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A CLASS OF p -VALENT MEROMORPHIC FUNCTIONS DEFINED BY THE LUI–SRIVASTAVA OPERATOR

ПРО КЛАС p -ВАЛЕНТНИХ МЕРОМОРФНИХ ФУНКЦІЙ, ЩО ВИЗНАЧЕНІ ОПЕРАТОРОМ ЛУІ–ШРІВАСТАВИ

We introduce a subclass of p -valent meromorphic functions involving the Lui–Srivastava operator and investigate various properties of this subclass. We also indicate the relationships between various results presented in the paper with the results obtained in earlier works.

Введено підклас p -валентних мероморфних функцій, що визначаються оператором Луї–Шривастави, та вивчено різноманітні властивості цього підкласу. Також вказано співвідношення між різноманітними результатами, що отримані в роботі, та результатами, отриманими раніше.

1. Introduction. Let Σ_p denote the class of all meromorphic functions f of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p}, \quad p \in \mathbb{N} = \{1, 2, \dots\}, \quad (1.1)$$

which are analytic and p -valent in the punctured disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$. Let $\Sigma_p^*(\lambda)$ denote the class of all meromorphic p -valent starlike of order λ ($0 \leq \lambda < p$) in U .

For functions $f \in \Sigma_p$, given by (1.1), and $g \in \Sigma_p$ defined by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p}, \quad p \in \mathbb{N},$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

For complex parameters $\alpha_1, \dots, \alpha_q$ and β_1, \dots, β_s ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$; $j = 1, 2, \dots, s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ by (see, for example, [9, p.19])

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k z^k}{(\beta_1)_k \dots (\beta_s)_k k!},$$

$$q \leq s + 1, \quad q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad z \in U,$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ , by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1, & \nu = 0, \quad \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \\ \theta(\theta - 1) \dots (\theta + \nu - 1), & \nu \in \mathbb{N}, \quad \theta \in \mathbb{C}. \end{cases}$$

Corresponding to the function $h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$, defined by

$$h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z^{-p} {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z),$$

we consider a linear operator

$$H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s): \Sigma_p \rightarrow \Sigma_p,$$

which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) = h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z).$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) = z^{-p} + \sum_{k=1}^{\infty} \Gamma_{p,q,s}(\alpha_1) a_k z^{k-p}, \quad (1.2)$$

where

$$\Gamma_{p,q,s}(\alpha_1) = \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k k!}. \quad (1.3)$$

If, for convenience, we write

$$H_{p,q,s}[\alpha_1] = H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s),$$

then one can easily verify from the definition (1.2) that (see [5])

$$z(H_{p,q,s}[\alpha_1]f(z))' = \alpha_1 H_{p,q,s}[\alpha_1 + 1]f(z) - (\alpha_1 + p)H_{p,q,s}[\alpha_1]f(z). \quad (1.4)$$

The linear operator $H_{p,q,s}[\alpha_1]$ was investigated recently by Liu and Srivastava [5] and Aouf [2].

In particular, for $q = 2$, $s = 1$, $\alpha_1 > 0$, $\beta_1 > 0$ and $\alpha_2 = 1$, we obtain the linear operator

$$H_p(\alpha_1, 1; \beta_1)f(z) = \ell_p(\alpha_1, \beta_1)f(z),$$

which was introduced and studied by Liu and Srivastava [4].

We note that,

$$H_{p,2,1}(n+p, 1; 1)f(z) = D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z), \quad n > -p,$$

where D^{n+p-1} is the differential operator studied by Uralegaddi and Somanatha [10] and Aouf [1].

Making use of the operator $H_{p,q,s}[\alpha_1]$, we now introduce a subclass of the function class Σ_p as follows:

we say that a function $f \in \Sigma_p$ is in the class $\Omega_{p,q,s}(\alpha_1; \lambda)$, if it satisfies the following inequality:

$$\operatorname{Re} \left\{ \alpha_1 \frac{H_{p,q,s}[\alpha_1 + 1]f(z)}{H_{p,q,s}[\alpha_1]f(z)} - (\alpha_1 + p) \right\} < -\lambda, \quad 0 \leq \lambda < p, \quad p \in \mathbb{N},$$

or, in view of (1.4), if it satisfies the following inequality:

$$\operatorname{Re} \left\{ \frac{z(H_{p,q,s}[\alpha_1]f(z))'}{H_{p,q,s}[\alpha_1]f(z)} \right\} < -\lambda, \quad 0 \leq \lambda < p, \quad p \in \mathbb{N}.$$

2. Main results. In order to establish our main results, we need the following lemma.

Lemma 2.1 [3]. *Let $w(z)$ be a non-constant analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0 , then we have*

$$z_0 w'(z_0) = \zeta w(z_0), \quad (2.1)$$

where $\zeta \geq 1$ is a real number.

Theorem 2.1. *Let $\alpha_1 \geq 0$ and $0 \leq \lambda < p$, then*

$$\Omega_{p,q,s}(\alpha_1 + 1; \lambda) \subset \Omega_{p,q,s}(\alpha_1; \lambda).$$

Proof. Let $f \in \Omega_{p,q,s}(\alpha_1 + 1; \lambda)$, then

$$\operatorname{Re} \left\{ \frac{z(H_{p,q,s}[\alpha_1 + 1]f(z))'}{H_{p,q,s}[\alpha_1 + 1]f(z)} \right\} < -\lambda, \quad z \in U. \quad (2.2)$$

We have to show that implies the inequality

$$\operatorname{Re} \left\{ \frac{z(H_{p,q,s}[\alpha_1]f(z))'}{H_{p,q,s}[\alpha_1]f(z)} \right\} < -\lambda, \quad z \in U. \quad (2.3)$$

Define the function $w(z)$ in U by

$$\frac{z(H_{p,q,s}[\alpha_1]f(z))'}{H_{p,q,s}[\alpha_1]f(z)} = -\frac{p + (p - 2\lambda)w(z)}{1 - w(z)}. \quad (2.4)$$

Clearly, $w(z)$ is analytic in U and $w(0) = 0$. Using the identity (1.4), (2.4) may be written as

$$\alpha_1 \frac{H_{p,q,s}[\alpha_1 + 1]f(z)}{H_{p,q,s}[\alpha_1]f(z)} = \frac{\alpha_1 - [\alpha_1 + 2(p - \lambda)]w(z)}{1 - w(z)}. \quad (2.5)$$

Differentiating (2.5) logarithmically with respect to z and using (1.4), we obtain

$$\begin{aligned} & \frac{z(H_{p,q,s}[\alpha_1 + 1]f(z))'}{H_{p,q,s}[\alpha_1 + 1]f(z)} + \lambda = \\ & = -(p - \lambda) \left\{ \frac{1 + w(z)}{1 - w(z)} + \frac{2zw'(z)}{(1 - w(z))(\alpha_1 - [\alpha_1 + 2(p - \lambda)]w(z))} \right\}. \end{aligned} \quad (2.6)$$

We claim that $|w(z)| < 1$ in U . For otherwise, there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Applying Lemma 2.1 to $w(z)$ at the point z_0 , we have $z_0 w'(z_0) = \zeta w(z_0)$ where $\zeta \geq 1$. So, (2.6) yields

$$\operatorname{Re} \left\{ \frac{z(H_{p,q,s}[\alpha_1 + 1]f(z_0))'}{H_{p,q,s}[\alpha_1 + 1]f(z_0)} + \lambda \right\} =$$

$$= -(p - \lambda) \operatorname{Re} \left\{ \frac{1 + w(z_0)}{1 - w(z_0)} + \frac{2\zeta w(z_0)}{(1 - w(z_0))(\alpha_1 - [\alpha_1 + 2(p - \lambda)]w(z_0))} \right\} \geq \\ \geq \frac{p - \lambda}{2(\alpha_1 + p - \lambda)} > 0,$$

which contradicts the inequality (2.2). Hence, $|w(z)| < 1$ in U and it follows that $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$.

Theorem 2.1 is proved.

Theorem 2.2. Let $\delta > 0$ and $f(z) \in \Sigma_p$ satisfy the following inequality:

$$\operatorname{Re} \left\{ \frac{z(H_{p,q,s}[\alpha_1]f(z))'}{H_{p,q,s}[\alpha_1]f(z)} \right\} < -\lambda + \frac{p - \lambda}{2(\delta + \lambda p - \lambda)}, \quad z \in U. \quad (2.7)$$

Then the function $\mathcal{F}_{\delta,p}(f)$ defined by

$$\mathcal{F}_{\delta,p}(f)(z) = \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} f(t) dt, \quad \delta > 0, \quad (2.8)$$

belongs to $\Omega_{p,q,s}(\alpha_1; \lambda)$.

Proof. From (2.8), we readily have

$$z(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z))' = \delta H_{p,q,s}[\alpha_1]f(z) - (\delta + p)H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z). \quad (2.9)$$

Using the identity (1.4) and (2.9), condition (2.7) may be written as

$$\operatorname{Re} \left\{ \alpha_1 \frac{H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z)}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} \frac{\frac{z(H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z))' + \delta + p}{H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z)}}{\frac{z(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z))' + \delta + p}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)}} - \alpha_1 - p \right\} < \\ < -\lambda + \frac{p - \lambda}{2(\delta + \lambda p - \lambda)}. \quad (2.10)$$

We have to prove that $\mathcal{F}_{\delta,p}(f) \in \Omega_{p,q,s}(\alpha_1; \lambda)$ implies the inequality

$$\operatorname{Re} \left\{ \frac{z(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z))'}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} \right\} < -\lambda, \quad 0 \leq \lambda < p, \quad z \in U. \quad (2.11)$$

Consider the function $w(z)$ in U defined by

$$\frac{z(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z))'}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} = -\frac{p + (p - 2\lambda)w(z)}{1 - w(z)}, \quad 0 \leq \lambda < p, \quad z \in U. \quad (2.12)$$

Clearly, $w(z)$ is analytic and $w(0) = 0$. (2.12) may be written as

$$\alpha_1 \frac{H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z)}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} = \frac{\alpha_1 - [\alpha_1 + 2(p - \lambda)]w(z)}{1 - w(z)}. \quad (2.13)$$

Differentiating (2.13) logarithmically with respect to z and using (2.12), we obtain

$$\begin{aligned} & \frac{z(H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z))'}{H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z)} = \\ & = -\frac{p + (p - 2\lambda)w(z)}{1 - w(z)} - \frac{2(p - \lambda)zw'(z)}{(1 - w(z))(\alpha_1 - [\alpha_1 + 2(p - \lambda)]w(z))}. \end{aligned} \tag{2.14}$$

Using (2.12)–(2.14) and (2.10), we get

$$\begin{aligned} & \alpha_1 \frac{H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z)}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} \frac{\frac{z(H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z))'}{H_{p,q,s}[\alpha_1 + 1]\mathcal{F}_{\delta,p}(f)(z)} + \delta + p}{\frac{z(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z))'}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} + \delta + p} - \alpha_1 - p + \lambda = \\ & = -(p - \lambda) \left\{ \frac{1 + w(z)}{1 - w(z)} + \frac{2zw'(z)}{(1 - w(z))(\delta - [\delta + 2(p - \lambda)]w(z))} \right\}. \end{aligned} \tag{2.15}$$

The remaining part of the proof is similar to that of Theorem 2.1.

Theorem 2.2 is proved.

According to Theorem 2.2, we have the following corollary.

Corollary 2.1. *If $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$, then the function $\mathcal{F}_{\delta,p}(f)$ defined (2.8) also belongs to $\Omega_{p,q,s}(\alpha_1; \lambda)$.*

Theorem 2.3. *If $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$ if and only if the function g defined by*

$$g(z) = \frac{\alpha_1}{z^{\alpha_1+p}} \int_0^z t^{\alpha_1+p-1} f(t) dt, \quad \alpha_1 > 0, \tag{2.16}$$

belongs to $\Omega_{p,q,s}(\alpha_1 + 1; \lambda)$.

Proof. From (2.16), we have

$$z(\mathcal{F}_{\delta,p}(f)(z))' = \alpha_1 f(z) - (\alpha_1 + p)\mathcal{F}_{\delta,p}(f)(z). \tag{2.17}$$

Using identity (1.4) and (2.17), hence

$$H_{p,q,s}[\alpha_1]f(z) = H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z) \tag{2.18}$$

and the result follows.

To prove Theorem 2.4, we need the following lemmas.

Lemma 2.2 [8]. *The function $(1 - z)^\gamma = \exp(\gamma \log(1 - z))$, $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, is univalent in U if and only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disc $|\gamma + 1| \leq 1$.*

Lemma 2.3 [7]. *Let $q(z)$ be univalent in U and let $Q(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

- (1) $Q(z)$ is starlike (univalent) in U ,
- (2) $\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0, z \in U$.

If $g(z)$ is analytic in U , with $p(0) = q(0)$, $p(U) \subset D$, and

$$\theta(g(z)) + zg'(z)\phi(g(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \quad (2.19)$$

then $g(z) \prec q(z)$, and $q(z)$ is the best dominant of (2.19).

Theorem 2.4. Let $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$ and $\gamma \in \mathbb{C}^*$ and satisfy either

$$|2\gamma(p - \lambda) - 1| \leq 1 \quad \text{or} \quad |2\gamma(p - \lambda) + 1| \leq 1. \quad (2.20)$$

Then

$$(z^p H_{p,q,s}[\alpha_1]f(z))^\gamma \prec (1 - z)^{2\gamma(p-\lambda)} = q(z), \quad (2.21)$$

and $q(z)$ is the best dominant.

Proof. Set

$$g(z) = (z^p H_{p,q,s}[\alpha_1]f(z))^\gamma, \quad z \in U, \quad (2.22)$$

then $g(z)$ is analytic in U with $g(0) = 1$. Differentiating (2.22) logarithmically with respect to z , we obtain

$$\frac{zg'(z)}{g(z)} = \gamma \left[\frac{z(H_{p,q,s}[\alpha_1]f(z))'}{H_{p,q,s}[1]f(z)} + p \right]. \quad (2.23)$$

Since $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$, this is equivalent to

$$\frac{z(H_{p,q,s}[\alpha_1]f(z))'}{H_{p,q,s}[\alpha_1]f(z)} \prec -\frac{p + (p - 2\lambda)z}{1 - z}, \quad (2.24)$$

from (2.23), (2.24) can be rewritten as

$$-p + \frac{zg'(z)}{\gamma g(z)} \prec -p + \frac{z((1 - z)^{2\gamma(p-\lambda)})'}{(1 - z)^{2\gamma(p-\lambda)}}. \quad (2.25)$$

On the other hand, if we take

$$q(z) = (1 - z)^{2\gamma(p-\lambda)}, \quad \theta(z) = -p, \quad \phi(w) = \frac{1}{\gamma w}$$

in Lemma 2.3, then $q(z)$ is univalent by the condition (2.20) and Lemma 2.2. It is easy to see that $q(z)$, $\theta(w)$, and $\phi(w)$ satisfy the conditions of Lemma 2.3. Since

$$Q(z) = zq'(z)\phi(q(z)) = -\frac{2(p - \lambda)z}{1 - z}$$

is univalent starlike in U and

$$h(z) = \theta(q(z)) + Q(z) = -\frac{p + (p - 2\lambda)z}{1 - z},$$

from (2.25) and Lemma 2.3, then

$$g(z) \prec (1 - z)^{2\gamma(p-\lambda)} = q(z)$$

and the function $(1 - z)^{2\gamma(p-\lambda)}$ is the best dominant.

Theorem 2.4 is proved.

Corollary 2.2. *Let $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$. Then*

$$\operatorname{Re} \left\{ (z^p H_{p,q,s}[\alpha_1] f(z))^\gamma \right\} > 2^{2\gamma(p-\lambda)}, \quad z \in U,$$

where γ is a real number and $\gamma \in \left[-\frac{1}{2(p-\lambda)}, 0 \right)$. The result is sharp.

Proof. From Theorem 2.4, we have

$$\operatorname{Re} \left\{ (z^p H_{p,q,s}[\alpha_1] f(z))^\gamma \right\} = \operatorname{Re} \left\{ (1 - w(z))^{2\gamma(p-\lambda)} \right\}, \quad z \in U, \tag{2.26}$$

where $w(z)$ is analytic in U , $w(0) = 0$, and $|w(z)| < 1$ for $z \in U$. In view of

$$\operatorname{Re}(t^b) \geq (\operatorname{Re} t)^b, \quad \operatorname{Re}(t) > 0, \quad 0 < b \leq 1,$$

(2.26) yields

$$\operatorname{Re} \left\{ (z^p H_{p,q,s}[\alpha_1] f(z))^\gamma \right\} \geq \left\{ \operatorname{Re} \left(\frac{1}{1 - w(z)} \right)^{-2\gamma(p-\lambda)} \right\} > 2^{2\gamma(p-\lambda)}, \quad z \in U,$$

for $-1 \leq 2\gamma(p-\lambda) < 0$. To see that the bound $2^{2\gamma(p-\lambda)}$ cannot be increased, we consider the function $f(z)$ which satisfies

$$z^p H_{p,q,s}[\alpha_1] f(z) = (1 - z)^{2(p-\lambda)}, \quad 0 \leq \lambda < p, \quad z \in U.$$

We easily have $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$ and

$$\operatorname{Re} \left\{ (z^p H_{p,q,s}[\alpha_1] f(z))^\gamma \right\} \rightarrow 2^{2\gamma(p-\lambda)} \quad \text{as} \quad \operatorname{Re}(z) \rightarrow -1.$$

Corollary 2.2 is proved.

3. Convolution conditions. We give some necessary and sufficient condition in terms of convolution operator for meromorphic functions to be in the classes $\mathcal{S}_p^*(\lambda)$ and $\Omega_{p,q,s}(\alpha_1; \lambda)$.

Lemma 3.1. *The function $f(z) \in \Sigma_p$ belongs to the class $\Sigma \mathcal{S}_p^*(\lambda)$ ($0 \leq \lambda < p$) if and only if*

$$z^p \left[f(z) * \frac{\left(1 - \frac{1 - e^{-i\theta} + 2(p-\lambda)}{2(p-\lambda)} z \right)}{z^p(1-z)^2} \right] \neq 0, \quad 0 < \theta < 2\pi, \quad z \in U.$$

Proof. A function $f(z) \in \Sigma \mathcal{S}_p^*(\lambda)$ if and only if

$$\frac{zf'(z)}{f(z)} \neq -\frac{p + (p - 2\lambda)e^{i\theta}}{1 - e^{i\theta}}, \quad 0 < \theta < 2\pi, \quad z \in U,$$

which is equivalent to

$$z^p \left[\left(1 - e^{i\theta} \right) z f'(z) + \left[p + (p - 2\lambda)e^{i\theta} \right] f(z) \right] \neq 0, \quad 0 < \theta < 2\pi, \quad z \in U. \tag{3.1}$$

Since

$$f(z) = f(z) * \frac{1}{z^p(1-z)} \quad \text{and} \quad zf'(z) = f(z) * \left(\frac{(p+1)z-p}{z^p(1-z)^2} \right).$$

Therefore, we may write (3.1) as

$$\begin{aligned} & z^p \left[(1 - e^{i\theta}) zf'(z) + [p + (p - 2\lambda)e^{i\theta}] f(z) \right] = \\ & = z^p \left\{ f(z) * \left[\frac{1 - e^{i\theta}}{z^p(1-z)^2} + \frac{p + (p - 2\lambda)e^{i\theta}}{z^p(1-z)} \right] \right\} = \\ & = 2(p - \lambda)e^{i\theta} z^p \left[f(z) * \frac{\left(1 - \frac{1 - e^{-i\theta} + 2(p - \lambda)}{2(p - \lambda)} z \right)}{z^p(1-z)^2} \right] \neq 0. \end{aligned}$$

Lemma 3.1 is proved.

Theorem 3.1. *A necessary and sufficient condition for the function $f(z)$ defined by (1.1) to be in the class $\Omega_{p,q,s}(\alpha_1; \lambda)$ is that*

$$1 - \sum_{k=1}^{\infty} \frac{(1 - e^{-i\theta})k - 2(p - \lambda)}{2(p - \lambda)} \Gamma_{p,q,s}(\alpha_1) a_k z^k \neq 0, \quad 0 < \theta < 2\pi, \quad z \in U,$$

where $\Gamma_{p,q,s}(\alpha_1)$ is given by (1.3).

Proof. From Lemma 3.1, we find that $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$ if and only if

$$z^p \left[H_{p,q,s}[\alpha_1]f(z) * \frac{\left(1 - \frac{1 - e^{-i\theta} + 2(p - \lambda)}{2(p - \lambda)} z \right)}{z^p(1-z)^2} \right] \neq 0, \quad 0 < \theta < 2\pi, \quad 0 \leq \lambda < p, \quad z \in U. \quad (3.2)$$

From (1.2), the left-hand side of (3.2) may be written as

$$\begin{aligned} & z^p \left[H_{p,q,s}[\alpha_1]f(z) * \frac{\left(1 - \frac{1 - e^{-i\theta} + 2(p - \lambda)}{2(p - \lambda)} z \right)}{z^p(1-z)^2} \right] = \\ & = 1 - \sum_{k=1}^{\infty} \frac{(1 - e^{-i\theta})k - 2(p - \lambda)}{2(p - \lambda)} \Gamma_{p,q,s}(\alpha_1) a_k z^k \neq 0. \end{aligned}$$

Theorem 3.1 is proved.

Remarks. 3.1. Putting $q = 2$, $s = \alpha_2 = \beta_1 = 1$ and $\alpha_1 = n + p$ ($n > -p$) in Theorem 2.1, we obtain the results obtained by Aouf [1] (Theorem 1).

3.2. Putting $q = 2$, $s = \alpha_2 = \beta_1 = 1$, $\delta = c - p + 1$ ($c > p - 1$) and $\alpha_1 = n + p$ ($n > -p$) in Theorem 2.2, we obtain the results obtained by Aouf [1] (Theorem 2).

3.3. Putting $q = 2$, $s = \alpha_2 = \beta_1 = 1$ and $\alpha_1 = n + p(n > -p)$ in Theorem 2.3, we improve the result obtained by Aouf [1] (Theorem 2.3).

3.4. Taking $q = 2$, $s = \alpha_2 = 1$ and $\alpha_1, \beta_1 > 0$ in Theorems 2.1–2.4, Corollaries 2.1 and 2.2, respectively, we obtain the results obtained by Liu and Owa [6] (Theorems 2.2, 2.3, 2.5, 2.9, Corollaries 2.4 and 2.10, respectively).

3.5. Taking $e^{-i\theta} = -x$ in Lemma 3.1, we obtain the results obtained by Liu and Owa [6] (Lemma 3.1).

3.6. Taking $q = 2$, $s = \alpha_2 = 1$, $\alpha_1, \beta_1 > 0$ and $e^{-i\theta} = -x$ in Theorem 3.1, we obtain the result obtained by Liu and Owa [6] (Theorem 3.2).

1. *Aouf M. K.* New criteria for multivalent meromorphic starlike functions of order α // Proc. Jap. Acad. A. – 1993. – **69**. – P. 66–70.
2. *Aouf M. K.* Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function // Comput. Math. and Appl. – 2008. – **55**, № 3. – P. 494–509.
3. *Jack I. S.* Functions starlike and convex of order α // J. London Math. Soc. – 1971. – **2**, № 3. – P. 469–474.
4. *Liu J.-L., Srivastava H. M.* A linear operator and associated families of meromorphically multivalent functions // J. Math. Anal. and Appl. – 2000. – **259**. – P. 566–581.
5. *Liu J.-L., Srivastava H. M.* Classes of meromorphically multivalent functions associated with the generalized hypergeometric function // Math. Comput. Modelling. – 2004. – **39**. – P. 21–34.
6. *Liu J.-L., Owa S.* On a classes of meromorphic p -valent starlike functions involving certain linear operators // Int. J. Math. and Math. Sci. – 2002. – **32**. – P. 271–280.
7. *Miller S. S., Mocanu P. T.* On some classes of first-order differential subordinations // Mich. Math. J. – 1985. – **32**, № 2. – P. 185–195.
8. *Robertson M. S.* Certain classes of starlike functions // Mich. Math. J. – 1985. – **32**, № 2. – P. 135–140.
9. *Srivastava H. M., Karlsson P. W.* Multiple Gaussian hypergeometric series. – New York etc.: Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, 1985.
10. *Uralegaddi B. A., Somanatha C.* Certain classes of meromorphic multivalent functions // Tamkang J. Math. – 1992. – **23**. – P. 223–231.

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