by an idempotent [4].

For an endomorphism α of ring R, Hong, Kim, and Kowak [7] called R an *α-skew Armendariz ring* if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x,\alpha]$, $f(x)g(x) = 0$ then $a_i \alpha^i b_j = 0$ for each i and j. Some properties of Armendariz rings have been studied in [9–11].

In [2], the notions of α-Abelian, α-semicommutative, α-reduced, α-symmetric and α-Armendariz rings have been introduced which generalize Abelian, semicommutative, reduced, symmetric and Armendariz rings. Aghayev et al. defined a ring R is called *α-Abelian* if, for any a, $b \in R$, and any idempotent $e \in r$, $ea = ae$ and $ab = 0$ if and only if $a\alpha(b) = 0$ and a ring R is called *α-semicommutative* if, for any a, $b \in R$, $ab = 0$ implies $aRb = 0$ and $ab = 0$ if and only if $a\alpha(b) = 0$. A ring R is called *α-reduced* if, for any a, $b \in R$, $ab = 0$ implies $aR \cap Rb = 0$ and $ab = 0$ if and only if $a\alpha(b) = 0$. A ring R is called *α-symmetric* if, for any $a, b, c \in R$, $abc = 0$ implies $acb = 0$ and $ab = 0$ if and only if $a\alpha(b) = 0$.

They proved that α-semicommutative, α-reduced, α-symmetric and α-Armendariz rings are α-Abelian. For a right principally projective ring R, they also proved the following conditions on α-reduced of a ring R are equivalent:

$$\begin{pmatrix} \alpha\text{-symmetric} & \Leftrightarrow & \alpha\text{-semicommutative} \\ \Updownarrow & & \Updownarrow \\ \alpha\text{-Abelian} & \Leftrightarrow & \alpha\text{-Armendariz} \end{pmatrix}.$$

In this paper we introduce the concepts of *α-Abelian, α-semicommutative, α-reduced, α-symmetric and α-skew Armendariz left ideals* and investigate their properties. Moreover, we prove that if there exists a classical right quotient ring Q of a ring R consisting of central elements and I is α-semicommutative left ideal of R, then QI is α-semicommutative left ideal of $Q(R)$.

Similarly we prove that if I is a left ideal of a ring R and Δ is a multiplicatively closed subset of R consisting of central elements and I is α-semicommutative left ideal of R, then $\Delta^{-1}I$ is α-semicommutative left ideal of $\Delta^{-1}R$.

2. Semicommutative and skew Armendariz ideals. In this section the notion of an α-Abelian left ideals is introduced as a generalization of Abelian left ideals. We recall that a left ideal I of R is called Abelian if for any $a, b \in R$ and any idempotent $e \in R$, $ea - ae \in r_R(I)$. Now we have the following definition.

Definition 2.1. *A left ideal I of R is called α-Abelian if, for any a, $b \in R$ and any idempotent $e \in R$, we have the following conditions*:

1) $ea - ae \in r_R(I)$,

2) $ab \in r_R(I)$ *if and only if* $a\alpha(b) \in r_R(I)$.

So a left ideal I is Abelian if and only if it is 1-Abelian. The following example shows that there exists an Abelian left ideal, but it is not α-Abelian left ideal.

***Example* 2.1.** Let R be the ring $\mathbb{Z} \bigoplus \mathbb{Z}$ with the usual componentwise operation. It is clear that R is an Abelian ring. Let $\alpha\colon R \to R$ be defined by $\alpha((a, b)) = (b, a)$. Then $(1, 0)(0, 1) = 0$, but $(1, 0)\alpha((0, 1)) \neq 0$. Hence R is not α-Abelian. If ideal $I = R$ then $r_R(I) = 0$ and then I is an Abelian left ideal, but it is not an α-Abelian left ideal.

Definition 2.2. *A left ideal I of R is called semicommutative if, for any a, $b \in R$, $ab \in r_R(I)$ then $aRb \subseteq r_R(I)$.*

Definition 2.3. *A left ideal I of R is called α-semicommutative if, for any a, $b \in R$ we have the following conditions*:

1) $ab \in r_R(I)$ *then* $aRb \subseteq r_R(I)$,

2) $ab \in r_R(I)$ *if and only if* $a\alpha(b) \in r_R(I)$.

So a left ideal I is semicommutative if and only if it is 1-semicommutative.

In general the reverse implication in the above definition does not hold by the following example which also shows that there exist an endomorphism α of a ring R and left ideal I of R such that I is semicommutative but is not α-semicommutative.

***Example* 2.2.** Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider a ring $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ with the usual addition and multiplication. If $I = \mathbb{Z}_2 \bigoplus 0$ be a left ideal of R then $r_R(I) = 0 \bigoplus \mathbb{Z}_2$. Now, let $\alpha\colon R \to R$ be defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphism of R. It is clear that I is semicommutative left ideal. For $a = (1, 0)$ and $b = (0, 1) \in R, ab = (0, 0) \in r_R(I)$ but $a\alpha(b) = (1, 0) \notin r_R(I)$.

Lemma 2.1. *If the left ideal I of R is α-semicommutative, then I is α-Abelian.*

Proof. If e is an idempotent in R, then $e(1-e) = 0 \in r_R(I)$. Since I is α-semicommutative, we have $e\alpha(1-e) = 0 \in r_R(I)$ for any $a \in R$ and so $ea - eae \in r_R(I)$. Similarly, $(1-e)e = 0 \in r_R(I)$. Since I is α-semicommutative $(1-e)ae = 0 \in r_R(I)$. So $ae - eae \in r_R(I)$. Therefore, $ae - ea \in r_R(I)$. Thus I is α-Abelian.

Lemma 2.1 is proved.

The following example shows that the condition $\alpha(1) = 1$ in Lemma 2.1 is not superfluous.

***Example* 2.3.** Let $\mathbb{Z}$ be the ring of integers. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a,\, b,\, c \in \mathbb{Z} \right\}.$$

If $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \;\middle|\; b \in \mathbb{Z} \right\}$ be an right ideal of R then $l_R(I) = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \;\middle|\; b, c \in \mathbb{Z} \right\}$. Let $\alpha\colon R \to R$ be defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. For $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R$, if $AB \in l_R(I)$ then we obtain $aa' = 0$, and so $a = 0$ or $a' = 0$. This implies $AR\alpha(B) \subseteq l_R(I)$ and thus I is α-semicommutative. Note that $\alpha(1) \neq 1$ and I is not Abelian.

Definition 2.4. *A left ideal I of R is called α-skew Armendariz if the following conditions are satisfied*:

1) *for any* $f(x) = \sum_{i=0}^{n} a_i x^i$ *and* $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x,\alpha], f(x)g(x) \in r_{R[x,\alpha]}(I[x])$ *implies* $a_i\alpha^i(b_j) \in r_R(I)$,

2) $ab \in r_R(I)$ *if and only if* $a\alpha(b) \in r_R(I)$.

We introduce an α-skew Armendariz left ideal in the following example.

***Example* 2.4.** Let R be an α-skew Armendariz ring and consider

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,\, b \in R \right\}.$$

It is clear that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \;\middle|\; b \in R \right\}$ is the left ideal of S. Let $f(x) = A_0 + A_1 x + \ldots + A_n x^n$ and $g(x) = B_0 + B_1 x + \ldots + B_m x^m \in S[x,\alpha]$, where $A_i = \begin{pmatrix} a_{0i} & a_{1i} \\ 0 & a_{0i} \end{pmatrix}$, $B_j = \begin{pmatrix} b_{0j} & b_{1i} \\ 0 & b_{0j} \end{pmatrix}$ for $i = 0, \ldots, n$, $j = 0, \ldots, m$ such that $f(x)g(x) \in r_{R[x,\alpha]}(I[x])$. Let

$$f(x) = \begin{pmatrix} \alpha_0(x) & \alpha_1(x) \\ 0 & \alpha_0(x) \end{pmatrix}, \qquad g(x) = \begin{pmatrix} \beta_0(x) & \beta_1(x) \\ 0 & \beta_0(x) \end{pmatrix},$$

$$\alpha_0(x) = a_{00} + a_{01}x + \ldots + a_{0n}x^n, \qquad \beta_0(x) = b_{00} + b_{01}x + \ldots + b_{0m}x^m.$$

Since $f(x)g(x) \in r_{R[x,\alpha]}(I[x])$ thus for any $h(x) = \begin{pmatrix} 0 & \gamma(x) \\ 0 & 0 \end{pmatrix} \in I[x]$, that $\gamma(x) = \gamma_0 + \gamma_1 x + \ldots + \gamma_t x^t$, $\gamma(x)f(x)g(x) = 0$. Thus $\gamma(x)\alpha_0(x)\beta_0(x) = 0$. Since I is α-skew Armendariz left ideal hence $\gamma_k\alpha^k(a_{0i}\alpha^i(b_{0j})) = 0$ for all $k = 0, \ldots, t$, $i = 0, \ldots, n$ and $j = 0, \ldots, m$. If set $k = 0$, then $\gamma_0(a_{0i}\alpha^i(b_{0j})) = 0$. Since $\gamma_0 \in R$ is arbitrary, thus $\begin{pmatrix} 0 & \gamma_0 \\ 0 & 0 \end{pmatrix} \in I$. Therefore $a_{0i}\alpha^i(b_{0j}) \in r_R(I)$, and hence $A_i\alpha^i(B_j) \in r_R(I)$. Now we consider

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \in S, \qquad \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \in I, \qquad \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \in r_R(I).$$

Thus $\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1a_2 & a_1b_2 + a_2b_1 \\ 0 & a_1a_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and hence $ca_1a_2 = 0$. Since I is α-skew Armendariz, $ca_1\alpha(a_2) = 0$. Thus $\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \in r_R(I)$ if and only if

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \alpha \left(\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \right) \in r_R(I).$$

Therefore I is an α-skew Armendariz left ideal.

Proposition 2.1. *If I is an α-skew Armendariz left ideal of R and for some a, b, $c \in R$ and some integer $n \geq 1$, $ab \in r_R(I)$ and $ac^n\alpha^n(b) \in r_R(I)$, then $acb \in r_R(I)$.*

Proof. Consider $f(x) = a(1 - cx)$, $g(x) = (1 + cx + \ldots + c^{n-1}x^{n-1})b \in R[x, \alpha]$, $f(x)g(x) = ab - ac^n\alpha^n(b)x^n \in r_{R[x,\alpha]}(I[x])$. Since I is an α-skew Armendariz left ideal of R, so $acb \in r_R(I)$.

Proposition 2.1 is proved.

Next, we show that every α-skew Armendariz left ideal of R is an α-Abelian left ideal.

Proposition 2.2. *If I is an α-skew Armendariz left ideal of R, then I is an α-Abelian left ideal.*

Proof. Assume that I is an α-skew Armendariz left ideal of R. Consider $e = e^2 \in R$ and let $a = e, b = (1 - e), c = er(1 - e)$ with $r \in R$. Then clearly $ab \in r_R(I)$ and $c^2 = 0 \in r_R(I)$ and hence $ac^2\alpha^2(b) \in r_R(I)$ and then by Proposition 2.1, $acb \in r_R(I)$. So $er - ere \in r_R(I)$. Let $a_1 = 1 - e, b_1 = e$ and $c_1 = (1 - e)re$, we also have $a_1b_1c_1 \in r_R(I)$. So $re - ere \in r_R(I)$. Then $re - er \in r_R(I)$.

Proposition 2.2 is proved.

Theorem 2.1. *Let R be a ring and I, J be left ideals of R. If $I \subseteq J$ and J/I is an α-skew Armendariz left ideal of R/I, then J is an α-skew Armendariz left ideal of R.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j \in R[x, \alpha]$ such that $f(x)g(x) \in r_{R[x,\alpha]}(J[x])$. Then $\sum_{i=0}^{n} \bar{a}_ix^i \sum_{j=0}^{m} \bar{b}_jx^j \in r_{R/I[x,\alpha]}(J/I)[x])$. Thus $\bar{a}_i\alpha^i(\bar{b}_j) \in r_{R/I}(J/I)$. Hence $a_i\alpha^i(b_j) \in r_R(J)$. Therefore J is an α-skew Armendariz left ideal of R.

Theorem 2.1 is proved.

The following is an immediate corollary of Theorem 2.1.

Corollary **2.1.** *Let R be a ring and I an left ideal of R. If R/I is α-skew Armendariz then R is an α-skew Armendariz ring.*

A ring R is called *locally finite* if every finite subset of R generates a finite semigroup multiplicatively. Finite rings are clearly locally finite and the algebraic closure of a finite field is locally finite but it is not finite.

Proposition 2.3. *Let R be a locally finite ring and I be an α-skew Armendariz left ideal of R. Then I is α-semicommutative left ideal of R.*

Proof. Let $ab \in r_R(I)$ with a, $b \in R$. For any $r \in R$, since R is locally finite there exist integers $m, k \geq 1$ such that $r^m = r^{m+k}$. So we obtain inductively $r^m = r^mr^k = r^{2k} = \ldots \ldots = r^mr^{mk} = r^{m(k+1)}$, put $h = k + 1$ then $r^m = r^{mh}$ with $h \geq 2$. Notice that $r^{(h-1)m} = r^{(h-2)m}r^m = r^{(h-2)m}r^{mh} = r^{2(h-2)m} = (r^{(h-1)m})^2$. Hence $r^{(h-1)m}$ is an idempotent and so by Proposition 2.2, $ar^{(h-1)m} - r^{(h-1)m}a \in r_R(I)$ and $abr^{(h-1)m} - r^{(h-1)m}ab \in r_R(I)$. Thus

$r^{(h-1)m}ab \in r_R(I)$. On the other hand by Proposition 2.2, $ar^{(h-1)m} - r^{(h-1)m}a \in r_R(I)$, so $ar^{(h-1)m}b - r^{(h-1)m}ab \in r_R(I)$, and hence $ar^{(h-1)m}b \in r_R(I)$. Since I is an α-skew Armendariz left ideal of R so $ar^{(h-1)m}\alpha^{(h-1)m}(b) \in r_R(I)$, and by Proposition 2.1, we imply that $arb \in r_R(I)$ for all $r \in R$.

Proposition 2.3 is proved.

Let α be an endomorphism of a ring R and $M_n(R)$ be the $(n \times n)$-matrix over ring R and $\overline{\alpha}\colon M_n(R) \longrightarrow M_n(R)$ defined by $\overline{\alpha}(a_{ij}) = (\alpha(a_{ij}))$. Then $\overline{\alpha}$ is an endomorphism of $M_n(R)$. It is obvious that, the restriction of $\overline{\alpha}$ to $D_n(R)$ is an endomorphism of $D_n(R)$, where $D_n(R)$ is the diagonal $(n \times n)$-matrix ring over R. We also denote $\overline{\alpha}\mid_{D_n(R)}$ by $\overline{\alpha}$.

Proposition 2.4. *Let α be an endomorphism of a ring R. Then $D_n(I)$ is an $\overline{\alpha}$-skew Armendariz left ideal of $D_n(R)$ if I is an α-skew Armendariz left ideal for any n.*

Proof. Let $f(x) = A_0 + A_1x + \ldots + A_px^p$ and $g(x) = B_0 + B_1x + \ldots + B_qx^q \in D_n(R)[x, \overline{\alpha}]$ satisfying $f(x)g(x) \in r_{D_n(R)[x,\overline{\alpha}]}(D_n(I)[x])$, where

$$A_i = \begin{pmatrix} a_{11}^{(i)} & 0 & \ldots & 0 \\ 0 & a_{22}^{(i)} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & a_{nn}^{(i)} \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} b_{11}^{(j)} & 0 & \ldots & 0 \\ 0 & b_{22}^{(j)} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & b_{nn}^{(j)} \end{pmatrix}.$$

Then from $f(x)g(x) \in r_{D_n(R)[x,\overline{\alpha}]}(D_n(I)[x])$, it follows that

$$\left(\sum_{i=0}^{p} a_{ss}^{(i)}x^i\right)\left(\sum_{j=0}^{q} b_{ss}^{(j)}x^j\right) \in r_{R[x,\alpha]}(I[x]),$$

for each $1 \le s \le n$. Since I is an α-skew Armendariz left ideal of R, then $a_{ss}^{(i)}\alpha^i(b_{ss}^{(j)}) \in r_R(I)$ for any $1 \le i \le p$ and $1 \le j \le q$. Therefore

$$A_i\alpha^i(B_j) = \begin{pmatrix} a_{11}^{(i)}\alpha^i(b_{11}^{(j)}) & 0 & \ldots & 0 \\ 0 & a_{22}^{(i)}\alpha^i(b_{22}^{(j)}) & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & a_{nn}^{(i)}\alpha^i(b_{nn}^{(j)}) \end{pmatrix} \in r_{D_n(R)}(D_n(I)).$$

Thus it shows that $D_n(I)$ is an $\overline{\alpha}$-skew Armendariz left ideal of $D_n(R)$.

Proposition 2.4 is proved.

Every endomorphism σ of rings R and S can be extended to the endomorphism of rings $R[x]$ and $S[x]$ defined by $\sum_{i=0}^{m} a_ix^i \to \sum_{j=0}^{m} \sigma(a_i)x^i$, which we also denote by σ.

Proposition 2.5. *Let $\sigma\colon R \to S$ be a ring isomorphism, I_1 be an ideal of R and I_2 be an ideal of S with $\sigma(I_1) = I_2$. If I_2 is an $\sigma\alpha\sigma^{-1}$-skew Armendariz left ideal of ring S, then I_1 is an α-skew Armendariz left ideal of R.*

Proof. Let $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{m}b_jx^j\in R[x,\alpha]$ such that $f(x)g(x)\in$ $\in r_{R[x,\alpha]}(I_1[x])$. We set $f_1(x)=\sigma(f(x))=\sum_{i=0}^{m}\sigma(a_i)x^i=\sum_{i=0}^{m}a_i'x^i$ and $g_1(x)=\sigma(g(x))=$ $=\sum_{j=0}^{m}\sigma(b_j)x^j=\sum_{j=0}^{m}b_j'x^j\in S[x,\sigma\alpha\sigma^{-1}]$. First we shall show $f(x)g(x)\in r_{R[x,\alpha]}(I_1[x])$ implies that $f_1(x)g_1(x)\in r_{S[x,\sigma\alpha\sigma^{-1}]}(I_2[x])$. Let $I_1[x]f(x)g(x)=0$. From the definition of $f_1(x)$ and $g_1(x)$, we have $\sigma(I_1[x]f(x)g(x))=I_2[x]f_1(x)g_1(x)=0$. From the fact that I_2 is an $\sigma\alpha\sigma^{-1}$-skew Armendariz left ideal of ring S, we have $a_i'(\sigma\alpha\sigma^{-1})^ib_j'\in r_S(I_2)$. So that $(\sigma\alpha\sigma^{-1})^t=\sigma\alpha^t\sigma^{-1}$ we obtain $a_i'(\sigma\alpha^i\sigma^{-1})b_j'\in r_S(I_2)$. Since $\sigma(a_i)=a_i'$ and $\sigma(b_j)=b_j'$ and $\sigma(I_1)=I_2$, then $\sigma(I_1a_i\alpha^i(b_j))=0$. Since σ is an isomorphism, then $a_i\alpha^i(b_j)\in r_R(I_1)$. Clearly the other condition in definition is hold. Hence I_1 is an α-skew Armendariz left ideal of R.

Proposition 2.5 is proved.

As a result, we shall show that, under certain condition, the left ideals of the subring of upper triangular skew matrices over a ring R have an α-skew Armendariz structure.

Let $E_{ij}=(e_{st}\colon 1\le s,t\le n)$ denotes unit $(n\times n)$-matrices over ring R, in which $e_{ij}=1$ and $e_{st}=0$ when $s\ne i$ or $t\ne j$, $0\le i,\ j\le n$ for $n\ge 2$. If $V=\sum_{i=1}^{n-1}E_{i,i+1}$, then $V_n(R)=RI_n+RV+\ldots+RV^{n-1}$ is the subring of upper triangular skew matrices.

Corollary 2.2. *Suppose that α is an endomorphism of a ring R, $\theta\colon V_n(R)\to\dfrac{R[x]}{(x^n)}$ be a ring isomorphism, I_1 is a left ideal of $V_n(R)$ and I_2 is a left ideal of $\dfrac{R[x]}{(x^n)}$. If I_2 is an $\overline{\alpha}$-skew Armendariz left ideal of $\dfrac{R[x]}{(x^n)}$ and $\theta(I_1)=I_2$, then I_1 is an $\overline{\alpha}$-skew Armendariz left ideal of $V_n(R)$.*

Proof. Assume that I_2 is an $\overline{\alpha}$-skew Armendariz left ideal of $\dfrac{R[x]}{(x^n)}$ and define

$$\theta\colon V_n(R)\to\frac{R[x]}{(x^n)}$$

by

$$\theta\big(r_0I_n+r_1V+\ldots+r_{n-1}V^{n-1}\big)=r_0+r_1x+\ldots+r_{n-1}x^{n-1}+(x^n).$$

Now we have I_1 is a $\theta^{-1}\overline{\alpha}\theta$-skew Armendariz left ideal of $V_n(R)$ and that

$$\theta^{-1}\overline{\alpha}\theta(r_0I_n+r_1V+\ldots+r_{n-1}V^{n-1})=\overline{\alpha}(r_0I_n+r_1V+\ldots+r_{n-1}V^{n-1}),$$

which means that I_1 is an $\overline{\alpha}$-skew Armendariz left ideal of $V_n(R)$.

Corollary 2.2 is proved.

Recall that a ring is reduced if it has no nonzero nilpotent elements. In [2], α-reduced ring is introduced. A ring R is α-reduced, if for any $a,\,b\in R$

1) $ab=0$ implies $aR\cap Rb=0$,

2) $ab=0$ if and only if $a\alpha(b)=0$.

In this work we define reduced and α-reduced left ideals.

Definition 2.5. *A left ideal I of R is called reduced, if for any $a,\,b\in R$, $ab\in r_R(I)$, then $aR\cap Rb\subseteq r_R(I)$.*

Definition 2.6. *A left ideal I of R is called α-reduced, if for any $a,\,b\in R$, we have the following conditions*:

1) $ab \in r_R(I)$ *then* $aR \cap Rb \subseteq r_R(I)$,

2) $ab \in r_R(I)$ *if and only if* $a\alpha(b) \in r_R(I)$.

So the left ideal I is reduced if and only if it is 1-reduced.

Lemma 2.2. *If I is an α-reduced left ideal of R, then I is an α-semicommutative.*

Proof. Suppose $ab \in r_R(I)$ for any $a,\, b \in R$. Since I is an α-reduced left ideal of R then $aR \cap Rb \subseteq r_R(I)$. Because $aRb \subseteq aR \cap Rb$, then $aRb \subseteq r_R(I)$. Therefore I is an α-semicommutative.

Now by Lemma 2.2 we have the following lemma.

Lemma 2.3. *If I is an α-reduced left ideal of R, then I is α-Abelian.*

Proposition 2.6. *Let α be an endomorphism of a ring R and I be an α-reduced left ideal of R. Then I is an α-skew Armendarize left ideal.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x,\alpha]$ such that $f(x)g(x) \in$ $\in r_{R[x,\alpha]}(I[x])$. Then for each $h \in I$, $h\Big(\sum_{i+j=l} a_i \alpha^i(b_j)\Big) = 0$. Thus $\sum_{i+j=l} a_i \alpha^i(b_j) \in r_R(I)$ for $l = 0, \ldots, m+n$. So $ha_0b_0 = 0$. Thus $ha_0b_1b_0 = 0$, since I is α-semicommutative and $h(a_1\alpha(b_0) + a_0b_1) = 0$. Multiplying by b_0 on the right we have $h(a_1\alpha(b_0) + a_0b_1)b_0 = 0$. So we have $h(a_1\alpha(b_0)b_0) = 0$. Thus $h(a_1\alpha^2(b_0)) = 0$, and then $h(a_1\alpha(b_0)) = 0$, since I is α-reduced. Thus $ha_0b_1 = 0$. Assume that $s \geq 1$ and $h(a_i\alpha^i(b_j)) = 0$ for all i and j with $i + j \leq s$. Note that $h(a_0b_{s+1} + a_1\alpha(b_s) + \ldots + a_{s+1}\alpha^{s+1}(b_0)) = 0$, where a_i and b_j are 0 if $i > n$ and $j > m$. Multiplying by $\alpha^s(b_0)$ on the right yields

$$h(a_0b_{s+1}\alpha^s(b_0) + a_1\alpha(b_s)\alpha^s(b_0) + \ldots + a_{s+1}\alpha^{s+1}(b_0)\alpha^s(b_0)) = 0.$$

Since I is α-semicommutative and $h(a_i\alpha^i(b_0)) = 0$ for $i \leq s$, it follows that $h(a_iR\alpha^i(b_0)) = 0$. Thus $h(a_{s+1}\alpha^{s+1}(b_0)\alpha^s(b_0)) = h(a_{s+1}\alpha(\alpha^s(b_0))\alpha^s(b_0)) = 0$, which implies $h(a_{s+1}\alpha^{s+1}(b_0)) = 0$ by assumption. So

$$h(a_0b_{s+1} + a_1\alpha(b_s) + \ldots + a_s\alpha^s(b_1)) = 0.$$

Analogously, multiplying by $\alpha^{s-1}(b_1)$ on the right yields

$$h(a_0b_{s+1}\alpha^{s-1}(b_1) + a_1\alpha(b_s)\alpha^{s-1}(b_1) + \ldots + a_s\alpha^s(b_1)\alpha^{s-1}(b_1)) = 0.$$

The similar argument as the above reveals that $h(a_s\alpha^s(b_1)\alpha^{s-1}(b_1)) = 0$. Thus $h(a_s\alpha^s(b_1)) = 0$. Continuing this process, we have $ha_s\alpha^s(b_1) = \ldots = ha_1\alpha(b_s) = ha_0b_{s+1} = 0$. So we prove that $ha_i\alpha^i(b_j) = 0$ for all i and j with $i + j \leq s + 1$. By the induction principle, $ha_i\alpha^i(b_j) = 0$ for every i and j.

Proposition 2.6 is proved.

Definition 2.7. *A left ideal I of R is called symmetric, if for any $a,\, b,\, c \in R$, $abc \in r_R(I)$, then $acb \in r_R(I)$.*

Definition 2.8. *A left ideal I of R is called α-symmetric, if for any $a,\, b,\, c \in R$,*

1) $abc \in r_R(I)$ *then* $acb \in r_R(I)$,

2) $ab \in r_R(I)$ *if and only if* $a\alpha(b) \in r_R(I)$.

So the left ideal I is symmetric if and only if it is 1-symmetric.

Proposition 2.7. *If I is an α-symmetric left ideal of R, then I is an α-semicommutative.*

Proof. Suppose $ab \in r_R(I)$, for any $a,\, b \in R$. Thus $abr \in r_R(I)$, for any $r \in R$. So $arb \in r_R(I)$, since I is α-symmetric. Therefore I is an α-semicommutative.

Now by Proposition 2.7 we have the following corollary.

***Corollary* 2.3.** *If I is an α-symmetric left ideal of R, then I is an α-Abelian.*

There exists an α-Abelian right ideal which are also α-semicommutative, α-reduced and α-symmetric.

***Example* 2.5.** Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^{2\times 2}$ the full (2×2)-matrix ring over $\mathbb{Z}$,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \,\middle|\, a \equiv d(\mathrm{mod} 2),\ b \equiv 0(\mathrm{mod} 2) \right\}$$

and

$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \mathbb{Z}^{2\times 2} \,\middle|\, b \equiv 0(\mathrm{mod} 2) \right\}$$

be the right ideal of R. We have

$$l_R(I) = \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \,\middle|\, d \equiv 0(\mathrm{mod} 2), b \equiv 0(\mathrm{mod} 2) \right\}.$$

We define $\alpha\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. 0, 1 are only idempotents in R and for any $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in R$ and $B = \begin{pmatrix} c & e \\ 0 & h \end{pmatrix} \in R$, $AB \in l_R(I)$ if and only if $ac = 0$. Since $\mathbb{Z}$ is domain we have $a = 0$ or $c = 0$. If $a = 0$, then

$$A\alpha(B) = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \alpha\left(\begin{pmatrix} c & e \\ 0 & h \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & dh \end{pmatrix} \in l_R(I).$$

If $c = 0$, then

$$A\alpha(B) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \alpha\left(\begin{pmatrix} 0 & e \\ 0 & h \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & dh \end{pmatrix} \in l_R(I).$$

On the other hand if $A\alpha(B) \in l_R(I)$ therefore $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \alpha\left(\begin{pmatrix} c & e \\ 0 & h \end{pmatrix}\right) \in l_R(I)$, then $\begin{pmatrix} ac & bh \\ 0 & dh \end{pmatrix} \in l_R(I)$. So $ac = 0$ and similarly we have $AB \in l_R(I)$. Therefore I is α-Abelian right ideal of R. Now we show that I is α-semicommutative right ideal of R. For any $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $B = \begin{pmatrix} c & e \\ 0 & h \end{pmatrix}$ and $C = \begin{pmatrix} g & k \\ 0 & m \end{pmatrix} \in R$, let $AB \in l_R(I)$ thus $ac = 0$ and so $acg = agc = 0$, since $a, c, g \in \mathbb{Z}$. We have

$$ACB = \begin{pmatrix} agc & age + akh + bhm \\ 0 & dmh \end{pmatrix} = \begin{pmatrix} 0 & age + akh + bhm \\ 0 & dmh \end{pmatrix} \in l_R(I).$$

I is α-symmetric right ideal of R, since $ABC \in l_R(I)$ iff $acg = 0$, iff $agc = 0$. Therefore $ACB \in l_R(I)$. Now we show that I is α-reduced right ideal of R. Let $AB \in l_R(I)$, then $ac = 0$. Thus $a = 0$

or $c = 0$. Now if $X \in AR \cap RB$, then there exist $K = \begin{pmatrix} k_1 & k_2 \\ 0 & k_3 \end{pmatrix} \in R$ and $G = \begin{pmatrix} g_1 & g_2 \\ 0 & g_3 \end{pmatrix} \in R$, such that $X = AK = GB$. But $AK = \begin{pmatrix} ak_1 & ak_2 + bk_3 \\ 0 & dk_3 \end{pmatrix}$ and $GB = \begin{pmatrix} g_1 c & g_1 e + g_2 h \\ 0 & g_3 h \end{pmatrix}$. Thus $g_1 c = ak_1$. If $a = 0$, then $X = \begin{pmatrix} 0 & bk_3 \\ 0 & dk_3 \end{pmatrix} \in l_R(I)$ and if $c = 0$, then $X = \begin{pmatrix} 0 & g_2 h \\ 0 & g_3 h \end{pmatrix} \in l_R(I)$. Therefore I is α-reduced right ideal of R.

Recall that $r_R(\bigoplus I_i) = \bigcap r_R(I_i)$. Now we have the next proposition.

Proposition 2.8. *For any index set Γ, if I_i is an α-Abelian left ideal of R for each $i \in \Gamma$, then $\bigoplus_{i\in\Gamma} I_i$ is an α-Abelian left ideal of R.*

Theorem 2.2. *Suppose that I is left ideal a ring R and Δ is a multiplicatively closed subset of R consisting of central regular elements. We have the following conditions*:

1. *If I is an α-semicommutative left ideal of R, then $\Delta^{-1}I$ is an α-semicommutative left ideal of $\Delta^{-1}R$.*
2. *If I is an α-symmetric left ideal of R, then $\Delta^{-1}I$ is an α-symmetric left ideal of $\Delta^{-1}R$.*
3. *If I is an α-reduced left ideal of R, then $\Delta^{-1}I$ is an α-reduced left ideal of $\Delta^{-1}R$.*

Proof. We employ the method used in the proof of [8] (Proposition 3.1). For instance, we prove (1). Let $\beta\gamma \in r_{\Delta^{-1}R}(\Delta^{-1}I)$ with $\beta = u^{-1}a$, $\gamma = v^{-1}b$, $u,v \in \Delta$ and $a,b \in R$. Since Δ is contained in the center of R, we have $0 = \Delta^{-1}I\beta\gamma = \Delta^{-1}Iu^{-1}av^{-1}b = \Delta^{-1}Iab(uv)^{-1}$. So $Iab = 0$. It follows that $arb \in r_R(I)$ for all $r \in R$, since I is an α-semicommutative left ideal of R. Now for $\delta = w^{-1}r$ with $w \in \Delta$ and $r \in R$, $\Delta^{-1}I\beta\delta\gamma = \Delta^{-1}Iarb(uwv)^{-1} = 0$. Thus $\beta\delta\gamma \in r_{\Delta^{-1}R}(\Delta^{-1}I)$. Now suppose that $\beta\gamma \in r_{\Delta^{-1}R}(\Delta^{-1}I)$. Therefore $0 = \Delta^{-1}I\beta\gamma = {} = \Delta^{-1}Iu^{-1}av^{-1}b = \Delta^{-1}Iab(uv)^{-1}$ iff $Iab = 0$, iff $Ia\alpha(b) = 0$, iff $\Delta^{-1}Ia\alpha(b)(uv)^{-1} = 0$, iff $\beta\alpha(\gamma) \in r_{\Delta^{-1}R}(\Delta^{-1}I)$, since I is an α-semicommutative left ideal of R and α is endomorphism of R and $\alpha(\gamma) = v^{-1}\alpha(b)$. Hence $\Delta^{-1}I$ is an α-semicommutative left ideal of $\Delta^{-1}R$.

Theorem 2.2 is proved.

A ring of R is called right Ore if given $a,b \in R$ with b regular there exist a_1, $b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is a right Ore ring if and only if there exists a classical right quotient ring of R.

Theorem 2.3. *Suppose that there exists a classical right quotient Q of a ring R consisting of central elements. We have the following conditions*:

1. *I is an α-semicommutative left ideal of R if and only if QI is an α-semicommutative left ideal of Q.*
2. *I is an α-symmetric left ideal of R if and only if QI is an α-symmetric left ideal of Q.*
3. *I is an α-left reduced ideal of R if and only if QI is an α-reduced left ideal of Q.*

Proof. For instance, we prove (1). Let $\beta\gamma \in r_Q(QI)$ with $\beta = u^{-1}a$, $\gamma = v^{-1}b, u, v \in R$ and a, $b \in R$. Since Q is contained in the center of R, we have $0 = QI\beta\gamma = QIu^{-1}av^{-1}b = {} = QIab(uv)^{-1}$, so $Iab = 0$. It follows that $arb \in r_R(I)$ for all $r \in R$, since I is an α-semicommutative ideal of R. Now for $\delta = w^{-1}r$ with $w \in R$ and $r \in R$, $QI\beta\delta\gamma = QIarb(uwv)^{-1} = {} = 0$. Thus $\beta\delta\gamma \in r_Q(QI)$. Now suppose that $\beta\gamma \in r_Q(QI)$. Therefore $0 = QI\beta\gamma = QIu^{-1}av^{-1}b = {} = QIab(uv)^{-1}$ iff $Iab = 0$, iff $Ia\alpha(b) = 0$, iff $QIa\alpha(b)(uv)^{-1} = 0$, iff $\beta\alpha(\gamma) \in r_Q(QI)$, since I is an α-semicommutative ideal of R and α is endomorphism of R and $\alpha(\gamma) = v^{-1}\alpha(b)$. Hence QI is an α-semicommutative left ideal of Q.

Theorem 2.3 is proved.

Let α be an automorphism of a ring R. Suppose that there exists a classical right quotient Q of a ring R. Then for any $b^{-1}a \in Q$, where $a,\ b \in R$ with b regular the induced map $\overline{\alpha} : Q(R) \to Q(R)$ defined by $\overline{\alpha}(b^{-1}a) = (\alpha(b))^{-1}\alpha(a)$ is also an automorphism.

Proposition 2.9. *Suppose that there exists a classical right quotient Q of a ring R consisting of central elements. If I is α-semicommutative left ideal of R, then I is α-skew Armendariz left ideal of R if and only if QI is $\overline{\alpha}$-skew Armendariz left ideal of Q.*

Proof. Suppose that I is α-skew Armendariz. Let $f(x) = s_0^{-1}a_0 + s_1^{-1}a_1x + \ldots + s_m^{-1}a_mx^m$ and $g(x) = t_0^{-1}b_0 + t_1^{-1}b_1x + \ldots + \ t_n^{-1}b_nx^n \in QI[x, \overline{\alpha}]$ such that $f(x)g(x) \in r_{QI[x,\overline{\alpha}]}(QI[x])$. Let C be a left denominator set. There exist $s, t \in C$ and $a'_i, b'_j \in R$ such that $s_i^{-1}a_i = s^{-1}a'_i$ and $t_j^{-1}b_j = t^{-1}b'_j$ for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. Then $s^{-1}(a'_0 + a'_1x + \ldots + a'_mx^m)t^{-1}(b'_0 + b'_1x + \ldots + b'_nx^n) \in$ $\in r_{QI[x,\overline{\alpha}]}(QI[x])$. It follows that $(a'_0 + a'_1x + \ldots + a'_mx^m)t^{-1}(b'_0 + b'_1x + \ldots + b'_nx^n) \in r_{QI[x,\overline{\alpha}]}(QI[x])$. Thus $(a'_0t^{-1} + a'_1(\alpha(t))^{-1}x + \ldots + a'_m(\alpha^m(t))^{-1}x^m)(b'_0 + b'_1x + \ldots + b'_nx^n) \in r_{QI[x,\overline{\alpha}]}(QI[x])$. For $a'_i(\alpha^i(t))^{-1}$, $i = 0, 1, \ldots, n$, there exist $t' \in C$ and $a''_i \in R$ such that $a'_i(\alpha^i(t))^{-1} = t'^{-1}a''_i$. Hence $t'^{-1}(a''_0 + a''_1x + \ldots + a''_mx^m)(b'_0 + b'_1x + \ldots + b'_nx^n) \in r_{QI[x,\overline{\alpha}]}(QI[x])$. We have $(a''_0 +$ $+ a''_1x + \ldots + a''_mx^m)(b'_0 + b'_1x + \ldots + b'_nx^n) \in r_{R[x,\alpha]}(I[x])$. Since I is α-skew Armendariz, so $a''_i\alpha^i(b'_j) \in r_{R[x,\alpha]}(I[x])$ for all i and j. Since I is α-semicommutative, by Theorem 2.3, QI is α-semicommutative. Then $t'^{-1}a''_i\alpha^i(b'_j) \in r_Q(QI)$. So $a'_i\overline{\alpha}^i(t^{-1}b'_j) = (a'_i(\alpha^i(t)))^{-1}\alpha^i(b'_j) =$ $= ((t'^{-1}a''_i)\alpha^i(b'_j)) \in r_Q(QI)$. Similarly we have $(s_i^{-1}a'_i)\overline{\alpha}^i(t_j^{-1}b'_j) = (s_i^{-1}a'_i)\overline{\alpha}^i(t^{-1}b'_j) \in r_Q(QI)$. Let $\beta\gamma \in r_Q(QI)$ with $\beta = u^{-1}a$, $\gamma = v^{-1}b$, $u, v \in R$ and $a, b \in R$. Therefore $0 = QI\beta\gamma =$ $= QIu^{-1}av^{-1}b = QIab(uv)^{-1}$ iff $Ia\alpha(b) = 0$, iff $QIa\alpha(b)(uv)^{-1} = 0$, iff $QIa\alpha(b)u^{-1}(\alpha(v))^{-1} =$ $= 0$, iff $QI(u^{-1}a)((\alpha(v))^{-1}\alpha(b)) = 0$, iff $QI\beta\overline{\alpha}(\gamma) = 0$, since I is α-skew Armendariz and α is an automorphism of R and Q is contained in the center of R. Thus QI is α-skew Armendariz. The converse is clear.

Proposition 2.9 is proved.

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